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TWO FAMILIES OF GRAPHS SATISFYING THE CYCLE BASIS INTERPOLATION PROPERTY

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The length of a cycle basis of a graph G is the sum of the lengths of its cycles. Let c^-, c^+ be the lengths of the minimal and maximal cycle basis, respectively. Then G has the cycle basis interpolation property (cbip) if for all integers c, $c^- \leqslant c \leqslant c^+$, there exists a cycle basis of length c. We construct two families of graphs with the cbip, namely snake-graphs and kite-graphs.

1. Introduction

Let G be a simple, undirected and connected graph with p vertices and q edges. All cycles, the disjoint unions of cycles in G and an empty set form a vector space over the field of integers modulo 2, where the addition corresponds to the symmetric difference of edge sets. A basis of this cycle space is called a *cycle basis*. It is well known that when G is connected, the dimension $m = m(G)$ of the cycle space of G satisfies $m = q - p + 1$.

The length of a cycle C is the number of edges in it and is denoted by $l(C)$. The length of a cycle basis $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ is the sum of the lengths of its cycles:

$$l(c) = \sum_{i=1}^{m} l(C_i).$$

Let $c^- = c^-(G)$ and $c^+ = c^+(G)$ be the minimum and maximum length, respectively, of a cycle basis of G. Then G has the *cycle basis interpolation property* (cbip) if for all integers c, $c^- \leqslant c \leqslant c^+$, G has a cycle basis of length c.

For the symmetric difference of cycles we use the notation $C_1 \oplus C_2$, $\sum C_i$. Terminology not given here can be found in [1].

Some elementary properties of cycle bases relevant for studying graphs with the cbip can be found in [2]. Two families of graphs with the cbip are known, namely complete graphs K_p and wheels W_p ([2]). We prove that snake-graphs and kite-graphs also satisfy the cbip. The proofs are based on the following lemma:

Lemma 1.1 [2]. *If m cycles $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_m\}$ generate all cycles of some cycle basis $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$, then $\mathscr{C}'$ is also a cycle basis.*

2. Snake-graphs

It is clear that graphs K_p and W_p have a triangular cycle basis. Studying the cbip of these and some other graphs, the authors of [2] conjecture that:

Conjecture 1. Every biconnected graph which has a triangular cycle basis has the cbip.

The snake graphs described in this section form one more family of graphs having a triangular cycle basis and satisfying the cbip.

Definition. *Snake-graph* S_m, $m \geqslant 2$, is a maximal outerplanar graph whose cycle graph G_{S_m} is a path of m vertices (*Fig. 1*).

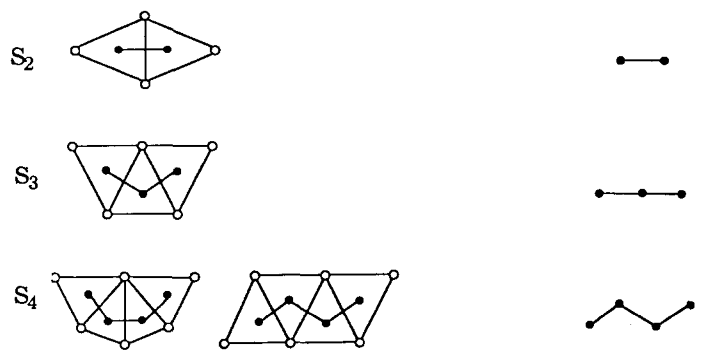

Fig. 1. The snake-graphs S_m and their cycle graphs $\mathcal{G}_{S_m}$ for $m = 2, 3, 4$.

It is clear that every S_m has a triangular cycle basis as the minimal basis. To identify a cycle basis of the particular length c, we first characterize m cycles of the total length c and then show that they generate a triangular cycle basis of S_m (Lemma 1.1) (*Fig. 2*).

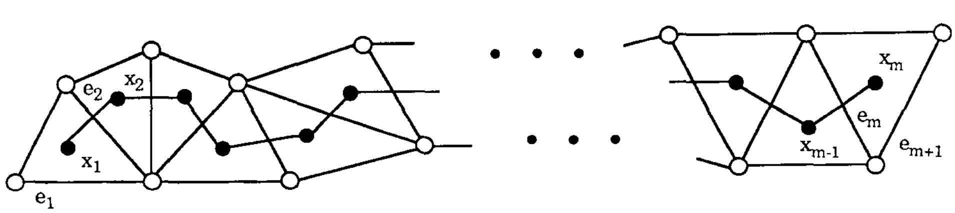

Fig. 2. The snake-graph S_m with its cycle graph $\mathcal{G}_{S_m}$ (the path $(x_1, x_2, \ldots, x_m)$).

