Some Remarks on C^*-Convexity

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ABSTRACT

Recently the study of completely positive maps has become important to the results of Brown, Douglas, and Fillmore on Ext($\mathcal{A}$), $\mathcal{A}$ a C^*-algebra. Attempts to solve questions related to Ext have often turned into questions about the matrix algebras M_n. In this paper we wish to discuss a notion of C^*-convexity related to completely positive linear maps, to state some facts about C^*-convexity, and to ask some questions about C^*-convexity. To a large degree, the tone of this paper is expository.

Recently the study of completely positive maps has become important to the results of Brown, Douglas, and Fillmore on Ext($\mathcal{A}$), $\mathcal{A}$ a C^*-algebra. Attempts to solve questions related to Ext have often turned into questions about the matrix algebras M_n. In this paper we wish to discuss a notion of C^*-convexity related to completely positive linear maps, to state some facts about C^*-convexity, and to ask some questions about C^*-convexity. To a large degree, the tone of this paper is expository.

We shall let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on a (separable) Hilbert space $\mathcal{H}$, and let M_n denote the algebra of complex $n\times n$ matrices. Our general references are [8] and [14].

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DEFINITION 1. A set $\mathcal{K}\subseteq\mathcal{L}(\mathcal{H})$ is *C^*-convex* if $\{x_1,\ldots,x_n\}\subseteq\mathcal{K}$, and $\{A_1,\ldots,A_n\}\subseteq\mathcal{L}(\mathcal{H})$ with $\sum_{i=1}^n A_i^*A_i=1$ implies that $\sum_{i=1}^n A_i^*x_iA_i\in\mathcal{K}$.

Let us give some examples of C^*-convex sets. Notice that $\{x\}$ is C^*-convex if and only if $x=\lambda 1$ for some scalar λ.

EXAMPLE 1. Let $\mathcal{P}=\{T: 0\leqslant T\leqslant 1\}$; then $\mathcal{P}$ is C^*-convex, for clearly $0\leqslant\sum A_i^*T_iA_i$, but $T_i\leqslant 1\Rightarrow A_i^*T_iA_i\leqslant A_i^*A_i$, so $\sum A_i^*T_iA_i\leqslant\sum A_i^*A_i=1$.

EXAMPLE 2. Let $\mathcal{K}$ be the set of $T\in\mathcal{L}(\mathcal{H})$ such that the numerical radius $w(T)\leqslant r$, for $r>0$. Then for $z\in\mathcal{H}$, $\|z\|=1$, we have $|\langle\sum A_i^*T_iA_iz,z\rangle| \leqslant\sum|\langle A_i^*T_iA_iz,z\rangle|=\sum|\langle T_iA_iz,A_iz\rangle|\leqslant r\sum\|A_iz\|^2=r$.

EXAMPLE 3. Let $\mathcal{B}_R=\{T:\|T\|\leqslant R\}$; then on $\mathcal{L}(\mathcal{H})\otimes M_n$, we have

$$\left\|\begin{bmatrix} A_1^* & \cdots & A_n^* \\ & & \\ & 0 & \end{bmatrix}_{n\times n}\begin{bmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_n\end{bmatrix}_{n\times n}\begin{bmatrix} A_1 & & \\ \vdots & 0 & \\ A_n & & \end{bmatrix}_{n\times n}\right\|$$

$$=\left\|\begin{bmatrix}\sum A_i^*\,T_iA_i & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{bmatrix}\right\|$$

$$\leqslant\left\|\begin{bmatrix} A_1^* & \cdots & A_n^* \\ & & \\ & 0 & \end{bmatrix}\right\|\left\|\begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_n\end{bmatrix}\right\|\left\|\begin{bmatrix} A_1 & & \\ \vdots & 0 & \\ A_n & & \end{bmatrix}\right\|$$

$$\leqslant R\left\|\begin{bmatrix}\sum A_i^*A_i & & 0\\ & & \\ 0 & & 0\end{bmatrix}\right\|=R.$$

Before the next example, we remind the reader that a linear map φ between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be *completely positive* if for all n, the map $\varphi_n=\varphi\otimes\mathrm{id}_n: \mathcal{A}\otimes M_n\to\mathcal{B}\otimes M_n$ is positive. A well-known theorem of Stinespring characterizes completely positive maps.

THEOREM 2 [16]. *The linear map $\varphi:\mathcal{A}\to\mathcal{B}$ $[\subseteq\mathcal{L}(\mathcal{H})]$ is completely positive if and only if there is a *-representation π of $\mathcal{A}$ on $\mathcal{L}(\mathcal{K})$, for some Hilbert space $\mathcal{K}$, and a map $V: \mathcal{H}\to\mathcal{K}$ such that $\varphi(A)=V^*\pi(A)V$ for all $A\in\mathcal{A}$.*

EXAMPLE 4. In [2,3], Arveson defined the nth matrix range of an operator T, denoted $\mathcal{W}_n(T)$, as $\mathcal{W}_n(T)=\{\varphi(T):\varphi$ is a completely positive map from $C^*(T)\to M_n$ with $\varphi(1)=1\}$. We claim that $\mathcal{W}_n(T)$ is C^*-convex; for if $\varphi_1,\ldots,\varphi_n$ are completely positive maps as above, and $\Sigma A_i^*A_i=1$, then $\sigma(A)=\Sigma A_i^*\varphi_i(A)A_i$ is another such completely positive map. By Stinespring's theorem it is evident that each summand $A_i^*\varphi_iA_i$ is completely positive, and it is easy to see that the sum of completely positive maps is completely positive. By Stinespring's theorem each $\varphi_i=V_i^*\pi_iV_i$, and $\varphi_i(1)=1$ implies $V_i^*V_i=1$; but then $\sigma(1)=\Sigma A_i^*V_i^*V_iA_i=\Sigma A_i^*(1)A_i=1$. Thus if $T_1,\ldots,T_n\in\mathcal{W}_n(T)$, then $T_i=\varphi_i(T)$, where φ_i are as above, and $\Sigma A_i^*T_iA_i=\sigma(T)\in\mathcal{W}_n(T)$.

We remark that Examples 1, 2, and 3 are all special cases of Example 4 [2,13].

REMARK 1. If $\mathcal{K}$ is C^*-convex, then $\mathcal{K}$ is convex in the usual sense.

