Some Remarks on $C^*$-Convexity

Richard I. Loeb

Department of Mathematics
Wayne State University
Detroit, Michigan 48202

and

Vern I. Paulsen*

Department of Mathematics
University of Houston
Houston, Texas 77004

Submitted by Chandler Davis

ABSTRACT

Recently the study of completely positive maps has become important to the results of Brown, Douglas, and Fillmore on $\text{Ext}(\mathcal{A})$, $\mathcal{A}$ a $C^*$-algebra. Attempts to solve questions related to $\text{Ext}$ have often turned into questions about the matrix algebras $M_n$. In this paper we wish to discuss a notion of $C^*$-convexity related to completely positive linear maps, to state some facts about $C^*$-convexity, and to ask some questions about $C^*$-convexity. To a large degree, the tone of this paper is expository.

We shall let $\mathcal{L}^b(\mathcal{H})$ denote the algebra of bounded linear operators on a (separable) Hilbert space $\mathcal{H}$, and let $M_n$ denote the algebra of complex $n \times n$ matrices. Our general references are [8] and [14].

*Research conducted with partial support of the NSF.
DEFINITION 1. A set $\mathcal{K} \subseteq \mathcal{L}(\mathcal{K})$ is C*-convex if $\{x_1, \ldots, x_n\} \subseteq \mathcal{K}$, and $\{A_1, \ldots, A_n\} \subseteq \mathcal{L}(\mathcal{K})$ with $\Sigma_{i=1}^n A_i^* A_i = 1$ implies that $\Sigma_{i=1}^n A_i^* x_i A_i \in \mathcal{K}$.

Let us give some examples of C*-convex sets. Notice that $\{x\}$ is C*-convex if and only if $x = \lambda 1$ for some scalar $\lambda$.

**Example 1.** Let $\mathcal{B} = \{T: 0 < T < 1\}$; then $\mathcal{B}$ is C*-convex, for clearly $0 < \Sigma A_i^* T_i A_i$, but $T_i < 1 \Rightarrow A_i^* T_i A_i < A_i^* A_i$, so $\Sigma A_i^* T_i A_i < \Sigma A_i^* A_i = 1$.

**Example 2.** Let $\mathcal{K}$ be the set of $T \in \mathcal{L}(\mathcal{K})$ such that the numerical radius $w(T) < r$, for $r > 0$. Then for $z \in \mathcal{K}$, $\|z\| = 1$, we have $|\langle \Sigma A_i^* T_i A_i z, z \rangle| < \Sigma |\langle T_i A_i z, A_i z \rangle| < r \Sigma \|A_i z\|^2 = r$.

**Example 3.** Let $\mathcal{B}_R = \{T: \|T\| < R\}$; then on $\mathcal{L}(\mathcal{K}) \otimes M_n$, we have

\[
\left[\begin{array}{ccc}
A_1^* & \cdots & A_n^*
\end{array}\right] \left[\begin{array}{cccc}
T_1 & & & 0 \\
& \ddots & & \\
0 & & \ddots & \\
& & 0 & T_n
\end{array}\right]_{n \times n} \left[\begin{array}{c}
A_1 \\
\vdots \\
A_n
\end{array}\right]_{n \times n} = \left[\begin{array}{c}
\Sigma A_i^* T_i A_i \\
\vdots \\
0
\end{array}\right]_{n \times n} \left[\begin{array}{c}
A_1 \\
\vdots \\
A_n
\end{array}\right]_{n \times n} \left[\begin{array}{c}
T_1 \\
\vdots \\
T_n
\end{array}\right]_{n \times n} \left[\begin{array}{c}
A_1 \\
\vdots \\
A_n
\end{array}\right]_{n \times n}
\]

Before the next example, we remind the reader that a linear map $\varphi$ between C*-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be completely positive if for all $n,$
the map $\varphi_n = \varphi \otimes \text{id}_n : \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n$ is positive. A well-known theorem of Stinespring characterizes completely positive maps.

**Theorem 2** [16]. The linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is completely positive if and only if there is a *-representation $\pi$ of $\mathcal{A}$ on $\mathcal{L}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, and a map $V : \mathcal{H} \to \mathcal{K}$ such that $\varphi(A) = V^* \pi(A) V$ for all $A \in \mathcal{A}$.

**Example 4.** In [2,3], Arveson defined the $n$th matrix range of an operator $T$, denoted $\mathcal{M}_n(T)$, as $\mathcal{M}_n(T) = \{\varphi(T) : \varphi$ is a completely positive map from $C^*(T) \to M_n$ with $\varphi(1) = 1\}$. We claim that $\mathcal{M}_n(T)$ is $C^*$-convex; for if $\varphi_1, \ldots, \varphi_n$ are completely positive maps as above, and $\sum A_i A_i^* = 1$, then $\psi(A) = \sum A_i \varphi_i(A_i)$ is another such completely positive map. By Stinespring's theorem it is evident that each summand $A_i \varphi_i A_i$ is completely positive, and it is easy to see that the sum of completely positive maps is completely positive. By Stinespring's theorem each $\varphi_1 = V_i^* \pi_i V_i$, and $\varphi_i(1) = 1$ implies $V_i^* V_i = 1$; but then $\sigma(1) = \sum A_i^* V_i^* V_i A_i = \sum A_i^* (1) A_i = 1$. Thus if $T_1, \ldots, T_n \in \mathcal{M}_n(T)$, then $\psi = \varphi_i(T)$, where $\varphi_i$ are as above, and $\sum A_i^* T_i A_i = \sigma(T) \in \mathcal{M}_n(T)$.

We remark that Examples 1, 2, and 3 are all special cases of Example 4 [2,13].

**Remark 1.** If $\mathcal{K}$ is $C^*$-convex, then $\mathcal{K}$ is convex in the usual sense.

**Remark 2.** If $\mathcal{K}$ is $C^*$-convex and $K \in \mathcal{K}$, then $\{V^* K V : V^* V = 1\} \subseteq \mathcal{K}$ and $\{V K V^* : V V^* = 1\} \subseteq \mathcal{K}$. In particular, if $\mathcal{K}$ is $C^*$-convex, $K \in \mathcal{K}$, and $L$ is unitarily equivalent to $K$, then $L \in \mathcal{K}$.

