On Generalized Invariants of Injective Nonsingular Module Algebras

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0. INTRODUCTION

Let $H$ be a Hopf algebra over a field $k$, let $H^0$ be its finite dual, and let $A$ be an $H$-module algebra. The invariants of $A$, $A^H$, are \{ $a \in A | h \cdot a = a$ for all $h \in H$ \}, while its $\sigma$-invariants, $A_\sigma$, are \{ $a \in A | h \cdot a = \langle \sigma, h \rangle a$ for all $h \in H$ \}, where $\sigma \in G(H^0)$, the set of group-like elements of $H^0$. It is easy to see that $A_\sigma = \rho^{-1}(A^\text{fin} \otimes k \sigma)$, where $\rho$ is the coaction induced by the given action and $A^\text{fin}$ is an appropriate subalgebra of $A$. In the language of Lie algebras $0 \neq A_\sigma$ would be termed a weight space. Naturally, generalized invariants are $A_V = \rho^{-1}(A^\text{fin} \otimes V)$, where $V$ is a subspace of $H^0$. Of particular interest is $A_V$ for $V = kG(H^0)$ which equals the sum of all one-dimensional $H$-submodules of $A$.

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In Section 2 we show (consequences of Proposition 2.5 and Theorem 2.7) that if \( A \) is an injective and nonsingular \( A^H \)-module then so is \( A_v \) for any finite-dimensional \( V \subset H^0 \). If, moreover, \( V \subset C \subset H^0 \), where \( C \) is a pointed subcoalgebra of \( H^0 \), and \( A_v \) is finitely generated over \( A^H \) for all \( \sigma \in G(C) \), then so is \( A_v \).

In Section 3 we show that such \( A \) are abundant. In fact if \( A \) is \( H \)-semiprime and \( H \)-commutative then its \( H \)-quotient ring \( Q \) is a nonsingular and injective module over its invariants (Theorem 3.8). (If the \( H \)-commutativity condition is removed we show in Theorem 3.7 that the same conclusion holds over the central invariants.) Another feature of \( Q \) is that its \( \alpha \)-invariants are normalizing elements of \( Q \) (Corollary 3.2), and have a very simple form. They are cyclic modules over the invariants (Theorem 3.5). Consequently, results of Section 2 apply and we conclude in Theorem 3.8 that generalized invariants \( Q_v \) are finitely generated injective and projective modules over the invariants for all finite-dimensional \( V \subset C \subset H^0 \), where \( C \) is a pointed subcoalgebra of \( H^0 \).

In Section 4 we apply these results in some special cases. The first are \( H \)-module algebras in the so called “Yetter–Drinfeld” category, \( \mathcal{YD} \). We prove in Theorems 4.3 and 4.4 certain commutativity relations among elements in \( A_{k(G(H))} = \sum_{a \in G(H)} A_v \) and obtain additional results when \( H \) is a quasitriangular Hopf algebra. The second class of special cases are finite-dimensional Hopf algebras with special emphasis on \( A = H \) under the adjoint action (Theorem 4.8, Theorem 4.9, and Corollary 4.11). The final example is \( H = A = O_q(SL_n) \) where we show in Theorem 4.14 certain properties of its Martindale ring of quotients. Some generalized invariants are computed with respect to the pointed sub-Hopf algebra \( U_q(sl_n) \subset (O_q(SL_n))^0 \).

1. PRELIMINARIES

The basic references are [Sw, Mo].

Let \( H \) be a Hopf algebra over a field \( k \) with a comultiplication \( \Delta \), counit \( \varepsilon \), and antipode \( S \). For a left \( H \)-comodule, \((M, \eta)\), write

\[
\eta(m) = \sum m_{-1} \otimes m_0 \in H \otimes M
\]

and for a right \( H \)-comodule, \((M, \rho)\), write

\[
\rho(m) = \sum m_0 \otimes m_1 \in M \otimes H.
\]

Given \( m \in M \), \( M \) a left (right) comodule, set \( L(m) = sp_\lambda(m_{-1}) \) and \( R(m) = sp_\lambda(m_1) \). For an \( H \)-module \( M \), let

\[
M^H = \{ m \in M \mid h \cdot m = \varepsilon(h)m, \text{ all } h \in H \}.
\]
For a left $H$-comodule $(M, \rho)$, let

$$M^\text{co}H = \{ m \in M \mid \rho(m) = 1 \otimes m \}.$$ 

An algebra $A$ is called an $H$-module algebra if $A$ is an $H$-module and

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b), \quad \text{for all } h \in Ha, \quad b \in A.$$ 