REMARK 2. If $\mathcal{K}$ is C^*-convex and $K\in\mathcal{K}$, then $\{V^*KV: V^*V=1\}\subseteq\mathcal{K}$ and $\{VKV^*: VV^*=1\}\subseteq\mathcal{K}$. In particular, if $\mathcal{K}$ is C^*-convex, $K\in\mathcal{K}$, and L is unitarily equivalent to K, then $L\in\mathcal{K}$.

REMARK 3. From Remark 2, it is easy to see that the segment $[0,A]=\{T: 0\leqslant T\leqslant A\}$ is not in general C^*-convex, although the segment is convex in the usual sense [10]. For example, if

$$A=\begin{pmatrix}1&0\\0&0\end{pmatrix},$$

then

$$B=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\[4pt]\frac{1}{2}&\frac{1}{2}\end{bmatrix}$$

is unitarily equivalent to A, but $B\nleqslant A$ (and also $A\nleqslant B$).

REMARK 4. The notion of C^*-convexity is unchanged by translation by a fixed scalar, that is, $\mathcal{K}+\alpha 1$ is C^*-convex if and only if $\mathcal{K}$ is. Thus if $\alpha 1 \in \mathcal{K}$, then for purposes of C^*-convexity we can assume $0 \in \mathcal{K}$. However, translation by scalars is apparently the only allowable translation in the study of C^*-convexity, in contrast with the usual study of convexity [10].

DEFINITION 3. If $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$, let $\mathrm{MCL}(\mathcal{S})$ denote the smallest norm-closed C^*-convex set containing $\mathcal{S}$.

REMARK 5. It is easy to see that $\mathrm{MCL}(\mathcal{S}) = \bigcap \mathcal{K}$, the intersection taken over all norm-closed C^*-convex sets $\mathcal{K} \supseteq \mathcal{S}$. Note also that $\mathrm{MCL}(\mathcal{S}-\lambda) = \mathrm{MCL}(\mathcal{S})-\lambda$, and that $\mathrm{MCL}(\mathcal{S}^*) = \mathrm{MCL}(\mathcal{S})^*$.

LEMMA 4. *If $\mathcal{K}$ is C^*-convex, so is its norm closure.*

Proof. Let $T_1, \ldots, T_n \in \overline{\mathcal{K}}$, and let $A_1, \ldots, A_n$ be such that $\sum A_i^* A_i = 1$. Then for each i, $A_i^* A_i \leq 1$, so $\|A_i\| \leq 1$. Let $\varepsilon > 0$, and for each i, let S_i be an element of $\mathcal{K}$ with $\|T_i - S_i\| \leq \varepsilon/n$. Then by hypothesis we have $\sum A_i^* S_i A_i \in \mathcal{K}$, and furthermore $\|\sum A_i^* T_i A_i - \sum A_i^* S_i A_i\| = \|\sum A_i^*(T_i - S_i)A_i\| \leq \sum \|A_i^*(T_i - S_i)A_i\| \leq \sum \|A_i^*\| \|T_i - S_i\| \|A_i\| \leq \sum \varepsilon/n = \varepsilon$. Hence $\sum A_i^* T_i A_i \in \overline{\mathcal{K}}$, as was to be proved. ∎

Let $\overline{\mathrm{co}}\, \mathcal{S}$ denote the (usual) closed convex hull of $\mathcal{S}$.

LEMMA 5. $\mathrm{MCL}(\mathcal{S}) = \mathrm{MCL}(\overline{\mathrm{co}}\, \mathcal{S})$, *for* $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$.

Proof. Clearly $\mathcal{S} \subseteq \mathrm{MCL}(\mathcal{S})$; but $\mathrm{MCL}(\mathcal{S})$ is closed by definition, and convex by Remark 1 above. Thus $\overline{\mathrm{co}}\, \mathcal{S} \subseteq \mathrm{MCL}(\mathcal{S})$, and hence $\mathrm{MCL}(\overline{\mathrm{co}}\, \mathcal{S}) \subseteq \mathrm{MCL}(\mathcal{S})$. On the other hand, $\mathcal{S} \subseteq \overline{\mathrm{co}}\, \mathcal{S}$; therefore $\mathrm{MCL}(\mathcal{S}) \subseteq \mathrm{MCL}(\overline{\mathrm{co}}\, \mathcal{S})$. ∎

The following results are consequences of the deep theory concerning Ext [4], but are readily obtainable from first principles.

LEMMA 6. *Let $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is infinite dimensional. If $\mathcal{S}$ contains a compact operator, then* $0 \in \mathrm{MCL}(\mathcal{S})$.

Proof. Let K be a compact operator in $\mathcal{S}$, and let U denote a unilateral shift. We have that $U^{n*} K U^n \in \mathrm{MCL}(\mathcal{S})$ for all positive integers n, while $\|U^{n*} K U^n\| \to 0$ as $n \to +\infty$. Thus, $0 \in \mathrm{MCL}(\mathcal{S})$ by closure. ∎

We recall that if $\lambda \in \sigma_{1e}(T)$, then there exists an orthonormal sequence of vectors $\{x_n\}$ such that $\|(T-\lambda)x_n\| \to 0$ as $n \to +\infty$.

LEMMA 7. *If* $T \in \mathcal{L}(\mathcal{H})$, $\mathcal{H}$ *infinite dimensional, and* $\lambda \in \sigma_{le}(T)$, then $\lambda \in \mathrm{MCL}(T)$.

Proof. Let $\{x_n\}$ be an orthonormal sequence with $\|(T-\lambda)x_n\| < 1/2^n$, let $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$, and let V be an isometry such that $Ve_n = x_n$. We have that $V^*(T-\lambda)V \in \mathrm{MCL}(T-\lambda)$, and we claim that $V^*(T-\lambda)V$ is compact. Indeed, if P_m denotes the projection onto the span of $\{e_1, \dots, e_m\}$, then $V^*(T-\lambda)VP_m$ is finite rank and $\|V^*(T-\lambda)V - V^*(T-\lambda)VP_m\| \to 0$ as $m \to +\infty$, since $\|V^*(T-\lambda)V(1-P_m)(\sum_{i=1}^{\infty}\alpha_i e_i)\| \leqslant \|(T-\lambda)V(\sum_{i=m+1}^{\infty}\alpha_i e_i)\| = \|\sum_{i=m+1}^{\infty}\alpha_i(T-\lambda)x_i\| \leqslant \sum_{i=m+1}^{\infty}|\alpha_i|\,\|(T-\lambda)x_i\| \leqslant (\sum_{i=m+1}^{\infty}|\alpha_i|^2)^{1/2}(\sum_{i=m+1}^{\infty}\|(T-\lambda)x_i\|^2)^{1/2} \leqslant \|\sum_{i=1}^{\infty}\alpha_i e_i\|/3\times 4^m$. Thus, we have that $\mathrm{MCL}(T-\lambda)$ contains a compact operator. Hence, by Lemma 6, $0 \in \mathrm{MCL}(T-\lambda)$ and so $\lambda \in \mathrm{MCL}(T)$ by Remark 5. ■

We recall that the essential spectrum of T, $\sigma_e(T)$, satisfies $\sigma_e(T) = \sigma_{1e}(T) \cup \overline{\sigma_{1e}(T^*)}$ [8].