**Remark 3.** From Remark 2, it is easy to see that the segment $[0, A] = \{T : 0 < T < A\}$ is not in general $C^*$-convex, although the segment is convex in the usual sense [10]. For example, if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is unitarily equivalent to $A$, but $B \notin A$ (and also $A \notin B$).
Remark 4. The notion of \( C^* \)-convexity is unchanged by translation by a fixed scalar, that is, \( \mathcal{K} + \alpha 1 \) is \( C^* \)-convex if and only if \( \mathcal{K} \) is. Thus if \( \alpha 1 \in \mathcal{K} \), then for purposes of \( C^* \)-convexity we can assume \( 0 \in \mathcal{K} \). However, translation by scalars is apparently the only allowable translation in the study of \( C^* \)-convexity, in contrast with the usual study of convexity [10].

**Definition 3.** If \( \mathcal{S} \subseteq \mathcal{L}(\mathcal{K}) \), let \( \text{MCL}(\mathcal{S}) \) denote the smallest norm-closed \( C^* \)-convex set containing \( \mathcal{S} \).

Remark 5. It is easy to see that \( \text{MCL}(\mathcal{S}) = \cap \mathcal{K} \), the intersection taken over all norm-closed \( C^* \)-convex sets \( \mathcal{K} \supseteq \mathcal{S} \). Note also that \( \text{MCL}(\mathcal{S}^*) = \text{MCL}(\mathcal{S})^* \).

**Lemma 4.** If \( \mathcal{K} \) is \( C^* \)-convex, so is its norm closure.

*Proof.* Let \( T_1, \ldots, T_n \in \mathcal{K} \), and let \( A_1, \ldots, A_n \) be such that \( \sum A_n^* A_i = 1 \). Then for each \( i \), \( A_i^* A_i \leq 1 \), so \( \| A_i \| < 1 \). Let \( \varepsilon > 0 \), and for each \( i \), let \( S_i \) be an element of \( \mathcal{K} \) with \( \| T_i - S_i \| < \varepsilon/n \). Then by hypothesis we have \( \sum A_i^* S_i A_i \in \mathcal{K} \), and furthermore \( \| \sum A_i^* T_i A_i - \sum A_i^* S_i A_i \| \leq \varepsilon \| \sum A_i^* (T_i - S_i) A_i \| < \varepsilon/n = \varepsilon \). Hence \( \sum A_i^* T_i A_i \in \mathcal{K} \), as was to be proved.

Let \( \overline{\text{co}} \mathcal{S} \) denote the (usual) closed convex hull of \( \mathcal{S} \).

**Lemma 5.** \( \text{MCL}(\mathcal{S}) = \text{MCL}(\overline{\text{co}} \mathcal{S}) \), for \( \mathcal{S} \subseteq \mathcal{L}(\mathcal{K}) \).

*Proof.* Clearly \( \mathcal{S} \subseteq \text{MCL}(\mathcal{S}) \); but \( \text{MCL}(\mathcal{S}) \) is closed by definition, and convex by Remark 1 above. Thus \( \overline{\text{co}} \mathcal{S} \subseteq \text{MCL}(\mathcal{S}) \), and hence \( \text{MCL}(\overline{\text{co}} \mathcal{S}) \subseteq \text{MCL}(\mathcal{S}) \). On the other hand, \( \mathcal{S} \subseteq \overline{\text{co}} \mathcal{S} \); therefore \( \text{MCL}(\mathcal{S}) \subseteq \text{MCL}(\overline{\text{co}} \mathcal{S}) \).

The following results are consequences of the deep theory concerning \( \text{Ext} \) [4], but are readily obtainable from first principles.

**Lemma 6.** Let \( \mathcal{S} \subseteq \mathcal{L}(\mathcal{K}) \), where \( \mathcal{K} \) is infinite dimensional. If \( \mathcal{S} \) contains a compact operator, then \( 0 \in \text{MCL}(\mathcal{S}) \).

*Proof.* Let \( K \) be a compact operator in \( \mathcal{S} \), and let \( U \) denote a unilateral shift. We have that \( U^n*KU^n \in \text{MCL}(\mathcal{S}) \) for all positive integers \( n \), while \( \| U^n*KU^n \| \to 0 \) as \( n \to +\infty \). Thus, \( 0 \in \text{MCL}(\mathcal{S}) \) by closure.

We recall that if \( \lambda \in \sigma_{12}(T) \), then there exists an orthonormal sequence of vectors \( \{ x_n \} \) such that \( \| (T - \lambda)x_n \| \to 0 \) as \( n \to +\infty \).
Lemma 7. If \( T \in \mathcal{L}(\mathcal{K}) \), \( \mathcal{K} \) infinite dimensional, and \( \lambda \in \sigma_{le}(T) \), then \( \lambda \in \text{MCL}(T) \).

Proof. Let \( \{x_n\} \) be an orthonormal sequence with \( \|(T-\lambda)x_n\| < 1/2^n \), let \( \{e_n\} \) be an orthonormal basis for \( \mathcal{K} \), and let \( V \) be an isometry such that \( Ve_n = x_n \). We have that \( V^*(T-\lambda)V \in \text{MCL}(T-\lambda) \), and we claim that \( V^*(T-\lambda)V \) is compact. Indeed, if \( P_m \) denotes the projection onto the span of \( \{e_1, \ldots, e_m\} \), then \( V^*(T-\lambda)V P_m \) is finite rank and \( \|V^*(T-\lambda)V - V^*(T-\lambda)V P_m\| \to 0 \) as \( m \to +\infty \), since \( \|V^*(T-\lambda)V(1-P_m)(\Sigma_{i=m+1}^\infty \alpha_i e_i)\| < \|(T-\lambda)V(\Sigma_{i=m+1}^\infty \alpha_i e_i)\| \leq \Sigma_{i=m+1}^\infty |\alpha_i| \|(T-\lambda)x_i\| \leq (\Sigma_{i=m+1}^\infty |\alpha_i|^2^{1/2})(\Sigma_{i=m+1}^\infty \|(T-\lambda)x_i\|^2)^{1/2} < \Sigma_{i=m+1}^\infty \alpha_i ||/3 \times 4^m \). Thus, we have that \( \text{MCL}(T-\lambda) \) contains a compact operator. Hence, by Lemma 6, \( 0 \in \text{MCL}(T-\lambda) \) and so \( \lambda \in \text{MCL}(T) \) by Remark 5.

We recall that the essential spectrum of \( T, \sigma_e(T) \), satisfies \( \sigma_e(T) = \sigma_{1e}(T) \cup \overline{\sigma_{1e}(T^*)} \) [8].

Lemma 8. If \( T \in \mathcal{L}(\mathcal{K}) \), \( \mathcal{K} \) separable and infinite dimensional, and \( \lambda \in \sigma_e(T) \), then \( \lambda \in \text{MCL}(T) \).