Let us denote by t_i a triangle of S_m corresponding to the vertex x_i in G_{S_m}, $i = 1, 2, 3, \ldots, m$.

These triangles form the minimal cycle basis $\mathcal{B}_m^-$ of the graph S_m

$$\mathcal{B}_m^- = \{t_1, \ldots, t_m\} \quad \text{and} \quad l(\mathcal{B}_m^-) = 3m.$$

Define $C^{k,p} = t_k \oplus t_{k+1} \oplus \cdots \oplus t_p$, $1 \leqslant k \leqslant p \leqslant m$. $C^{k,p}$ is a cycle of the length $p-k+3$: $l(C^{k,p}) = p-k+3$.

Consider a set of cycles

$$\mathcal{B}_m^+ = \{C^{1,m}, C^{2,m}, \ldots, C^{\lceil m/2 \rceil, m}, C^{1, \lceil m/2 \rceil}, C^{1, \lceil m/2 \rceil +1}, \ldots, C^{1,m-1}\}.$$

$\mathcal{B}_\mathrm{m}^+$ consists of m cycles, which generate $\mathcal{B}_m^-$:

$$t_k = \begin{cases} C^{k,m} \oplus C^{k+1,m} & k = 1, 2, \ldots, \lceil m/2 \rceil - 1 \\ C^{1,k-1} \oplus C^{1,k} & k = \lceil m/2 \rceil + 1, \ldots, m \\ C^{1,m} \oplus C^{\lceil m/2 \rceil, m} \oplus C^{1, \lceil m/2 \rceil} & k = \lceil m/2 \rceil. \end{cases} \tag{1}$$

Hence $\mathcal{B}_m^+$ is a cycle basis (Lemma 1.1).

Let us denote e_i = the common edge of t_{i-1} and t_i, $i = 2, 3, \ldots, m$; e_1 = an edge of t_1 different from e_2; e_{m+1} = an edge of t_m different from e_m.

$\mathcal{B}_m^+$ is a maximal cycle basis, because it can be constructed by the Stepanec–Zykov method with the set of edges $e_1, e_2, \ldots, e_m$ ([3]).

Enumerate the elements of $\mathcal{B}_m^+$ as following:

$$\begin{array}{ll} C_1 = C^{1,m} & \\ & l(C_1) = m+2 \\ C_2 = C^{2,m} & \\ & l(C_2) = l(C_3) = m+1 \\ C_3 = C^{1,m-1} & \\ C_4 = C^{3,m} & \\ & l(C_4) = l(C_5) = m \\ C_5 = C^{1,m-2} & \\ \vdots & \vdots \\ C_{2i} = C^{i+1,m} & \\ & l(C_{2i}) = L(C_{2i+1}) = m+2-i \\ C_{2i+1} = C^{1,m-i} & \\ \vdots & \end{array} \tag{2}$$

For m odd we obtain

$$\begin{array}{ll} C_{m-1} = C^{\lceil m/2 \rceil, m} & \\ C_m = C^{1, \lceil m/2 \rceil} & l(C_{m-1}) = l(C_m) = (m+5)/2 \end{array}$$

and for m even:

$$\begin{array}{ll} C_{m-2} = C^{\lceil m/2 \rceil, m} = C^{m/2,m} & \\ C_{m-1} = C^{1, \lceil m/2 \rceil + 1} = C^{1, m/2+1} & l(C_{m-2}) = l(C_{m-1}) = (m+6)/2 \\ C_m = C^{1,m/2} & l(C_m) = (m+4)/2. \end{array}$$

Note that

$$\begin{array}{ll} C_1 \oplus C_2 = t_1 & C_1 \oplus C_3 = t_m \\ C_2 \oplus C_4 = t_2 & C_3 \oplus C_5 = t_{m-1} \\ C_4 \oplus C_6 = t_3 & C_5 \oplus C_7 = t_{m-2} \\ \vdots & \vdots \\ C_{2i-2} \oplus C_{2i} = t_i \quad \text{for} & C_{2i+1} \oplus C_{2i+3} = t_{m-i} \quad \text{for} \\ i = 1, 2, \ldots, [m/2] & i = 0, 1, \ldots, \lceil m/2 \rceil - 2. \end{array} \tag{3}$$

It is easy to see that the set of cycles $C_1, C_2, \ldots, C_k$ $(k \geq 3)$ generates triangles $t_1, t_2, \ldots, t_{\lfloor k/2 \rfloor}$ and $t_m, t_{m-1}, \ldots, t_{m-(\lceil k/2 \rceil - 2)}$

Theorem 1. *The snake-graphs S_m, $m \geq 2$ have the* cbip.