LEMMA 8. *If* $T \in \mathcal{L}(\mathcal{H})$, $\mathcal{H}$ *separable and infinite dimensional, and* $\lambda \in \sigma_e(T)$, *then* $\lambda \in \mathrm{MCL}(T)$.

Proof. If $\lambda \in \sigma_{1e}(T)$, we are done by Lemma 7. Otherwise, $\bar{\lambda} \in \sigma_{1e}(T^*)$, and so by Lemma 7, $\bar{\lambda} \in \mathrm{MCL}(T^*) = \mathrm{MCL}(T)^*$ and we are done. ■

LEMMA 9. *If* $T \in M_n$ *and* $\lambda \in \sigma(T)$, *then* $\lambda \in \mathrm{MCL}(T)$.

Proof. There exists a unitary U such that the $(1,1)$ entry of U^*TU is λ. If $E_{i,j}$ denote the usual matrix units, then we have that $\lambda = \sum_{j=1}^{n} E_{1,j}^* U^*TUE_{1,j} \in \mathrm{MCL}(T)$. ■

We shall show in Remark 11 that for $\mathcal{H}$ separable and infinite dimensional and $T \in \mathcal{L}(\mathcal{H})$, it is possible for $\lambda \in \sigma(T)$, while $\lambda \notin \mathrm{MCL}(T)$.

The importance of Lemmas 8 and 9 is that any closed C^*-convex set necessarily contains a scalar. Note also that in the finite dimensional case any C^*-convex set contains a scalar.

DEFINITION 10. For $S, T \in \mathcal{L}(\mathcal{H})$, the C^*-segment connecting S and T, denoted $S(S,T)$, is defined to be the set $\{A^*SA + B^*TB : A^*A + B^*B = 1\}$.

Examples show that $S(S,T)$ is not, in general, C^*-convex. Thus let

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad T = 0.$$

Then $S(S,T)$ consists entirely of rank-1 matrices; yet it contains S and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, whose midpoint is rank-2; so it is not even convex. However, we shall show (Theorems 15 and 16) that, as in ordinary convexity, if a set contains the C^*-segments joining each pair of elements in the set, then the set is C^*-convex.

Definition 11. For $S,T\in\mathcal{L}(\mathcal{H})$, the C^*-convex segment connecting S and T, denoted $\mathrm{MS}(S,T)$, is defined to be the set $\{\sum_{i=1}^m A_i^*SA_i + \sum_{j=1}^n B_j^*TB_j : \sum_i A_i^*A_i + \sum_j B_j^*B_j = 1\}$.

Lemma 12. *For $S,T\in\mathcal{L}(\mathcal{H})$, $MS(S,T)$ is C^*-convex and contains both S and T.*

Proof. Clearly S and T belong to $\mathrm{MS}(S,T)$. Let $x_1,\dots x_n\in\mathrm{MS}(S,T)$, so each $x_k=\sum_{i=1}^{m_k}A_i^{(k)*}SA_i^{(k)}+\sum_{j=1}^{n_k}B_j^{(k)*}TB_j^{(k)}$. If $\sum_{k=1}^n U_k^*U_k=1$, then $\sum_{k=1}^n U_k^*x_kU_k=\sum_k U_k^*\left(\sum_{i=1}^{m_k}A_i^{(k)*}SA_i^{(k)}+\sum_{j=1}^{n_k}B_j^{(k)*}TB_j^{(k)}\right)U_k=\sum_k\sum_{i=1}^{m_k}U_k^*A_i^{(k)*}SA_i^{(k)}U_k+\sum_k\sum_{j=1}^{n}U_k^*B_j^{(k)*}TB_j^{(k)}U_k$. But looking at the coefficients, we have $\sum_k\sum_{i=1}^{m_k}U_k^*A_i^{(k)*}A_i^{(k)}U_k+\sum_k\sum_{j=1}^{n_k}U_k^*B_j^{(k)}B_j^{(k)}U_k=\sum_k U_k^*(\sum_{i=1}^{m_k}A_i^{(k)*}A_i^{(k)}+\sum_{j=1}^{n_k}B_j^{(k)*}B_j^{(k)})U_k=\sum_k U_k^*(1)U_k=1$, completing the proof. ■

Lemma 13. *If $T\geqslant 0$, then $\{S:0\leqslant S\leqslant T\}\subseteq S(0,T)$.*

Proof. It suffices to show that if $0\leqslant S\leqslant T$, then there is an operator A with $S=A^*TA$ and $A^*A\leqslant 1$, i.e., $S=A^*TA+\sqrt{1-A^*A}\,0\sqrt{1-A^*A}\in\mathrm{MS}(0,T)$. However, since $0\leqslant S\leqslant T$, then $0\leqslant\sqrt{S}\leqslant\sqrt{T}$. By a theorem of Douglas [7] there is an operator A with $\|A\|\leqslant 1$ such that $\sqrt{S}=\sqrt{T}\,A$. Then $S=(\sqrt{S})^*\sqrt{S}=(A^*\sqrt{T})(\sqrt{T}\,A)=A^*TA$, and $\|A\|\leqslant 1$ implies $A^*A\leqslant 1$. ■

Remark 6. From Remark 3, we see that the inclusion in Lemma 13 is, in general, strict. Furthermore, it is easy to see that $S(0,1)=\mathrm{MS}(0,1)=\mathcal{P}$ as given in Example 1.

Corollary 14. *Let $\mathcal{K}$ be C^*-convex, and suppose $0\in\mathcal{K}$. Then for $0\leqslant T\in\mathcal{K}$, we have $\{S:0\leqslant S\leqslant T\}\subseteq\mathcal{K}$.*

Remark 7. If $\mathcal{K}$ is C^*-convex and $0\in\mathcal{K}$, then for all $X\in\mathcal{K}$ and $\|A\|\leqslant 1$, we have $A^*XA\in\mathcal{K}$, for $A^*XA=A^*XA+\sqrt{1-A^*A}\,0\sqrt{1-A^*A}$.