Proof. If \( \lambda \in \sigma_{1e}(T) \), we are done by Lemma 7. Otherwise, \( \bar{\lambda} \in \sigma_{1e}(T^*) \), and so by Lemma 7, \( \bar{\lambda} \in \text{MCL}(T^*) = \text{MCL}(T)^* \) and we are done.

Lemma 9. If \( T \in M_n \) and \( \lambda \in \sigma(T) \), then \( \lambda \in \text{MCL}(T) \).

Proof. There exists a unitary \( U \) such that the \((1,1)\) entry of \( U^*TU \) is \( \lambda \). If \( E_{i,j} \) denote the usual matrix units, then we have that \( \lambda = \sum_{i=1}^n E_{i,i}^*U^*TE_{1,1} \in \text{MCL}(T) \).

We shall show in Remark 11 that for \( \mathcal{K} \) separable and infinite dimensional and \( T \in \mathcal{L}(\mathcal{K}) \), it is possible for \( \lambda \in \sigma(T) \), while \( \lambda \notin \text{MCL}(T) \).

The importance of Lemmas 8 and 9 is that any closed C*-convex set necessarily contains a scalar. Note also that in the finite dimensional case any C*-convex set contains a scalar.

Definition 10. For \( S, T \in \mathcal{L}(\mathcal{K}) \), the C*-segment connecting \( S \) and \( T \), denoted \( S(S, T) \), is defined to be the set \( \{A^*SA + B^*TB: A^*A + B^*B = 1\} \).

Examples show that \( S(S, T) \) is not, in general, C*-convex. Thus let

\[
S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T = 0.
\]
Then $S(S, T)$ consists entirely of rank-1 matrices; yet it contains $S$ and 
\[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix},
\]
whose midpoint is rank-2; so it is not even convex. However, we shall show (Theorems 15 and 16) that, as in ordinary convexity, if a set contains the $C^*$-segments joining each pair of elements in the set, then the set is $C^*$-convex.

**Definition 11.** For $S, T \in \mathcal{L}(\mathcal{H})$, the $C^*$-convex segment connecting $S$ and $T$, denoted $\text{MS}(S, T)$, is defined to be the set \{\[
\Sigma_{i=1}^n A_i^* S A_i + \Sigma_{i=1}^n B_i^* T B_i : \Sigma_i A_i^* A_i + \Sigma_i B_i^* B_i = 1\}\}.

**Lemma 12.** For $S, T \in \mathcal{L}(\mathcal{H})$, $\text{MS}(S, T)$ is $C^*$-convex and contains both $S$ and $T$.

**Proof.** Clearly $S$ and $T$ belong to $\text{MS}(S, T)$. Let $x_1, \ldots, x_n \in \text{MS}(S, T)$, so each $x_k = \Sigma_{i=1}^n A_i^* S A_i + \Sigma_{i=1}^n B_i^* T B_i$. If \[
\Sigma_{i=1}^n U_k = 1,
\]
then \[
\Sigma_{i=1}^n U_k x_k U_k = \Sigma_{i=1}^n U_k (\Sigma_{i=1}^n A_i^* S A_i + \Sigma_{i=1}^n B_i^* T B_i) U_k = \Sigma_{i=1}^n U_k A_i^* S A_i U_k + \Sigma_{i=1}^n U_k B_i^* T B_i U_k.
\]
But looking at the coefficients, we have \[
\Sigma_{i=1}^n U_k A_i^* S A_i U_k + \Sigma_{i=1}^n U_k B_i^* T B_i U_k = \Sigma_{i=1}^n U_k (\Sigma_{i=1}^n A_i^* A_i + \Sigma_{i=1}^n B_i^* B_i) U_k = \Sigma_{i=1}^n U_k (1) U_k = 1,
\]
completing the proof.

**Lemma 13.** If $T > 0$, then \{\[
S : 0 < S < T\} \subset \text{MS}(S, T).
\]

**Proof.** It suffices to show that if $0 < S < T$, then there is an operator $A$ with $S = A^* T A$ and $A^* A < 1$, i.e., $S = A^* T A + \sqrt{1 - A^* A}$ \[0 \sqrt{1 - A^* A} \in \text{MS}(S, T)\]. However, since $0 < S < T$, then $0 < \sqrt{S} < \sqrt{T}$. By a theorem of Douglas [7] there is an operator $A$ with $\|A\| < 1$ such that $\sqrt{S} = \sqrt{T} A$. Then $S = (\sqrt{S})^* \sqrt{S} = (A^* \sqrt{T})(\sqrt{T} A) = A^* T A$, and $\|A\| < 1$ implies $A^* A < 1$.

**Remark 6.** From Remark 3, we see that the inclusion in Lemma 13 is, in general, strict. Furthermore, it is easy to see that $S(0, 1) = \text{MS}(0, 1) = \emptyset$ as given in Example 1.

**Corollary 14.** Let $\mathcal{K}$ be $C^*$-convex, and suppose $0 \in \mathcal{K}$. Then for $0 < T \in \mathcal{K}$, we have \{\[
S : 0 < S < T\} \subset \mathcal{K}.
\]

**Remark 7.** If $\mathcal{K}$ is $C^*$-convex and $0 \in \mathcal{K}$, then for all $X \in \mathcal{K}$ and $\|A\| < 1$, we have $A^* X A \in \mathcal{K}$, for $A^* X A = A^* X A + \sqrt{1 - A^* A} \sqrt{1 - A^* A}$. 

This means that we can write $A^*XA = |A|U^*XU|A|$, where $A = U|A|$ is the polar decomposition of $A$, so $A^*XA = |A|Y|A|$, where $Y \in \mathcal{K}$.

**Theorem 15.** Let $\mathcal{K} \subseteq M_n$. Then $\mathcal{K}$ is $C^*$-convex if and only if $S(S, T) \subseteq \mathcal{K}$ for all $S$ and $T$ in $\mathcal{K}$.

**Proof.** If $\mathcal{K}$ is $C^*$-convex, then clearly $\mathcal{K}$ contains $S(S, T)$ for all $S$ and $T$ in $\mathcal{K}$.

To prove the converse, note that by Lemma 9 $\mathcal{K}$ contains a scalar, and since all of the above properties are preserved by translation by scalars, we may assume that $0 \in \mathcal{K}$.