The following identity appears in [C] for $H$-module algebras:

$$\langle h \cdot a, b \rangle = \sum h_1(aSh_2 \cdot b). \quad (1)$$

Definition 1.1. Let $(A, \eta)$ be a left $H$-comodule and a left $H$-module algebra. Then $A$ is $H$-commutative if

$$ab = \sum (a_{-1} \cdot b) a_0, \quad \text{for all } a, b \in A. \quad (2)$$

Sometimes we say $\langle A, \cdot, \eta \rangle$ is $H$-commutative. In [CW2] $H$-commutativity was termed quantum commutativity.

Example 1.2. Every commutative $H$-module algebra is $H$-commutative with $\eta$ defined by $\eta(a) = 1 \otimes a$, for all $a \in A$.

Example 1.3. $\langle H, \cdot_{adj}, \Delta \rangle$ is $H$-commutative where $h \cdot_{adj} x = \sum h_2 xSh_2$, for all $h, x \in H$.

More examples can be found in [CW1].

Lemma 1.4 [CW2]. If $\langle A, \cdot, \eta \rangle$ is $H$-commutative then $A^H$ and $A^{\text{co}H}$ are contained in $Z(A)$, the center of $A$.

A subcoalgebra $C$ of $H$ is a simple subcoalgebra if the only nonzero subcoalgebra of $C$ is $C$ itself. A nonzero element $\sigma \in H$ is called a grouplike element if $\Delta(\sigma) = \sigma \otimes \sigma$. The set of grouplike elements of $H$ is denoted by $G(H)$. The coalgebra $H$ is said to be pointed if its simple subcoalgebras are one dimensional. This implies that every irreducible $H$-comodule is one dimensional as well.

We denote by $H^0$ the finite dual of $H$. If $\sigma \in G(H^0)$ and $A$ is an $H$-module algebra, define the $\sigma$-invariants [BCM] by

$$A_\sigma = \{ a \in A \mid h \cdot a = \langle \sigma, h \rangle a, \text{for all } h \in H \},$$

where $\langle \cdot, \cdot \rangle$ denotes the evaluation of $H^*$ on $H$. If $H$ is finite dimensional then $M$ is a left $H$-module iff $M$ is a right $H^*$-comodule. Indeed, for a left $H$-module, $M$, define $\rho: M \to M \otimes H^*$ by $\rho(m) = \sum h_i \cdot m \otimes h^*_i$, where $(h_i)$ and $(h^*_i)$ are dual bases of $H$ and $H^*$, respectively. Conversely, if $(M, \rho)$ is a right $H$-comodule and $m \in M$ with $\rho(m) = \sum m_0 \otimes m_1$, then $M$ turns out to be a left $H$-module by defining $h \cdot m = \sum \langle m_0, h \rangle m_1$. 
If $H$ is infinite dimensional then such a correspondence is no longer possible; instead, for a left $H$-module $M$, define

$$M^\text{fin} = \{ m \in M | H \cdot m \text{ is finite dimensional} \}.$$ 

Then $M^\text{fin}$ is a right $H^0$-comodule [Mo, Lemma 1.6.4]. Of course if $H$ or $M$ are finite dimensional then $M^\text{fin} = M$.

From now on $\rho$ denotes the right comodule structure on $M^\text{fin}$ induced by the left action of $H$.

If $V \subset H^0$ and $M$ is a left $H$-module set

$$M_V = \{ m \in M^\text{fin} | \rho(m) \in M^\text{fin} \otimes V \} = \rho^{-1}(M^\text{fin} \otimes V).$$

It was observed in [CRW] that $A_V = \rho^{-1}(A^\text{fin} \otimes k\sigma)$; hence in particular $A_v = A^H$. Note that $a \in A$ generates a one-dimensional $H$-submodule if and only if $0 \neq a \in A_v$, some $\sigma \in G(H^0)$. Hence if $H^0$ is pointed then $A_{k G(H^0)} = \sum$ finite-dimensional irreducible $H$-subcomodules of $A$.

2. $A$ is an injective nonsingular $A^H$ module

Sometimes (see Section 3) an $H$-module algebra is an injective nonsingular $A^H$-module. We see how this affects $A_V$ for finite-dimensional $V \subset H^0$.