Proof. $\mathcal{B}_m^- = \{t_1, \ldots, t_m\}$ is a minimal cycle basis of S_m; hence $c^- = l(\mathcal{B}_m^-) = 3m$.
$\mathcal{B}_m^+ = \{C_1, C_2, \ldots, C_m\}$, described previously, is a maximal cycle basis of S_m.

$$c^+ = l(\mathcal{B}_m^+) = \begin{cases} (3m^2 + 8m + 1)/4 & m\text{-odd} \\ m(3m + 8)/4 & m\text{-even}. \end{cases}$$

Let c be an integer between c^- and c^+, and let k be the maximum number of consecutive cycles $C_1, C_2, \ldots, C_k$ from $\mathcal{B}_m^+$ that can occur in a cycle basis of length c:

$$k = \max_{1 \leq j < m} \left\{ j : \sum_{i=1}^{j} l(C_i) + 3(m - j) \leq c \right\}.$$

The cycle basis $\mathcal{B}$ of length c that we construct consists of k cycles $C_1, C_2, \ldots, C_k$ from $\mathcal{B}_m^+$, $m - k - 1$ triangles from $\mathcal{B}_m^-$ and a cycle C^* of length l, where $l = c - \sum_{i=1}^{k} l(C_i) - 3(m - k - 1)$ and $3 \leq l \leq l(C_{k+1})$. We proceed with regard to the values of m, k, and l.

Case A. $k = 0$.

$$\mathcal{B} = \{C^*, t_2, t_3, \ldots, t_m\}, \quad \text{where } C^* = C^{1,l-2}.$$

It is easy to check that $l(C^*) = l$ and $\mathcal{B}$ generates $\mathcal{B}_m^-$.

Case B. k is odd. There are two possibilities:

(i) If $k = 1$, then

$$\mathcal{B} = \{C_1, C^*, t_3, t_4, \ldots, t_m\}, \quad \text{where } C^* = C^{2,l-1}.$$

Of course, $l(C^*) = l$ and $\mathcal{B}$ generates $\mathcal{B}_m^-$. Namely,

$$t_2 = C^* \oplus t_3 \oplus t_4 \oplus \cdots \oplus t_{l-1}$$
$$t_1 = C_1 \oplus C^* \oplus t_l \oplus t_{l+1} \oplus \cdots \oplus t_m.$$

(ii) If $3 \leq k < m$, then

$$\mathcal{B} = \{C_1, C_2, \ldots, C_k, C^*, t_{\lceil k/2 \rceil + 2}, t_{\lceil k/2 \rceil + 3}, \ldots, t_{m - \lceil k/2 \rceil + 1}\} \quad \text{where} \quad C^* = C^{\lceil k/2 \rceil + 1, l + \lceil k/2 \rceil - 2}.$$

C^* is the sum of $l - 2$ triangles; hence $l(C^*) = l$.

$\mathcal{B}$ generates the elements of the minimal cycle basis $\mathcal{B}_m^-$ as follows:

$C_1, C_2, \ldots, C_k$ generate $t_1, t_2, \ldots, t_{\lfloor k/2 \rfloor}$ and $t_m, t_{m-1}, \ldots, t_{m - \lceil k/2 \rceil + 2}$.

Triangles $t_{\lceil k/2 \rceil + 2}, t_{\lceil k/2 \rceil + 3}, \ldots, t_{m - \lceil k/2 \rceil + 1}$ belong to $\mathcal{B}$.

$$t_{\lceil k/2 \rceil + 1} = C^* \oplus t_{\lceil k/2 \rceil + 2} \oplus t_{\lceil k/2 \rceil + 3} \oplus \cdots \oplus t_{l + \lceil k/2 \rceil - 2} \tag{4}$$

$$t_{\lceil k/2 \rceil} = (t_{\lfloor k/2 \rfloor + 1}) = C_1 \oplus t_1 \oplus \cdots \oplus t_{\lceil k/2 \rceil - 1} \oplus t_{\lceil k/2 \rceil + 1} \oplus \cdots \oplus t_m.$$

Note that $l \leq l(C_{k+1}) = m + 2 - (k+1)/2 = m + 2 - \lceil k/2 \rceil$. Hence $l + \lceil k/2 \rceil - 2 \leq m$ and all triangles in (4) either belong to $\mathcal{B}$ or can be generated by elements of $\mathcal{B}$.