To show that $\mathcal{K}$ is $C^*$-convex, we need to show that if $\{X_1, \ldots, X_n\} \subseteq \mathcal{K}$ and $\sum_{i=1}^n A_i^*A_i = 1$, then $\sum_{i=1}^n A_i^*X_iA_i$ is in $\mathcal{K}$. We shall prove by induction that if $\mathcal{K}$ contains every sum with $n-1$ terms, then $\mathcal{K}$ contains every sum with $n$ terms $(n \geq 3)$. We note that $\mathcal{K}$ contains every sum with 1 or 2 terms by hypothesis.

Given $\{X_1, \ldots, X_n\} \subseteq \mathcal{K}$ and $\sum_{i=1}^n A_i^*A_i = 1$, write $A_i = U_iP_i$ in canonical polar decomposition, so that $\sum_{i=1}^nP_i^2 = 1$. Furthermore, if $Y_i = U_i^*X_iU_i$, then by Remark 7, $Y_i \in \mathcal{K}$.

Let $P = (1 - P_n^2)^{1/2}$, so that $\sum_{i=1}^{n-1}P_i^2 = P^2$ and hence for $1 < i < n$, $P_i^2 < P^2$. We recall Douglas’ factorization [7]; for any $v \in \mathcal{K}$, $\|P_i v\| < \|Pv\|$, and so by setting $B_i(Pv)=P_i v$, we can define a contraction on the range of $P$ which can be extended by continuity to the closure of the range. The orthocomplement of the range of $P$ is the kernel of $P$, which is contained in the kernel of $P_1$, and for $v$ in the kernel of $P$ we set $B_1v = v/\sqrt{n-1}$ . Thus $B_1P = P_1$, and so $P_1 = PB_1^*$. For any vector of the form $Pv_1 + v_2$ where $v_2$ is in the kernel of $P$, since $P_i v_2 = 0$ for $1 < i < n$, we have that

$$
\left( \sum_{i=1}^n B_i^*B_i(Pv_1 + v_2), Pv_1 + v_2 \right) = \left( \sum_{i=1}^n B_i(Pv_1 + v_2), B_i(Pv_1 + v_2) \right) 
$$

$$
= \sum_{i=1}^{n-1} \left( P_i v_i + \frac{1}{\sqrt{n-1}} v_2, P_i v_i + \frac{1}{\sqrt{n-1}} v_2 \right) 
$$

$$
= \sum_{i=1}^{n-1} \left( P_i^2 v_1 + v_2, v_1 \right) + \frac{1}{n-1} \left( v_2, v_2 \right) 
$$

$$
= (P^2 v_1, v_1) + (v_2, v_2) = \|Pv_1 + v_2\|^2. 
$$
Since the vectors of the form $Pv_1 + v_2$ are dense in $\mathcal{K}$, we have that $\Sigma_{i=1}^{n-1} B_i^* B_i = 1$.

Finally, we may write $\Sigma_{i=1}^{n} A_i^* X_i A_i = \Sigma_{i=1}^{n} P_i Y_i P_i = P [\Sigma_{i=1}^{n-1} B_i^* Y_i B_i] P + P_n Y_n P_n$, where the term in brackets belongs to $\mathcal{K}$ by the inductive hypothesis. Thus, since $P^2 + P_n = 1$, we have written a sum with $n$ terms as something which lies on the matricial segment connecting two members of $\mathcal{K}$, and thus it is in $\mathcal{K}$. This completes the proof.

We remark that the above proof works for closed subsets of $\mathcal{L}(\mathcal{K})$, since by Lemma 9 they also contain scalars. However, the hypothesis of closure is unnecessary. This fact was pointed out to us by the referee, to whom the following is due:

**Theorem 16.** Let $\mathcal{K} \subseteq \mathcal{L}(\mathcal{K})$, infinite dimensional. Then $\mathcal{K}$ is $C^*$-convex if and only if $S(S, T) \subseteq \mathcal{K}$ for all $S$ and $T$ in $\mathcal{K}$.

**Proof.** It will be sufficient to show that if $S(S, T) \subseteq \mathcal{K}$ for all $S, T \in \mathcal{K}$, then $\mathcal{K}$ is $C^*$-convex, since the other implication is clear.

We begin by observing that if $U: \mathcal{K} \to \mathcal{K} \oplus \mathcal{K}$ is unitary, then $X = U^* (S \oplus T) U$ is in $S(S, T)$. For if $A_1 = (1 \oplus 0) U$, $A_2 = (0 \oplus 1) U$, then $A_1, A_2 \in \mathcal{L}(\mathcal{K})$, $A_1^* A_1 + A_2^* A_2 = 1$, and $X = A_1^* S A_1 + A_2^* T A_2$.

Thus, if $S(S, T) \subseteq \mathcal{K}$ for all $S$ and $T$ in $\mathcal{K}$, then by induction for any $\{X_1, \ldots, X_n\} \subseteq \mathcal{K}$ and unitary $U: \mathcal{K} \to \mathcal{K} \oplus \cdots \oplus \mathcal{K}$, we have $U^* (X_1 \oplus \cdots \oplus X_n) U \in \mathcal{K}$. Thus, given $\{A_1, \ldots, A_n\} \subseteq \mathcal{L}(\mathcal{K})$ with $\Sigma_{i=1}^{n} A_i^* A_i = 1$, and $\{X_1, \ldots, X_n\} \subseteq \mathcal{K}$, let $U: \mathcal{K} \to \mathcal{K} \oplus \cdots \oplus \mathcal{K}$ be unitary, and set $X = U^* (X_1 \oplus \cdots \oplus X_n) U \in \mathcal{K}$ and $A = U^* (A_1 \oplus \cdots \oplus A_n) \in \mathcal{L}(\mathcal{K})$. We have $\Sigma_{i=1}^{n} A_i^* X_i A_i = A^* X A \in \mathcal{K}$, since $A^* A = 1$. This completes the proof of the Theorem.

We have been unable to find one proof which works in both the infinite- and the finite-dimensional case.

**Remark 8.** If $\mathcal{K}$ is $C^*$-convex with $0 \in \mathcal{K}$, and $T = A_1^* X_1 A_1 + A_2^* X_2 A_2$ where $X_1, X_2 \in \mathcal{K}$ and $A_1^* A_1 + A_2^* A_2 = 1$, we can write $T = |A_1| Y_1 |A_1| + |A_2| Y_2 |A_2|$ where $Y_1, Y_2 \in \mathcal{K}$. But since $|A_1|^2 + |A_2|^2 = 1$, it follows that $|A_1|$ commutes with $|A_2|$. That is to say, $T$ can be written using commuting positive coefficients.

The next lemma shows that for $C^*$-convex sets, two coefficients usually suffice.