For completeness we review well known facts about injective and projective modules. A basic reference is [G, K a]. Recall:

Let $R$ be a ring with 1 and $M, N \in _R\mathcal{M}$ ($_R\mathcal{M}$ = the category of left $R$-modules). Then $M \subset N$ is an essential submodule of $N$ if $M \cap I \neq 0$ for every nonzero submodule $I$ of $N$. We write then $M \subset \text{ess } N$.

$M$ is a nonsingular $R$-module if for $E \subset \text{ess } R$ and $m \in M, E \cdot m = 0$ implies that $m = 0$.

**Proposition 2.1.** Let $R$ be a ring with 1 and let $M, N \in _R\mathcal{M}$. Then

1. If $M \subset N$ and $M$ is injective, then $M$ is a direct summand of $N$.
2. A direct summand of an injective (projective) $R$-module is injective (projective), respectively.
3. A product of injective (nonsingular) $R$-modules is injective (nonsingular), respectively.
4. If $M \subset N$ and both $M$ and $N/M$ are injective (projective) then $N$ is injective (projective), respectively.
5. Every $M \in _R\mathcal{M}$ can be essentially embedded in an injective module $I(M)$ (called the injective hull of $M$).
If \( M \subset N \) and \( N \) is injective, then \( I(M) \subset N \).

If \( M \subset_{\text{ess}} N \), and \( n \in N \), then \( E = n^{-1}M = \{ a \in R \mid na \in M \} \subset_{\text{ess}} R \).

If \( M \subset N \) then \( M \) is nonsingular if and only if \( N \) is nonsingular.

Let \( \phi : M \to N \), where \( M \) is injective and \( N \) is nonsingular; then \( \ker \phi \) is injective and \( M \cong \ker \phi \oplus L \), for some \( R \)-submodule \( L \) of \( M \).

If \( M \) is nonsingular then the map \( \alpha : \{ \phi \in \text{End}_R(I(M)) \mid \phi(M) \subset M \} \to \text{End}_R(M) \) defined by \( \alpha(\phi) = \phi_M \) is a ring isomorphism.

If \( R \) is a self-injective ring with 1 and \( N \) is finitely generated nonsingular \( R \)-module then \( N \) is a projective and injective \( R \)-module.

Let \( n < \infty \). If \( R \) is self-injective, \( N \) is a nonsingular \( n \)-generated \( R \)-module, and \( M \) is injective, then for every \( \phi \in \text{Hom}_R(M, N) \) we have \( M \cong \ker \phi \oplus \text{im} \phi \) and \( \text{im} \phi \) is an injective and projective \( m \)-generated \( R \)-module, where \( m \leq n \).

If \( M \) is an \( H \)-module and \( \phi \neq S \subset H \), set
\[
a_M(S) = \{ m \in M \mid S \cdot m = 0 \}.
\]

If \( A \) is an \( H \)-module algebra, then \( a_A(S) \) is obviously a left and right \( A^H \)-module.

Let \( A \) be an \( H \)-module algebra; in what follows \( A^H \) plays the role of \( R \) in Proposition 2.1.

**Lemma 2.2.** If \( A \) is an injective and nonsingular \( A^H \)-module, then \( a_A(S) \) is an injective \( A^H \)-module for all nonempty \( S \subset H \).

**Proof.** Set \( I = I(a_A(S)) \). Then by Proposition 2.1.6, \( I \subset A \). Let \( x \in I \) and \( E = x^{-1}(a_A(S)) \). Then \( E \subset_{\text{ess}} A^H \) by Proposition 2.1.5 and Proposition 2.1.7. Since \( Ex \subset a_A(S) \), we have
\[
0 = S \cdot (Ex) = E(S \cdot x).
\]
But \( A \) is a nonsingular \( A^H \)-module; hence by Proposition 2.1.8, \( S \cdot x = 0 \), and thus \( x \in a_A(S) \). We have shown that \( I = a_A(S) \) and hence the latter is injective.

**Remark 2.3.** Recall from linear algebra that if \( X \) is a vector space and \( V \) is a finite-dimensional subspace of \( X^* \) then \( V^* = \{ x \in X \mid \langle V, x \rangle = 0 \} \) is a subspace of \( X \) of finite codimension.