Case C. k is even.

(i) $k = 2$.

$$\mathcal{B} = \{C_1, C_2, t_3, t_4, \ldots, t_{m-1}, C^*\}, \quad \text{where } C^* = C^{m-l+3, m}.$$

Triangles t_1, t_2 and t_m are generated by $\mathcal{B}$:

$$t_m = C^* \oplus t_{m-l+3} \oplus t_{m-l+4} \oplus \cdots \oplus t_{m-1}$$

$$t_2 = C_2 \oplus t_3 \oplus t_4 \oplus \cdots \oplus t_m$$

$$t_1 = C_1 \oplus C_2.$$

(ii) $4 \leq k < m$.

$$\mathcal{B} = \{C_1, C_2, \ldots, C_k, C^*, t_{k/2+1}, t_{k/2+2}, \ldots, t_{m-1-k/2}\}, \quad \text{where} \quad C^* = C^{m+3-k/2-l, m-k/2}.$$

One can easily check that $l(C^*) = l$. We shall show how to generate $t_{m-k/2}$ and $t_{m-k/2+1}$. Other cycles of $\mathcal{B}_m^-$ either belong to $\mathcal{B}$ or are generated by $C_1, C_2, \ldots, C_k$.

$$t_{m-k/2} = C^* \oplus t_p \oplus t_{p+1} \oplus \cdots \oplus t_{m-k/2-1} \quad \text{where } p = m + 3 - k/2 - l$$

$$t_{m-k/2+1} = C_1 \oplus t_1 \oplus \cdots \oplus t_{m-k/2} \oplus t_{m-k/2+2} \oplus \cdots \oplus t_m.$$

The theorem is proved. □

All known families of graphs with the cbip have a triangular cycle basis. The triangles seem to play an important role in the cbip of graphs.

The graphs on the left side of *Fig. 3* do not satisfy the cbip. But if we "add" to them some more triangles, we obtain graphs (B) with the cbip. The graphs on the right side of *Fig. 3* form a family of the kite-graphs.

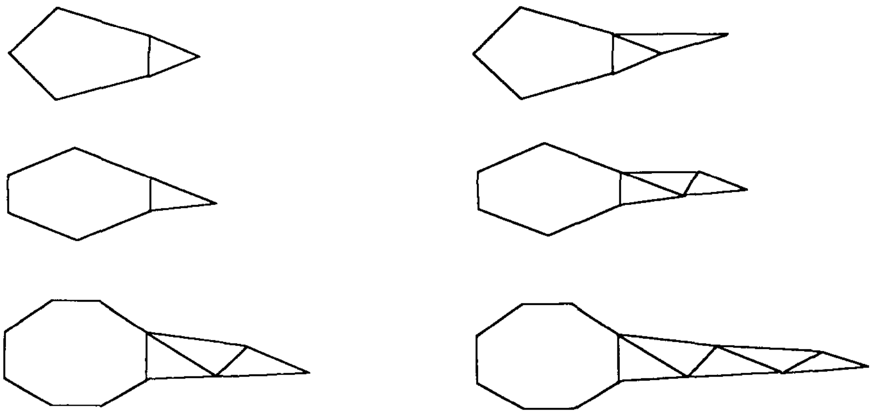

Fig. 3. Some graphs without (A) and with (B) the cbip.

3. Kite-graphs

Observe that every snake-graph has exactly two vertices of degree 2. By removing any such vertex from S_m, we obtain the graph S_{m-1}. The triangles of S_m with vertices of degree 2 will call *free-triangles* (they correspond to the end vertices of the cycle graph of S_m).

Definition. The kite-graph G_m is a graph obtained from the snake-graph S_{m-2} by replacing one of its free-triangles by the cycle C of length m.

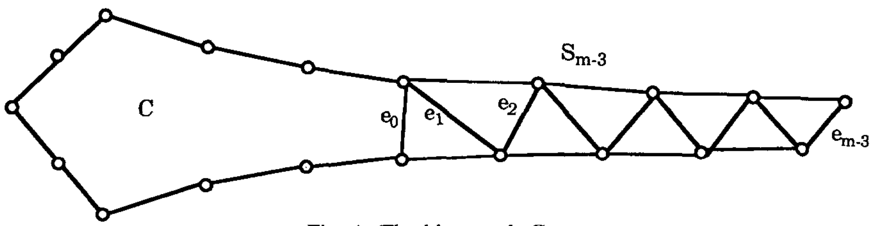

Fig. 4. The kite-graph G_m.