This means that we can write $A^*XA = |A|U^*XU|A|$, where $A = U|A|$ is the polar decomposition of A, so $A^*XA = |A|Y|A|$, where $Y \in \mathcal{K}$.

THEOREM 15. *Let* $\mathcal{K} \subseteq M_n$. *Then* $\mathcal{K}$ *is* C^**-convex if and only if* $S(S,T) \subseteq \mathcal{K}$ *for all* S *and* T *in* $\mathcal{K}$.

Proof. If $\mathcal{K}$ is C^*-convex, then clearly $\mathcal{K}$ contains $S(S,T)$ for all S and T in $\mathcal{K}$.

To prove the converse, note that by Lemma 9 $\mathcal{K}$ contains a scalar, and since all of the above properties are preserved by translation by scalars, we may assume that $0 \in \mathcal{K}$.

To show that $\mathcal{K}$ is C^*-convex, we need to show that if $\{X_1, \dots, X_n\} \subseteq \mathcal{K}$ and $\sum_{i=1}^n A_i^* A_i = 1$, then $\sum_{i=1}^n A_i^* X_i A_i$ is in $\mathcal{K}$. We shall prove by induction that if $\mathcal{K}$ contains every sum with $n-1$ terms, then $\mathcal{K}$ contains every sum with n terms ($n \geqslant 3$). We note that $\mathcal{K}$ contains every sum with 1 or 2 terms by hypothesis.

Given $\{X_1, \dots, X_n\} \subseteq \mathcal{K}$ and $\sum_{i=1}^n A_i^* A_i = 1$, write $A_i = U_i P_i$ in canonical polar decomposition, so that $\sum_{i=1}^n P_i^2 = 1$. Furthermore, if $Y_i = U_i^* X_i U_i$, then by Remark 7, $Y_i \in \mathcal{K}$.

Let $P = (1 - P_n^2)^{1/2}$, so that $\sum_{i=1}^{n-1} P_i^2 = P^2$ and hence for $1 \leqslant i < n$, $P_i^2 \leqslant P^2$. We recall Douglas' factorization [7]; for any $v \in \mathcal{K}$, $\|P_i v\| \leqslant \|Pv\|$, and so by setting $B_i(Pv) = P_i v$, we can define a contraction on the range of P which can be extended by continuity to the closure of the range. The orthocomplement of the range of P is the kernel of P, which is contained in the kernel of P_i, and for v in the kernel of P we set $B_i v = v/\sqrt{n-1}$. Thus $B_i P = P_i$, and so $P_i = PB_i^*$. For any vector of the form $Pv_1 + v_2$ where v_2 is in the kernel of P, since $P_i v_2 = 0$ for $1 \leqslant i < n$, we have that

$$\left(\sum_{i=1}^{n-1} B_i^* B_i (Pv_1 + v_2),\, Pv_1 + v_2 \right) = \left(\sum_{i=1}^{n-1} B_i(Pv_1 + v_2),\, B_i(Pv_1 + v_2) \right)$$

$$= \sum_{i=1}^{n-1} \left(P_i v_i + \frac{1}{\sqrt{n-1}} v_2,\, P_i v_i + \frac{1}{\sqrt{n-1}} v_2 \right)$$

$$= \sum_{i=1}^{n-1} \left((P_i^2 v_1, v_1) + \frac{1}{n-1} (v_2, v_2) \right)$$

$$= (P^2 v_1, v_1) + (v_2, v_2) = \|Pv_1 + v_2\|^2.$$

Since the vectors of the form Pv_1+v_2 are dense in $\mathcal{H}$, we have that $\sum_{i=1}^{n-1}B_i^*B_i=1$.

Finally, we may write $\sum_{i=1}^{n}A_i^*X_iA_i=\sum_{i=1}^{n}P_iY_iP_i=P[\sum_{i=1}^{n-1}B_i^*Y_iB_i]P+P_nY_nP_n$, where the term in brackets belongs to $\mathcal{K}$ by the inductive hypothesis. Thus, since $P^2+P_n^2=1$, we have written a sum with n terms as something which lies on the matricial segment connecting two members of $\mathcal{K}$, and thus it is in $\mathcal{K}$. This completes the proof. ∎

We remark that the above proof works for closed subsets of $\mathcal{L}(\mathcal{H})$, since by Lemma 9 they also contain scalars. However, the hypothesis of closure is unnecessary. This fact was pointed out to us by the referee, to whom the following is due:

THEOREM 16. *Let $\mathcal{K}\subseteq\mathcal{L}(\mathcal{H})$, infinite dimensional. Then $\mathcal{K}$ is C^*-convex if and only if $S(S,T)\subseteq\mathcal{K}$ for all S and T in $\mathcal{K}$.*

Proof. It will be sufficient to show that if $S(S,T)\subseteq\mathcal{K}$ for all $S,T\in\mathcal{K}$, then $\mathcal{K}$ is C^*-convex, since the other implication is clear.

We begin by observing that if $U:\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}$ is unitary, then $X=U^*(S\oplus T)U$ is in $S(S,T)$. For if $A_1=(1\oplus 0)U$, $A_2=(0\oplus 1)U$, then $A_1, A_2\in\mathcal{L}(\mathcal{H})$, $A_1^*A_1+A_2^*A_2=1$, and $X=A_1^*SA_1+A_2^*TA_2$.

Thus, if $S(S,T)\subseteq\mathcal{K}$ for all S and T in $\mathcal{K}$, then by induction for any $\{X_1,\ldots,X_n\}\subseteq\mathcal{K}$ and unitary $U:\mathcal{H}\to\mathcal{H}\oplus\cdots\oplus\mathcal{H}$, we have $U^*(X_1\oplus\cdots\oplus X_n)U\in\mathcal{K}$. Thus, given $\{A_1,\ldots,A_n\}\subseteq\mathcal{L}(\mathcal{H})$ with $\sum_{i=1}^{n}A_i^*A_i=1$, and $\{X_1,\ldots,X_n\}\subseteq\mathcal{K}$, let $U:\mathcal{H}\to\mathcal{H}\oplus\cdots\oplus\mathcal{H}$ be unitary, and set $X=U^*(X_1\oplus\cdots\oplus X_n)U\in\mathcal{K}$ and $A=U^*(A_1\oplus\cdots\oplus A_n)\in\mathcal{L}(\mathcal{H})$. We have $\sum_{i=1}^{n}A_i^*X_iA_i=A^*XA\in\mathcal{K}$, since $A^*A=1$. This completes the proof of the Theorem. ∎

We have been unable to find one proof which works in both the infinite- and the finite-dimensional case.