**Lemma 17.** Let $\mathcal{K}$ be a closed $C^*$-convex set. Let $\{X_i\}_{i=1}^{\infty}$ be a bounded subset of $\mathcal{K}$, and let $\Sigma_i^2 A_i^* A_i = 1$ (in norm). Suppose $\Sigma_i^2 A_i^* X_i A_i = T \in \mathcal{K}$. Then $T = A^* X A + B^* Y B$, where $Y \in \mathcal{K}$ and $A^* A + B^* B = 1$. 
SOME REMARKS ON C*-CONVEXITY

Proof. Let \( Z = \frac{1}{2} A_i^* A_i \); then \( 1 - Z > \frac{1}{2} \), so \( 1 - Z \) is positive and invertible; hence \( \sqrt{1 - Z} \) is also positive and invertible. Now let \( B_1 = \frac{1}{2} A_1 (\sqrt{1 - Z})^{-1} \), and for \( j > 2 \) let \( B_j = A_i (\sqrt{1 - Z})^{-1} \). Then \( \sum_{i=1}^{\infty} B_i^* B_i = (\sqrt{1 - Z})^{-1} (\frac{1}{2} A_i^* A_i + \sum_{i=1}^{\infty} A_i^* A_i) (\sqrt{1 - Z})^{-1} = \frac{1}{2} A_i^* A_i + 1 - A_i^* A_i) (\sqrt{1 - Z})^{-1} = (\sqrt{1 - Z})^{-1} (1 - Z) (\sqrt{1 - Z})^{-1} - 1. \) Then if \( Y = \sum_{i=1}^{\infty} B_i^* X_i B_i \), a simple norm estimate shows that \( Y \in \mathcal{K} \). Thus, we have that \( T = (\frac{1}{2} A_1^* X_i (\frac{1}{2} A_1) + \sqrt{1 - Z} Y \sqrt{1 - Z} \), as desired.

We remark that the closure of \( \mathcal{K} \) is not needed for finite sums, and if \( \sum_{i=1}^{\infty} A_i^* A_i = 1 \) in the strong operator topology, we can take \( \mathcal{K} \) to be strongly closed.

Remark 9. In view of the above results it is somewhat surprising that the sets \( \{ E z^* A_i^* X_i A_i : \sum A_i^* A_i = 1 \} \) where the infinite sums are taken to converge either in norm, strongly, or weakly, while C*-convex, are not necessarily equal to \( \text{MCL}(T) \). To see this one need only consider a positive operator \( T \) with trivial kernel and \( 0 \in \sigma_e(T) \). By Lemma 8, \( 0 \in \text{MCL}(T) \), while all of the elements of the above sets necessarily have trivial kernels. However, this problem does not occur in finite dimensions, as the following shows:

Lemma 18. Let \( \mathcal{K} = \{ X_1, \ldots, X_n \} \subseteq M_k \). Then \( \text{MCL}(\mathcal{K}) = \{ \sum_{i=1}^{\infty} A_i^* X_i A_i : \sum A_i^* A_i = 1 \text{ (in norm)}, X_i \in \mathcal{K} \} \).

Proof. Let \( \mathcal{E} = M_k \oplus \cdots \oplus M_k \) (\( n \) times); then \( \mathcal{E} \) is a C*-algebra with \( \| X_1 \oplus \cdots \oplus X_n \| = \max \| X_i \| \). Hence for \( X = X_1 \oplus \cdots \oplus X_n \in \mathcal{E} \), the set \( \mathcal{W}_k(X) \) is compact and C*-convex, and \( \mathcal{K} \subseteq \mathcal{W}_k(X) \). Thus, \( \text{MCL}(\mathcal{K}) \subseteq \mathcal{W}_k(X) \).

It is easy to see that any representation \( \pi \) of \( \mathcal{E} \) is of the form \( \pi = \pi_1 \oplus \cdots \oplus \pi_n \), with each \( \pi_i(X_i) = X_i \otimes 1_{\mathcal{K}} \) [14]. Since every completely positive map from \( C^*(X) \) to \( M_k \) can be extended to one from \( \mathcal{E} \) to \( M_k \), by Stinespring's theorem we conclude that \( \mathcal{W}_k(X) = \{ A_i^* X_i A_i : \sum A_i^* A_i = 1 \text{, strongly and } X_i \in \mathcal{K} \} \).

Thus, \( \mathcal{W}_k(X) \) is contained in the strongly closed C*-convex set generated by \( \mathcal{K} \). But by the finite-dimensionality of \( M_k \), all closures coincide, so \( \mathcal{W}_k(X) = \text{MCL}(\mathcal{K}) \), which concludes the proof.

Remark 10. This argument really shows that for bounded sets \( X, \{ \sum A_i^* X_i A_i : \sum A_i^* A_i = 1 \text{ strongly, } X_i \in X \text{ for all } i \} \) lies in every weakly (strongly) closed C*-convex set containing \( X \).
Lemma 19. Let $\mathcal{H}$ be a closed $C^*$-convex set contained in $M_n$, and let $T \in \mathcal{H}$. If $\mathbb{W}_n(T)$ denotes Arveson's $n$th matrix range of $T$ (as in Example 4), then $\mathbb{W}_n(T) \subseteq \mathcal{H}$.

Proof. Let $\varphi: M_n \rightarrow M_n$ be a completely positive map with $\varphi(1_n) = 1_n$. Then by Stinespring's theorem, $\varphi = V^* \pi V$, where $\pi$ is a $*$-representation of $M_n$ and $V^* V = 1$. But this forces $\pi(T) = T \otimes 1_{\mathcal{H}}$ and $V_\xi = (V_1 \xi, \ldots, V_n \xi, \ldots)$, where $V_i \in M_n$ and $\sum V_i^* V_i = 1_n$ (in norm) [14]. Then $\varphi(T) = V^* \pi(T) V = \sum V_i^* TV_i \in \mathcal{H}$.

Corollary 20. For $T \in M_n$, $\mathbb{W}_n(T) = \text{MCL}(T) = \{ \sum V_i^* TV_i: \sum V_i^* V_i = 1 \}$.

Proof. We have that $T \in \mathbb{W}_n(T)$, since $\text{id}: M_n \rightarrow M_n$ is completely positive, and by Lemma 19, $\mathbb{W}_n(T) \subseteq \text{MCL}(T)$. However, $\mathbb{W}_n(T)$ is compact [2], hence closed and $C^*$-convex (Example 3), so $\text{MCL}(T) \subseteq \mathbb{W}_n(T)$. The last inclusion was shown in the proof of Lemma 19.