**Lemma 2.4.** Let \( M \) be an \( H \)-module; then:

1. If \( J \) is a subspace of \( H \) of finite codimension then \( a_M(J) \subset M^{\text{fin}} \).
2. If \( V \subset H^0 \) is finite dimensional and \( J = V^* \), then \( M_V \subset a_M(J) \). In fact,
\[
M_V = \ker \phi, \quad \text{where} \quad \phi = (\text{id} \otimes \pi) \rho : a_M(J) \to M^{\text{fin}} \otimes (H^0/V),
\]
where \( \pi \) is the projection of \( H^0 \) on \( H^0/V \). If \( M = A \) is an \( H \)-module algebra then \( \phi \) is an \( A \)-module map.

\textbf{Proof.} 1. Let \( x \in a_m(J) \); then \( H \cdot x = (H/J) \cdot x \). Since \( H/J \) is finite dimensional, so is \( (H/J) \cdot x \). Hence \( x \in M^{\text{fin}} \).

2. By the remark above and by part 1, \( a_m(J) \subset M^{\text{fin}} \) and so \( \rho \) is defined on it. Let \( x \in M_V^i \); then \( \rho(x) = \sum x_0 \otimes x_1 \), where \( \{x_1\} \subset V \). Hence \( h \cdot x = \sum x_0 \langle x_1, h \rangle = 0 \), for all \( h \in J \). That is, \( x \in a_m(J) \). The rest is obvious. \( \square \)

Consequently,

\textbf{Proposition 2.5.} Let \( A \) be an \( H \)-module algebra and assume \( A \) is an injective and nonsingular \( A^H \)-module. If \( V \) is a finite-dimensional subspace of \( H^0 \), then \( A_V \) is an injective \( A^H \)-module. In particular \( A \sigma \) is an injective \( A^H \)-module for all \( \sigma \in G(H^0) \) and \( A^H \) is self-injective.

\textbf{Proof.} Let \( J \) and \( \phi \) be as in Lemma 2.4.2. Then by Lemma 2.2 \( a_m(J) \) is an injective \( A^H \)-module. Let \( \phi: a_m(J) \rightarrow A \otimes (H^0/V) \) be defined by \( \hat{\phi} = (i \otimes \text{id}) \phi \) where \( i \) is the inclusion map of \( A^{\text{fin}} \) in \( A \). Since \( A \otimes (H^0/V) \) as a left \( A^H \)-module is a product of copies of \( A \) and \( A \) is a nonsingular \( A^H \)-module, so is \( A \otimes (H^0/V) \). By Proposition 2.1.9 ker \( \hat{\phi} = A_V \) is an injective \( A^H \)-module. In particular, set \( V = k \sigma \), \( \sigma \in G(H^0) \); then \( A_v = A_V \). Hence \( A_v \) is an injective \( A^H \)-module for all \( \sigma \in G(H^0) \). For \( \sigma = e \), \( A_e = A^H \) is self-injective. \( \square \)

If \( W \subset V \) are left coideals of a coalgebra \( C \) then \( V/W \) is a left \( C \)-comodule with structure map \((\text{id} \otimes \pi)\Delta_C\). Set \( L(V/W) = \sum_{v \in V/W} L(\overline{v}) \).

\textbf{Lemma 2.6.} Let \( A \) be as in Proposition 2.5 and let \( K \subset V \) be finite-dimensional left coideals of \( H^0 \) such that \( \dim V/K = 1 \). Then \( L(V/K) = k \sigma \), \( \sigma \in G(H^0) \). If \( A_v \) is generated by \( n_v < \infty \) elements, then \( A_v / A_K \) is an injective and projective \( A^H \)-module generated over \( A^H \) by \( n_v \) elements.

\textbf{Proof.} Let \( \{v_i\}_{i \geq 1} \) be a basis of \( K \) and let \( v \in V - K \). Then \( B = \{v, v_i\}_{i \geq 1} \) is a basis of \( V \) and \( (\overline{v}) \) is a basis of \( V/K \). Let

\[ \Delta(v) = x \otimes v + \sum_{i \geq 1} x_i \otimes v_i, \quad x, x_i \in H^0, \quad x \neq 0. \]

Computing \((\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta\), using the fact that \( K \) is a left coideal of \( H^0 \) and that \( B \) is linearly independent, we get \( \Delta(x) = x \otimes x \), that is, \( x = \sigma \in G(H^0) \) and \( L(V/K) = k \sigma \).