REMARK 8. If $\mathcal{K}$ is C^*-convex with $0\in\mathcal{K}$, and $T=A_1^*X_1A_1+A_2^*X_2A_2$ where $X_1, X_2\in\mathcal{K}$ and $A_1^*A_1+A_2^*A_2=1$, we can write $T=|A_1|Y_1|A_1|+|A_2|Y_2|A_2|$ where $Y_1,Y_2\in\mathcal{K}$. But since $|A_1|^2+|A_2|^2=1$, it follows that $|A_1|$ commutes with $|A_2|$. That is to say, T can be written using commuting positive coefficients.

The next lemma shows that for C^*-convex sets, two coefficients usually suffice.

LEMMA 17. *Let $\mathcal{K}$ be a closed C^*-convex set. Let $\{X_i\}_{i=1}^{\infty}$ be a bounded subset of $\mathcal{K}$, and let $\sum_1^{\infty}A_i^*A_i=1$ (in norm). Suppose $\sum_1^{\infty}A_i^*X_iA_i=T\in\mathcal{K}$. Then $T=A^*X_1A+B^*YB$, where $Y\in\mathcal{K}$ and $A^*A+B^*B=1$.*

Proof. Let $Z=\frac{1}{2}A_1^*A_1$; then $1-Z\geq\frac{1}{2}$, so $1-Z$ is positive and invertible; hence $\sqrt{1-Z}$ is also positive and invertible. Now let $B_1=\frac{1}{2}A_1(\sqrt{1-Z})^{-1}$, and for $j\geq 2$ let $B_j=A_j(\sqrt{1-Z})^{-1}$. Then $\Sigma_1^\infty B_i^*B_i=(\sqrt{1-Z})^{-1}(\frac{1}{2}A_1^*A_1+\Sigma_2^\infty A_j^*A_j)(\sqrt{1-Z})^{-1}=(\sqrt{1-Z})^{-1}(\frac{1}{2}A_1^*A_1+1-A_1^*A_1)(\sqrt{1-Z})^{-1}=(\sqrt{1-Z})^{-1}(1-Z)(\sqrt{1-Z})^{-1}=1$. Then if $Y=\Sigma_1^\infty B_i^*X_iB_i$, a simple norm estimate shows that $Y\in\mathcal{K}$. Thus, we have that $T=(\frac{1}{2}A_1)^*X_1(\frac{1}{2}A_1)+\sqrt{1-Z}\,Y\sqrt{1-Z}$, as desired. ∎

We remark that the closure of $\mathcal{K}$ is not needed for finite sums, and if $\Sigma_{i=1}^\infty A_i^*A_i=1$ in the strong operator topology, we can take $\mathcal{K}$ to be strongly closed.

REMARK 9. In view of the above results it is somewhat surprising that the sets $\{\Sigma_{i=1}^\infty A_i^*TA_i:\Sigma A_i^*A_i=1\}$ where the infinite sums are taken to converge either in norm, strongly, or weakly, while C^*-convex, are not necessarily equal to MCL(T). To see this one need only consider a positive operator T with trivial kernel and $0\in\sigma_e(T)$. By Lemma 8, $0\in$MCL(T), while all of the elements of the above sets necessarily have trivial kernels.

However, this problem does not occur in finite dimensions, as the following shows:

LEMMA 18. *Let* $\mathcal{K}=\{X_1,\ldots,X_n\}\subseteq M_k$. *Then* $\mathrm{MCL}(\mathcal{K})=\{\Sigma_{i=1}^\infty A_i^*X_{j_i}A_i:\Sigma_{i=1}^\infty A_i^*A_i=1$ *(in norm)*, $X_{j_i}\in\mathcal{K}\}$.

Proof. Let $\mathcal{Q}=M_k\oplus\cdots\oplus M_k$ (n times); then $\mathcal{Q}$ is a C^*-algebra with $\|X_1\oplus\cdots\oplus X_n\|=\max\|X_i\|$. Hence for $X=X_1\oplus\cdots\oplus X_n\in\mathcal{Q}$, the set $\mathcal{W}_k(X)$ is compact and C^*-convex, and $\mathcal{K}\subseteq\mathcal{W}_k(X)$. Thus, $\mathrm{MCL}(\mathcal{K})\subseteq\mathcal{W}_k(X)$. It is easy to see that any representation π of $\mathcal{Q}$ is of the form $\pi=\pi_1\oplus\cdots\oplus\pi_n$, with each $\pi_i(X_i)=X_i\otimes 1_{\mathcal{K}_i}$ [14]. Since every completely positive map from $C^*(X)$ to M_k can be extended to one from $\mathcal{Q}$ to M_k, by Stinespring's theorem we conclude that $\mathcal{W}_k(X)=\{A_i^*X_{j_i}A_i:\Sigma A_i^*A_i=1$, strongly and $X_{j_i}\in\mathcal{K}\}$. Thus, $\mathcal{W}_k(X)$ is contained in the strongly closed C^*-convex set generated by $\mathcal{K}$. But by the finite-dimensionality of M_k, all closures coincide, so $\mathcal{W}_k(X)=\mathrm{MCL}(\mathcal{K})$, which concludes the proof.

REMARK 10. This argument really shows that for bounded sets X, $\{\Sigma_1^\infty A_i^*X_iA_i:\Sigma A_i^*A_i=1$ strongly, $X_i\in X$ for all $i\}$ lies in every weakly (strongly) closed C^*-convex set containing X.

LEMMA 19. *Let $\mathcal{K}$ be a closed C^*-convex set contained in M_n, and let $T\in\mathcal{K}$. If $\mathcal{W}_n(T)$ denotes Arveson's nth matrix range of T (as in Example 4), then $\mathcal{W}_n(T)\subseteq\mathcal{K}$.*

Proof. Let $\varphi: M_n\to M_n$ be a completely positive map with $\varphi(1_n)=1_n$. Then by Stinespring's theorem, $\varphi=V^*\pi V$, where π is a $*$-representation of M_n and $V^*V=1$. But this forces $\pi(T)=T\otimes 1_{\mathcal{K}}$ and $V\xi=(V_1\xi,\dots,V_n\xi,\dots)$, where $V_i\in M_n$ and $\Sigma V_i^*V_i=1_n$ (in norm) [14]. Then $\varphi(T)=V^*\pi(T)V=\Sigma V_i^*TV\in\mathcal{K}$. ■

COROLLARY 20. *For $T\in M_n$, $\mathcal{W}_n(T)=\mathrm{MCL}(T)=\{\Sigma V_i^*TV_i:\Sigma V_i^*V_i=1\}$.*

Proof. We have that $T\in\mathcal{W}_n(T)$, since id: $M_n\to M_n$ is completely positive, and by Lemma 19, $\mathcal{W}_n(T)\subseteq\mathrm{MCL}(T)$. However, $\mathcal{W}_n(T)$ is compact [2], hence closed and C^*-convex (Example 3), so $\mathrm{MCL}(T)\subseteq\mathcal{W}_n(T)$. The last inclusion was shown in the proof of Lemma 19. ■

By a result of [3] it follows that if S, T are in M_n and are irreducible, then S is unitarily equivalent to T if and only if $\mathrm{MCL}(S)=\mathrm{MCL}(T)$.