Remark 11. In view of Corollary 20 and Remark 9, it is perhaps reasonable to conjecture that for $\mathcal{H}$ separable and infinite dimensional, $\text{MCL}(T) \supseteq \{ \sum_{i=1}^\infty A_i^* T A_i: \sum_{i=1}^\infty A_i^* A_i = 1 \text{ strongly} \}$. This however is false, as the following example shows.

Let $T$ be the compact diagonal operator $T = (t_{i,j})$ with $t_{1,1} = \lambda$, $t_{i,j} = 0$ otherwise. Then, if $E_{i,j}$ denote the usual matrix units, we have $\sum_{i=1}^\infty E_{i,i}^* E_{1,j} = 1$ strongly, and $\sum_{i=1}^\infty E_{i,j}^* E_{1,j} = \lambda$ strongly. However, since $T$ is compact, every element of $\text{MCL}(T)$ will be compact. This example shows that one can have $\lambda \in \sigma(T)$, but $\lambda \notin \text{MCL}(T)$, and also $\lambda \notin \mathbb{W}_n(T)$ (the closed numerical range), while $\lambda \notin \text{MCL}(T)$.

We now introduce a notion of extreme points in $C^*$-convex sets.

Definition 21. $Z$ is a proper matrix combination of $\{ X_1, \ldots, X_n \}$ if $Z = \sum_i A_i^* X_i A_i$ where $\sum_i A_i^* A_i = 1$ and each $A_i$ is invertible.

Definition 22. Suppose $\mathcal{H}$ is a $C^*$-convex set. Then $Z \in \mathcal{H}$ is a $C^*$-extreme point of $\mathcal{H}$ if whenever $Z$ is a proper matricial combination of $\{ X_1, \ldots, X_n \} \subseteq \mathcal{H}$, then each $X_i$ is unitarily equivalent to $Z$.

It is easy to see that for $A_i$ scalars, Definition 22 reduces to the usual definition of an extreme point of a convex set, up to unitary equivalence.
Notice that by Lemma 17, we need only check sums with two terms for $C^*$-extremeness. Furthermore, since each $A_i$ is invertible, this means that in the polar decomposition $A_i = U_i P_i, U_i$ will be unitary. Thus, $Z$ is a $C^*$-extreme point of $\mathcal{K}$ if and only if whenever $Z$ is a proper matricial combination of $\{X_1, X_2\} \subseteq \mathcal{K}$ with positive, invertible coefficients, then each $X_i$ is unitarily equivalent to $Z$. Note also that necessarily the positive coefficients commute.

**Remark 12.** If $Z$ is an element in the $C^*$-convex set $\mathcal{K}$, then for any unitary $U, W = U^* Z U \in \mathcal{K}$; so we can write $Z = (\frac{1}{2} U) W (\frac{1}{2} U)^* + (\frac{1}{2} U) W (\frac{1}{2} U)^*$, that is $Z$ is a proper matrix combination of $W$. This phenomenon explains the unitary equivalence statement in the definition of $C^*$-extreme point. Only in the trivial case $\mathcal{K} = \{\lambda O_1\}$ is the unitary equivalence unnecessary.

**Remark 13.** Further, it follows that if $Z$ is a $C^*$-extreme point of the $C^*$-convex set $\mathcal{K}$, then for any $W$ unitarily equivalent to $Z$, we have that $W$ is also a $C^*$-extreme point of $\mathcal{K}$. Similarly, $-Z$ and $Z^*$ will also be $C^*$-extreme, in $-\mathcal{K}$ and $\mathcal{K}^*$, respectively.

**Proposition 23.** If $T$ is a $C^*$-extreme point of a $C^*$-convex subset $\mathcal{K}$ of $M_n$, then $T$ is a linear extreme point of $\mathcal{K}$.

**Proof.** Suppose not; then $T = tX + (1 - t)Y$, where $0 < t < 1, X \neq T$, and $Y \neq T$. By the $C^*$-extremity of $T$, $X$ and $Y$ are unitarily equivalent to $T$. Thus, $T$ is written as a proper linear combination of points in its unitary orbit. By [9], every operator in $M_n$ is linearly extreme in its unitary orbit. This contradiction completes the proof.

**Proposition 24.** Let $\mathcal{B}_1 = \{T: \|T\| < 1\} \subseteq \mathcal{L}(\mathcal{K})$; then the unitaries are $C^*$-extreme points of $\mathcal{B}_1$.

**Proof.** Let $U$ be unitary and suppose $U = P_1 X_1 P_1 + P_2 X_2 P_2$, with $\|X_i\| < 1, P_i > 0$ for $i = 1, 2$ and $P_1^2 + P_2^2 = 1$. Note that $\begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix}$ is unitary and that

$$
\begin{pmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix} = \begin{pmatrix} U & -P_1 X_1 P_2 + P_2 X_2 P_1 \\ -P_2 X_1 P_1 + P_1 X_2 P_2 & * \end{pmatrix}.
$$
Since $U$ is unitary and the norm of the product is not greater than 1, we have
that $0 = -P_1X_1P_2 + P_2X_2P_1 = -P_2X_1P_1 + P_1X_2P_2$. From the first expression
we see that $X_1 = P_1^{-1}P_2X_2P_1P_2^{-1}$, and from the second that
$X_2 = P_2^{-1}P_1X_1P_2P_1^{-1}$. Thus, we obtain $X_2P_2^2P_1^{-2} = P_2^{-2}P_1^2X_2$. Since $P_1$ and $P_2$ commute, we
see that $X_2$ commutes with $P_2^2P_1^{-2} = P_1^2(1-P_1^2)^{-1} = -1 + (1-P_1^2)^{-1} = -1 + P_2^{-2}$. Hence $X_2$ commutes with $P_2^{-2}$, and so by the spectral theorem
with $P_2$ and $P_1$.

This shows that $X_1 = P_1^{-1}P_2X_2P_1P_2^{-1} = X_2$, and hence that
$U = P_1X_1P_1 + P_2X_2P_2 = X_1 = X_2$. Thus $U$ is $C^*$-extreme.

**Corollary 25.** If $B_1 \subseteq M_n$, then the $C^*$-extreme points of $B_1$ are the
unitaries.

**Proof.** Since the linear extreme points of $B_1$ are the unitaries, we are
done by Propositions 23 and 24.

**Proposition 26.** Let $\mathcal{D} = \{T : 0 < T \leq 1\} \subseteq \mathcal{L}(\mathcal{H})$, $\mathcal{H}$ separable; then the
projections are $C^*$-extreme in $\mathcal{D}$.