It is clear that the minimal cycle basis $\mathscr{B}^-$ of the kite-graph G_m consists of the cycle C and $n-3$ triangles. Its length is equal to $c^- = m + 3(m-3) = 4m - 9$.

Let us denote by t_i the triangle of G_m induced by edges e_{i-1} and e_i, $i = 1, 2, \dots, m-3$.

The maximal cycle basis $\mathscr{B}^+$ of G_m can be obtained by the Stepanec–Zykov method with the set of edges $e_0, e_1, \dots, e_{m-3}$:

$$\mathscr{B}^+ = \{\{C_0, C_1, C_2, \dots, C_{m-3}\}, \quad \text{where } C_0 = C,$$
$$C_i = C \oplus t_1 \oplus \cdots \oplus t_i, \quad i = 1, 2, \dots, m-3.$$

$$l(C_i) = m + i, \quad i = 0, 1, \dots, m-3$$

$$c^+ = \sum_{i=0}^{m-3} (m+i) = m(m-2) + \sum_{i=0}^{m-3} i = m(m-2) + (m-3)(m-2)/2$$

$$c^+ = 3(m-1)(m-2)/2.$$

Let $c^- < c < c^+$ and $k = \max_{0 \leq j < m-3} \{j : \sum_{i=0}^{j} l(C_i) + 3(m-j-3) \leq c\}$.

We now describe the construction of a cycle basis of length c. The basis $\mathscr{B}$ consists of the $k+1$ longest cycles of $\mathscr{B}^+$ (namely $C_{m-3}, C_{m-4}, \ldots, C_{m-k-3}$), $m-k-4$ triangles of $\mathscr{B}^-$ and a cycle C^* of length l, where $l = c - \sum_{i=0}^{k} l(C_{m-3-i}) - 3(m-k-4)$ and $3 \leq l \leq l(C_{m-k-4}) = 2m-k-4$.

Consider two cases with regard to the value of l.

Case A. $m \leq l < 2m-k-4$.

$$\mathscr{B} = \{C_{m-3}, C_{m-4}, \ldots, C_{m-k-3}, C^*, t_1, t_2, \ldots, t_{m-k-4}\},$$
$$\text{where } C^* = C_0 \oplus t_1 \oplus t_2 \oplus \cdots \oplus t_{l-m}.$$

It is easy to see that $l(C^*) = l$. Furthermore,

$$C_0 = C^* \oplus t_1 \oplus \cdots \oplus t_{l-m},$$
$$t_i \in \mathscr{B}, \quad i = 1, 2, \ldots, m-k-4$$
$$t_i = C_{i-1} \oplus C_i, \quad i = m-k-2, \quad m-k-1, \ldots, m-3$$
$$t_{m-k-3} = C_{m-k-3} \oplus t_1 \oplus \cdots \oplus t_{m-k-4}.$$

Case B. $3 \leq l < m$.

$$\mathscr{B} = \{C_{m-3}, C_{m-4}, \ldots, C_{m-k-3}, C^*, t_2, t_3, \ldots, t_{m-k-3}\},$$
$$\text{where } C^* = t_1 \oplus t_2 \oplus \cdots \oplus t_{l-2}.$$

$\mathscr{B}$ generates all elements of $\mathscr{B}^-$, namely:

$$t_i = C_{i+1} + C_i, \quad i = m-k-2, \quad m-k-1, \ldots, m-3$$
$$t_i \in \mathscr{B} \quad i = 2, 3, \ldots, \quad m-k-3$$
$$t_1 = C^* \oplus t_2 \oplus t_3 \oplus \cdots \oplus t_{l-2}$$
$$C_0 = C_{m-k-3} \oplus t_1 \oplus t_2 \oplus \cdots \oplus t_{m-k-3}.$$

In both cases $l(C^*) = l$, $l(\mathscr{B}) = c$ and as $\mathscr{B}$ generates the cycles of the minimal cycle basis $\mathscr{B}^-$, $\mathscr{B}$ is also a cycle basis. Thus we have proved the following theorem:

Theorem. *The kite-graphs have the* cbip.

The kite-graphs do not have a triangular cycle basis, but still have many triangles. It would be interesting to find a family of (2-connected) graphs with a small number of triangles or without any triangle at all.

References

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