REMARK 11. In view of Corollary 20 and Remark 9, it is perhaps reasonable to conjecture that for $\mathcal{K}$ separable and infinite dimensional, $\mathrm{MCL}(T)\supseteq\{\Sigma_{i=1}^\infty A_i^*TA_i:\Sigma_1^\infty A_i^*A_i=1 \text{ strongly}\}$. This however is false, as the following example shows.

Let T be the compact diagonal operator $T=(t_{i,j})$ with $t_{1,1}=\lambda$, $t_{i,j}=0$ otherwise. Then, if $E_{i,j}$ denote the usual matrix units, we have $\Sigma_{j=1}^\infty E_{1,j}^*E_{1,j}=1$ strongly, and $\Sigma_{j=1}^\infty E_{1,j}^*TE_{1,j}=\lambda$ strongly. However, since T is compact, every element of $\mathrm{MCL}(T)$ will be compact. This example shows that one can have $\lambda\in\sigma(T)$, but $\lambda\notin\mathrm{MCL}(T)$, and also $\lambda\in\mathcal{W}_1(T)$ (the closed numerical range), while $\lambda\notin\mathrm{MCL}(T)$.

We now introduce a notion of extreme points in C^*-convex sets.

DEFINITION 21. Z is a *proper matrix combination* of $\{X_1,\dots,X_n\}$ if $Z=\Sigma_1^n A_i^*X_iA_i$ where $\Sigma_1^n A_i^*A_i=1$ and each A_i is invertible.

DEFINITION 22. Suppose $\mathcal{K}$ is a C^*-convex set. Then $Z\in\mathcal{K}$ is a *C^*-extreme point* of $\mathcal{K}$ if whenever Z is a proper matricial combination of $\{X_1,\dots,X_n\}\subseteq\mathcal{K}$, then each X_i is unitarily equivalent to Z.

It is easy to see that for A_i scalars, Definition 22 reduces to the usual definition of an extreme point of a convex set, up to unitary equivalence.

Notice that by Lemma 17, we need only check sums with two terms for C^*-extremeness. Furthermore, since each A_i is invertible, this means that in the polar decomposition $A_i = U_i P_i$, U_i will be unitary. Thus, Z is a C^*-extreme point of $\mathcal{K}$ if and only if whenever Z is a proper matricial combination of $\{X_1, X_2\} \subseteq \mathcal{K}$ with positive, invertible coefficients, then each X_i is unitarily equivalent to Z. Note also that necessarily the positive coefficients commute.

Remark 12. If Z is an element in the C^*-convex set $\mathcal{K}$, then for any unitary U, $W = U^*ZU \in \mathcal{K}$; so we can write $Z = (\frac{1}{2}U)W(\frac{1}{2}U)^* + (\frac{1}{2}U)W(\frac{1}{2}U)^*$, that is Z is a proper matrix combination of W. This phenomenon explains the unitary equivalence statement in the definition of C^*-extreme point. Only in the trivial case $\mathcal{K} = \{\lambda_0 1\}$ is the unitary equivalence unnecessary.

Remark 13. Further, it follows that if Z is a C^*-extreme point of the C^*-convex set $\mathcal{K}$, then for any W unitarily equivalent to Z, we have that W is also a C^*-extreme point of $\mathcal{K}$. Similarly, $-Z$ and Z^* will also be C^*-extreme, in $-\mathcal{K}$ and $\mathcal{K}^*$, respectively.

Proposition 23. *If T is a C^*-extreme point of a C^*-convex subset $\mathcal{K}$ of M_n, then T is a linear extreme point of $\mathcal{K}$.*

Proof. Suppose not; then $T = tX + (1-t)Y$, where $0 < t < 1$, $X \neq T$, and $Y \neq T$. By the C^*-extremity of T, X and Y are unitarily equivalent to T. Thus, T is written as a proper linear combination of points in its unitary orbit. By [9], every operator in M_n is linearly extreme in its unitary orbit. This contradiction completes the proof. ∎

Proposition 24. *Let $\mathcal{B}_1 = \{T : \|T\| \leq 1\} \subseteq \mathcal{L}(\mathcal{H})$; then the unitaries are C^*-extreme points of $\mathcal{B}_1$.*

Proof. Let U be unitary and suppose $U = P_1 X_1 P_1 + P_2 X_2 P_2$, with $\|X_i\| \leq 1$, $P_i > 0$ for $i = 1,2$ and $P_1^2 + P_2^2 = 1$. Note that $\begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix}$ is unitary and that

$$\begin{pmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{pmatrix}\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}\begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix} = \begin{pmatrix} U & -P_1X_1P_2 + P_2X_2P_1 \\ -P_2X_1P_1 + P_1X_2P_2 & * \end{pmatrix}.$$

Since U is unitary and the norm of the product is not greater than 1, we have that $0=-P_1X_1P_2+P_2X_2P_1=-P_2X_1P_1+P_1X_2P_2$. From the first expression we see that $X_1=P_1^{-1}P_2X_2P_1P_2^{-1}$, and from the second that $X_1=P_2^{-1}P_1X_1 P_2P_1^{-1}$. Thus, we obtain $X_2P_1^2P_2^{-2}=P_2^{-2}P_1^2X_2$. Since P_1 and P_2 commute, we see that X_2 commutes with $P_1^2P_2^{-2}=P_1^2(1-P_1^2)^{-1}=-1+(1-P_1^2)^{-1}=-1+P_2^{-2}$. Hence X_2 commutes with P_2^{-2}, and so by the spectral theorem with P_2 and P_1.