**Proof.** Let $P$ be a projection, and suppose $P = P_1X_1P_1 + P_2X_2P_2$, with
$P_1 > 0$, $0 < X_1 < 1$, and $P_2^2 + P_1^2 = 1$. If $z \in P^\mathcal{H}$, then $\langle X_1P_1z, P_1z \rangle + \langle X_2P_2z, P_2z \rangle = \langle (P_1X_1P_1 + P_2X_2P_2)z, z \rangle = \langle P_z, z \rangle = \langle z, z \rangle = \langle (P_1^2 + P_2^2)z, z \rangle$, and since $0 < X_i < 1$, $\langle X_iP_iz, P_iz \rangle = \langle P_1z, P_1z \rangle + \langle P_2z, P_2z \rangle$ for $i = 1, 2$. Thus, $X_i = 1$ on $P_1P^\mathcal{H}$. A similar calculation shows
that $X_i = 0$ on $P_2(P^\mathcal{H})$, and so $P_1P^\mathcal{H} \cap P_2(P^\mathcal{H}) = (0)$.

Since each $P_i$ is invertible, $P_1P^\mathcal{H}$ and $P_2(P^\mathcal{H})$ are closed subspaces with
dim$(P^\mathcal{H}) = \dim(P_1(P^\mathcal{H}))$ and $\dim((P^\mathcal{H}) = \dim(P_2(P^\mathcal{H}))$; further
$\mathcal{H} = P_1(P^\mathcal{H}) + P_2(P^\mathcal{H})$. Thus for each $i$, $\mathcal{H} = \mathcal{M}_i + \mathcal{M}_i$, where $\mathcal{M}_i$ is a closed subspace with $X_i = 1$, $\mathcal{M}_i$ is a closed subspace with $X_i = 0$, and $\mathcal{M}_i \cap \mathcal{M}_i = (0)$. Since each $X_i > 0$, it follows that each $X_i$ is an orthogonal projection, and the dimensions then imply that each $X_i$ is unitarily equivalent to $P$.

**Proposition 27.** Let $\mathcal{S} = \{T : -1 < T \leq 1\} \subseteq \mathcal{L}(\mathcal{H})$. Then $\mathcal{S}$ is $C^*$-convex,
and the $C^*$-extreme points of $\mathcal{S}$ belong to $\{E^{-1} : E > 0$ is a projection\}.

**Proof.** Since matrix combinations are scalar-order preserving, it is easy
to see that $\mathcal{S}$ is $C^*$-convex. For any $T \in \mathcal{S}$, we can write $T = T_1 \Theta T_2$ where $T_1 > 0$; let $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ be the corresponding decomposition of $\mathcal{K}$ and $P_1, P_2$
the corresponding projections. Let $Y = 2T_1 - 2T_2 + P_2 - P_1$; then $Y \in \mathcal{S}$. Fur-
ther, $T = \sqrt{\frac{1}{2}} Y \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}} (P_1 - P_2) \sqrt{\frac{1}{2}}$. If $T$ is $C^*$-extreme in $\mathcal{S}$, then $T$ is unitarily equivalent to $P_1 - P_2 = 2P_1 - 1$, so $T = 2E - 1$ where $E > 0$ is a projection.

**Proposition 28.** Let $\mathcal{P} = \{T : 0 \leq T \leq 1\} \subseteq \mathcal{L}(\mathcal{H})$. If $Q$ is a $C^*$-extreme point of $\mathcal{P}$, then $Q$ is a projection.

**Proof.** We claim that if $Q$ is $C^*$-extreme in $\mathcal{P}$, then $T = 2Q - 1$ is $C^*$-extreme in $\mathcal{P}$. For suppose we write $T = P_1 X_1 P_1 + P_2 X_2 P_2$ as a proper matrix combination of $\{X_1, X_2\} \subseteq \mathcal{S}$, where some $X_{i_0}$ is not unitarily equivalent to $T$; let $Y_i = (1 + X_i)/2$. Then $Q = P_1 Y_1 P_1 + P_2 Y_2 P_2$ is a representation of $Q$ as a proper matrix combination of elements of $\mathcal{P}$, but since $X_{i_0}$ is not unitarily equivalent to $T$, $Y_{i_0}$ is not unitarily equivalent to $Q$. This contradicts the $C^*$-extremity of $Q$, and hence $T$ is indeed $C^*$-extreme in $\mathcal{P}$. But then by Proposition 27, $2Q - 1 = T = 2E - 1$, where $E > 0$ is a projection, and hence $Q$ is a projection.

**Corollary 29.** $Q$ is $C^*$-extreme in $\mathcal{P}$ if and only if $Q$ is a projection.

**Remark 14.** Corollary 29 shows that the $C^*$-extreme points of $\mathcal{P}$ are identical with the usual extreme points [14]. Because of the unitary equivalence built into the definition of $C^*$-extreme points, the $C^*$-extreme points of $\mathcal{P}$ are completely specified by giving the dimension and codimension of the projection; for finite dimensions, only the dimension is needed, so in $M_n$, there are basically only $n + 1$ $C^*$-extreme points. Notice also that in $M_n$, every element of $\mathcal{P}$ is a matricial combination of at most two $C^*$-extreme points, namely 0 and 1, for if $0 \leq T \leq 1$, then $T = \sqrt{T} \sqrt{T} + \sqrt{1 - T} 0\sqrt{1 - T}$; as a linearly convex combination of extreme points, such an element $T$ will generally require many more than two extreme points.

**Question 1.** When are $X$ and $Y$ $C^*$-extreme points of $MS(X, Y)$? The case $(X, Y) = (0, 1)$ shows that there may be many others. The same difficulty occurs when $(X, Y) = (-1, 1)$.

**Remark 15.** Since a $C^*$-convex set is linearly convex, and since being linearly extreme is a unitary invariant, one expects that every $C^*$-extreme point is linearly extreme (see Proposition 23).
Remark 16. In $\mathcal{B} = \{x : \|x\| < 1 \}$, a $C^*$-extreme point must have norm 1, for if $0 < \|x\| < 1$, we can write

$$x = \frac{\sqrt{\|x\| + 1}}{2} \frac{x}{\|x\|} \sqrt{\frac{\|x\| + 1}{2}} + \frac{\sqrt{1 - \|x\|}}{2} \frac{-x}{\|x\|} \sqrt{\frac{1 - \|x\|}{2}}.$$ 

Hence $x$ will be extreme only if $\|x\| = 1$. Proposition 24 establishes much more.