This shows that $X_1=P_1^{-1}P_2X_2P_1P_2^{-1}=X_2$, and hence that $U=P_1X_1P_1+P_2X_2P_2=X_1=X_2$. Thus U is C^*-extreme. ∎

COROLLARY 25. *If $\mathcal{B}_1\subseteq M_n$, then the C^*-extreme points of $\mathcal{B}_1$ are the unitaries.*

Proof. Since the linear extreme points of $\mathcal{B}_1$ are the unitaries, we are done by Propositions 23 and 24. ∎

PROPOSITION 26. *Let $\mathcal{P}=\{T:0\leqslant T\leqslant 1\}\subseteq\mathcal{L}(\mathcal{H})$, $\mathcal{H}$ separable; then the projections are C^*-extreme in $\mathcal{P}$.*

Proof. Let P be a projection, and suppose $P=P_1X_1P_1+P_2X_2P_2$, with $P_i>0$, $0\leqslant X_i\leqslant 1$, and $P_1^2+P_2^2=1$. If $z\in P\mathcal{H}$, then $\langle X_1P_1z, P_1z\rangle+\langle X_2P_2z, P_2z\rangle=\langle(P_1X_1P_1+P_2X_2P_2)z, z\rangle=\langle Pz, z\rangle=\langle z, z\rangle=\langle(P_1^2+P_2^2)z, z\rangle=\langle P_1z, P_1z\rangle+\langle P_2z, P_2z\rangle$, and since $0\leqslant X_i\leqslant 1$, $\langle X_iP_iz, P_iz\rangle=\langle P_iz, P_iz\rangle$ for $i=1,2$. Thus, $X_i=1$ on $P_iP\mathcal{H}$. A similar calculation shows that $X_i=0$ on $P_i(P\mathcal{H})^{\perp}$, and so $P_iP\mathcal{H}\cap P_i(P\mathcal{H})^{\perp}=(0)$.

Since each P_i is invertible, $P_iP\mathcal{H}$ and $P_i(P\mathcal{H})^{\perp}$ are closed subspaces with $\dim(P\mathcal{H})=\dim(P_i(P\mathcal{H}))$ and $\dim((P\mathcal{H})^{\perp})=\dim(P_i(P\mathcal{H})^{\perp})$; further $\mathcal{H}=P_i(P\mathcal{H})+P_i(P\mathcal{H})^{\perp}$. Thus for each i, $\mathcal{H}=\mathcal{M}_i+\mathcal{N}_i$, where $\mathcal{M}_i$ is a closed subspace with $X_i=1$, $\mathcal{N}_i$ is a closed subspace with $X_i=0$, and $\mathcal{M}_i\cap\mathcal{N}_i=(0)$. Since each $X_i\geqslant 0$, it follows that each X_i is an orthogonal projection, and the dimensions then imply that each X_i is unitarily equivalent to P. ∎

PROPOSITION 27. *Let $\mathcal{S}=\{T: -1\leqslant T\leqslant 1\}\subseteq\mathcal{L}(\mathcal{H})$. Then $\mathcal{S}$ is C^*-convex, and the C^*-extreme points of $\mathcal{S}$ belong to $\{2E-1: E\geqslant 0$ is a projection$\}$.*

Proof. Since matrix combinations are scalar-order preserving, it is easy to see that $\mathcal{S}$ is C^*-convex. For any $T\in\mathcal{S}$, we can write $T=T_1\ominus T_2$ where $T_i\geqslant 0$; let $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$ be the corresponding decomposition of $\mathcal{H}$ and P_1, P_2 the corresponding projections. Let $Y=2T_1-2T_2+P_2-P_1$; then $Y\in\mathcal{S}$. Further, $T=\sqrt{\frac{1}{2}}\ Y\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{2}}\ (P_1-P_2)\sqrt{\frac{1}{2}}$. If T is C^*-extreme in $\mathcal{S}$, then T is unitarily equivalent to $P_1-P_2=2P_1-1$, so $T=2E-1$ where $E\geqslant 0$ is a projection. ∎

PROPOSITION 28. *Let $\mathcal{P}=\{T:0\leqslant T\leqslant 1\}\subseteq\mathcal{L}(\mathcal{H})$. If Q is a C^*-extreme point of $\mathcal{P}$, then Q is a projection.*

Proof. We claim that if Q is C^*-extreme in $\mathcal{P}$, then $T=2Q-1$ is C^*-extreme in $\mathcal{S}$. For suppose we write $T=P_1X_1P_1+P_2X_2P_2$ as a proper matrix combination of $\{X_1, X_2\}\subseteq\mathcal{S}$, where some X_{i_0} is not unitarily equivalent to T; let $Y_i=(1+X_i)/2$. Then $Q=P_1Y_1P_1+P_2Y_2P_2$ is a representation of Q as a proper matrix combination of elements of $\mathcal{P}$, but since X_{i_0} is not unitarily equivalent to T, Y_{i_0} is not unitarily equivalent to Q. This contradicts the C^*-extremity of Q, and hence T is indeed C^*-extreme in $\mathcal{S}$. But then by Proposition 27, $2Q-1=T=2E-1$, where $E\geqslant 0$ is a projection, and hence Q is a projection. ∎

COROLLARY 29. *Q is C^*-extreme in $\mathcal{P}$ if and only if Q is a projection.*

REMARK 14. Corollary 29 shows that the C^*-extreme points of $\mathcal{P}$ are identical with the usual extreme points [14]. Because of the unitary equivalence built into the definition of C^*-extreme points, the C^*-extreme points of $\mathcal{P}$ are completely specified by giving the dimension and codimension of the projection; for finite dimensions, only the dimension is needed, so in M_n there are basically only $n+1$ C^*-extreme points. Notice also that in M_n, every element of $\mathcal{P}$ is a matricial combination of at most two C^*-extreme points, namely 0 and 1, for if $0\leqslant T\leqslant 1$, then $T=\sqrt{T}\,1\sqrt{T}+\sqrt{1-T}\,0\sqrt{1-T}$; as a linearly convex combination of extreme points, such an element T will generally require many more than two extreme points.

QUESTION 1. When are X and Y C^*-extreme points of MS(X, Y)? The case $\{X, Y\}=\{0,1\}$ shows that there may be many others. The same difficulty occurs when $\{X, Y\}=\{-1,1\}$.

REMARK 15. Since a C^*-convex set is linearly convex, and since being linearly extreme is a unitary invariant, one expects that every C^*-extreme point is linearly extreme (see Proposition 23).