We now wish to make some comments about the matrix ranges $\mathbb{W}_n(T)$. We have previously observed that $\mathbb{W}_n(T)$ is a compact $C^*$-convex subset of $M_n$. Let $P_n$ be the linear map from $C^n \to C^{n+1}$ given by $P_n([x_1, \ldots, x_n]) = [x_1, \ldots, x_n, 0]$. 

Proposition 30. The matrix ranges of an operator $T$ satisfy 

$$\mathbb{W}_n(T) = P_n^* \mathbb{W}_{n+1}(T) P_n.$$ 

Proof. Let $\varphi : C^*(T) \to M_{n+1}$ be a completely positive map with $\varphi(1) = 1_{n+1}$. By Stinespring's theorem, $\varphi = V^* \pi V$, where $\pi$ is a $*$-representation of $C^*(T)$ on $\mathcal{E}(\mathcal{K})$ and $V : C^{n+1} \to \mathcal{K}$ with $V^* V = I_{n+1}$. Then $\tilde{\varphi} = P_n^* \varphi P_n = P_n^* (V^* \pi V) P_n = (VP_n)^* \pi (VP_n)$ is a completely positive map from $C^*(T) \to M_n$ and $\tilde{\varphi}(1) = P_n^* \varphi(1) P_n = P_n^* I_{n+1} P_n = I_n$. This shows that $P_n^* \mathbb{W}_{n+1}(T) P_n \subseteq \mathbb{W}_n(T)$.

Conversely, let $\varphi : C^*(T) \to M_n$ be completely positive with $\varphi(1) = 1$. Then $\tilde{\varphi} = P_n \varphi P_n^*$ is a completely positive map from $C^*(T) \to M_{n+1}$ with $\tilde{\varphi}(1) = P_n P_n^*$, which is a projection of rank $n$. Let $\sigma$ be a state on $C^*(T)$; then $\tilde{\sigma} : x \mapsto \sigma(x)(I_{n+1} - P_n P_n^*)$ is a positive linear map of $C^*(T)$ to an abelian $C^*$-algebra, and hence completely positive [1]. Then $\Psi = \tilde{\varphi} + \tilde{\sigma}$ is a completely positive map from $C^*(T) \to M_{n+1}$ with $\Psi(1) = \tilde{\varphi}(1) + \tilde{\sigma}(1) = P_n P_n^* + (I_{n+1} - P_n P_n^*) = I_{n+1}$. Then it is easy to see that $P_n^* \Psi P_n = P_n^* (\tilde{\varphi} + \tilde{\sigma}) P_n = P_n^* (P_n \varphi P_n^*) P_n + P_n^* [\sigma(I_{n+1} - P_n P_n^*)] P_n = \varphi$ and so $\mathbb{W}_n \subseteq P_n^* \mathbb{W}_{n+1} P_n$. 

The proof of the following result was pointed out to us by Norberto Salinas.

Proposition 31. A set $\mathcal{K} \subseteq M_n$ satisfies $\mathcal{K} = \mathbb{W}_n(T)$ for some separably acting $T$ if and only if $\mathcal{K}$ is compact and $C^*$-convex.

Proof. If $\mathcal{K} = \mathbb{W}_n(T)$, then $\mathcal{K}$ is compact and $C^*$-convex [3]. Conversely, if $\mathcal{K} \subseteq M_n$ is compact and $C^*$-convex, then $\mathcal{K}$ is hypoconvex in the sense of Salinas [15]. Hence, there is a separably acting operator $T$ with $R^*(T) = \mathcal{K}$, and $\mathbb{W}_n(T) = \mathcal{K}$. 


Comments. Recently, Hopenwasser, Moore, and the author have shown that the C*-extreme points of \( B \subseteq \mathcal{L}(\mathcal{H}) \) coincide with the linear extreme points.

In several papers, Salinas has introduced sets of \( n \times n \) matrices associated with an operator, for example the essential matricial spectrum. Furthermore, these sets are, in general, C*-convex. One of the useful results of Arveson is [1, 3.1.21], which states that for an operator \( T \), a point in the spectrum of \( T \) which lies on the boundary of the numerical range corresponds to a character (complex homomorphism) of \( C^*(T) \). There is reason to believe that a similar result holds for C*-extreme points, which would be extremely useful. We refer the reader to the work of Salinas for elaboration on this subject. Notice also that it is the finite-dimensional case which is of greatest interest.

Also, suppose \( A, B \in M_n \) are irreducible, i.e., \( C^*(A) = C^*(B) = M_n \). Then Arveson has shown that \( A \) is unitarily equivalent to \( B \) if and only if \( \mathcal{U}_n(A) = \mathcal{U}_n(B) \) [2,3]. Furthermore, since by Proposition 31 any C*-convex subset of \( M_n \) is the matrix range of some operator, we feel that the study of C*-convex sets would sharpen and/or make computationally feasible the results of Arveson on unitary equivalence of irreducible compact operators [3].

Let \( \mathcal{O}(X) = \{ U^*XU : U \text{ is unitary} \} \). It is not known if \( X \) is a proper matricial combination of points of \( \mathcal{O}(X) \). We have heard that A. M. Davie has done some work on this question. See [9] for some results on the linear extreme points of \( \mathcal{O}(X) \).

We should also remark that Davis [6] mentions C*-convexity, without the name and in another context; see p. 195.

It is our feeling that for compact C*-convex sets a form of Krein-Milman-type theorem should hold. At present we do not know how to establish this result. If \( T \) is normal, then the linear structure of \( \mathcal{U}_n(T) \) provides sufficient information to recapture much important information about \( T \), but if \( T \) is not normal complications arise. It is to resolve these complications that our interest in C*-convexity arose. Furthermore, it is hoped that the set \( \text{MCL}(T) \) will serve as a useful notion of an \"operator-valued spectrum\" in the spirit of the papers of Hadwin [17]. In a forthcoming paper we study \( \text{MCL}(T) \) in this context.

Finally, we would like to remark that most of this work carries over when \( \mathcal{L}(\mathcal{H}) \) is replaced by a more general C*-algebra. For example, the set \( \{ X \} \) would be C*-convex in the C*-algebra \( \mathcal{A} \) if and only if \( X \) is an element of the center of \( \mathcal{A} \).

We wish to thank the referee and Norberto Salinas for their many contributions to this paper. In particular, Remark 9 and Theorem 16 are due to the referee, and Norberto Salinas suggested Propositions 24 and 31.
REFERENCES


Received May 1979; revised 18 July 1979