REMARK 16. In $\mathfrak{B}_1=\{x: \|x\| \leq 1\}$, a C^*-extreme point must have norm 1, for if $0<\|x\|<1$, we can write

$$x=\sqrt{\frac{\|x\|+1}{2}}\ \frac{x}{\|x\|}\sqrt{\frac{\|x\|+1}{2}}+\sqrt{\frac{1-\|x\|}{2}}\ \frac{-x}{\|x\|}\sqrt{\frac{1-\|x\|}{2}}\ .$$

Hence x will be extreme only if $\|x\|=1$. Proposition 24 establishes much more.

We now wish to make some comments about the matrix ranges $\mathcal{W}_n(T)$. We have previously observed that $\mathcal{W}_n(T)$ is a compact C^*-convex subset of M_n. Let P_n be the linear map from $\mathbb{C}^n \to \mathbb{C}^{n+1}$ given by $P_n([x_1,\ldots,x_n]) =[x_1,\ldots,x_n,0]$.

PROPOSITION 30. *The matrix ranges of an operator T satisfy* $\mathcal{W}_n(T) =P_n^*\mathcal{W}_{n+1}(T)P_n$.

Proof. Let $\varphi: C^*(T)\to M_{n+1}$ be a completely positive map with $\varphi(1)= 1_{n+1}$. By Stinespring's theorem, $\varphi=V^*\pi V$, where π is a $*$-representation of $C^*(T)$ on $\mathfrak{L}(\mathcal{H})$ and $V:\mathbb{C}^{n+1}\to\mathcal{H}$ with $V^*V=I_{n+1}$. Then $\tilde{\varphi}=P_n^*\varphi P_n= P_n^*(V^*\pi V)P_n=(VP_n)^*\pi(VP_n)$ is a completely positive map from $C^*(T)\to M_n$,and $\tilde{\varphi}(1)=P_n^*\varphi(1)P_n=P_n^*I_{n+1}P_n=I_n$. This shows that $P_n^*\mathcal{W}_{n+1}(T)P_n\subseteq \mathcal{W}_n(T)$.

Conversely, let $\varphi: C^*(T)\to M_n$ be completely positive with $\varphi(1)=1$. Then $\tilde{\varphi}=P_n\varphi P_n^*$ is a completely positive map from $C^*(T)\to M_{n+1}$ with $\tilde{\varphi}(1)=P_nP_n^*$, which is a projection of rank n. Let σ be a state on $C^*(T)$; then $\tilde{\sigma}: x\to \sigma(x)(I_{n+1}-P_nP_n^*)$ is a positive linear map of $C^*(T)$ to an abelian C^*-algebra, and hence completely positive [1]. Then $\Psi=\tilde{\varphi}+\tilde{\sigma}$ is a completely positive map from $C^*(T)\to M_{n+1}$ with $\Psi(1)=\tilde{\varphi}(1)+\tilde{\sigma}(1)=P_nP_n^*+(I_{n+1}-P_nP_n^*) =I_{n+1}$. Then it is easy to see that $P_n^*\Psi P_n=P_n^*(\tilde{\varphi}+\tilde{\sigma})P_n=P_n^*(P_n\varphi P_n^*)P_n +P_n^*[\sigma(I_{n+1}-P_nP_n^*)]P_n=\varphi$ and so $\mathcal{W}_n\subseteq P_n^*\mathcal{W}_{n+1}P_n$. ∎

The proof of the following result was pointed out to us by Norberto Salinas.

PROPOSITION 31. *A set $\mathcal{K}\subseteq M_n$ satisfies $\mathcal{K}=\mathcal{W}_n(T)$ for some separably acting T if and only if $\mathcal{K}$ is compact and C^*-convex.*

Proof. If $\mathcal{K}=\mathcal{W}_n(T)$, then $\mathcal{K}$ is compact and C^*-convex [3]. Conversely, if $\mathcal{K}\subseteq M_n$ is compact and C^*-convex, then $\mathcal{K}$ is hypoconvex in the sense of Salinas [15]. Hence, there is a separably acting operator T with $R^n(T)=\mathcal{K}$, and $\mathcal{W}_n(T)=\mathcal{K}$. ∎

COMMENTS. Recently, Hopenwasser, Moore, and the author have shown that the C^*-extreme points of $B_1 \subseteq \mathcal{L}(\mathcal{H})$ coincide with the linear extreme points.

In several papers, Salinas has introduced sets of $n \times n$ matrices associated with an operator, for example the essential matricial spectrum. Furthermore, these sets are, in general, C^*-convex. One of the useful results of Arveson is [1, 3.1.2], which states that for an operator T, a point in the spectrum of T which lies on the boundary of the numerical range corresponds to a character (complex homomorphism) of $C^*(T)$. There is reason to believe that a similar result holds for C^*-extreme points, which would be extremely useful. We refer the reader to the work of Salinas for elaboration on this subject. Notice also that it is the finite-dimensional case which is of greatest interest.

Also, suppose $A, B \in M_n$ are irreducible, i.e., $C^*(A) = C^*(B) = M_n$. Then Arveson has shown that A is unitarily equivalent to B if and only if $\mathcal{W}_n(A) = \mathcal{W}_n(B)$ [2,3]. Furthermore, since by Proposition 31 any C^*-convex subset of M_n is the matrix range of some operator, we feel that the study of C^*-convex sets would sharpen and/or make computationally feasible the results of Arveson on unitary equivalence of irreducible compact operators [3].

Let $\mathcal{O}(X) = \{U^*XU : U \text{ is unitary}\}$. It is not known if X is a proper matricial combination of points of $\mathcal{O}(X)$. We have heard that A. M. Davie has done some work on this question. See [9] for some results on the linear extreme points of $\mathcal{O}(X)$.

We should also remark that Davis [6] mentions C^*-convexity, without the name and in another context; see p. 195.

It is our feeling that for compact C^*-convex sets a form of Krein-Milman-type theorem should hold. At present we do not know how to establish this result. If T is normal, then the linear structure of $\mathcal{W}_n(T)$ provides sufficient information to recapture much important information about T, but if T is not normal complications arise. It is to resolve these complications that our interest in C^*-convexity arose. Furthermore, it is hoped that the set MCL(T) will serve as a useful notion of an "operator-valued spectrum" in the spirit of the papers of Hadwin [17]. In a forthcoming paper we study MCL(T) in this context.

Finally, we would like to remark that most of this work carries over when $\mathcal{L}(\mathcal{H})$ is replaced by a more general C^*-algebra. For example, the set $\{X\}$ would be C^*-convex in the C^*-algebra $\mathcal{A}$ if and only if X is an element of the center of $\mathcal{A}$.

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