Let \( a \in A_V \). Then

\[ \rho(a) = a' \otimes v + \sum_{i \geq 1} a_i \otimes v_i. \]
Using the form of $\Delta(v)$ and the linear independence of $B$ we get: $\rho(a') = a' \otimes \sigma$. Define $\phi: A_v \to A_v$ by $\phi(a) = a'$ for all $a \in A_v$. Then $\phi$ is a left $A^H$-module map and $\ker \phi = A_K$. Hence $A_v/A_K \cong \text{im} \phi$. Moreover, by Proposition 2.5, $A_v$ and $A^H$ are injective. Apply Proposition 2.1.12 to $R = A^H$, $N = A_v$, $M = A_v$ and $\phi$ as above and we are done.

We are ready to prove the main theorem of this section:

**Theorem 2.7.** Let $H$ be a Hopf algebra, let $C$ be a pointed subcoalgebra of $H^0$, and let $W \subset V$ be finite-dimensional left coideals of $C$. Let $A$ be an $H$-module algebra so that:

1. $A$ is an injective and nonsingular $A^H$-module, and
2. For all $\sigma \in G(C) \cap L(V/W)$, $A_v$ is generated over $A^H$ by $n_\sigma < \infty$ elements.

Then $A_v/A_w$ is an injective and projective $A^H$-module generated by $nt$ elements where $n = \dim V/W$ and $t = \max(n_\sigma | \sigma \in G(C) \cap L(V/W))$. If $V$ is a finite-dimensional subspace of $C$ (not necessarily a coideal), then $A_v$ is finitely generated over $A^H$.

**Proof.** By induction, $n = \dim V/W$. This is obviously true for $n = 0$. Let $K$ be a left coideal with $W \subset K \subset V$ and let $V/K$ be a simple left comodule. Since $C$ is a pointed coalgebra, $\dim V/K = 1$. Since $L(V/K) \subset L(V/W)$, Lemma 2.6 applies to $V/K$ and so $A_v/A_K$ is an injective and projective $A^H$ module generated by $t$ elements. By induction, $A_K/A_w$ is an injective and projective $A^H$-module generated over $A^H$ by $(n - 1)t$ elements. Since $A_v/A_K \cong (A_v/A_w)/(A_K/A_w)$, we are done by the above and Proposition 2.1.4.

If $V$ is a finite-dimensional subspace of $C$ then $V^1 = \sum_{v \in V} R(v)$ is a left coideal contained in $V$ and $A_{V^1} = A_V$. By above $V_{V^1}$ is finitely generated over $A^H$. 

### 3. H-Quotient Algebras

In this section we prove that Theorem 2.7 applies to quotient rings of certain $H$-module algebras. $H$-quotient algebras of $H$-module algebras were introduced in [C]. The construction is an analogue of Martindale's ring of quotients which will be denoted by $\text{Mar}(A)$. We use in the construction the filter $\mathcal{F} = \{I | I$ is an $H$-stable ideal of $A$ with trivial left and right annihilators\}.

Specifically, let $I, J \in \mathcal{F}$ and $f: I \to A$, $f': J \to A$ be right $A$-module maps. Say that $f = f'$ if $f'|_K = f'|_K$ for some $K \in \mathcal{F}$. This is an equivalence
relation on the set of all such maps. Set \( q = [f] \) as the equivalence class of 
\( f \), and set \( Q_H(A) \) as the set of all such \( q \). Then \( Q_H(A) \) is an \( H \)-module 
algebra where the action of \( H \) on \( A \) is extended to \( Q_H(A) \) as follows: If 
\( q = [f], f: I \to A \), then \( h \cdot q = [h \cdot f] \) where 
\[
(h \cdot f): I \to A \quad \text{by} \quad a \mapsto \sum h_1 \cdot (f(Sh_2 \cdot a)).
\] 

The algebra \( A \subset Q_H(A) \) by \( a \mapsto L_a \) where \( L_a: A \to A \) is left multiplication by \( a \). Recall that \( q \in \text{Mar}(A) \) is an \( A \)-normalizing element if \( qA = Aq \).

Note that \( Q_H(A) \subset \text{Mar}(A) \) [Mo, p. 98]. It is easy to see that \( q \in \text{Mar}(A) \) belongs to \( Q_H(A) \) if and only if there exists an \( I \in \mathcal{F} \) so that 
\( qI \subset A \).

An algebra \( A \) is called \( H \)-semiprime if \( A \) contains no nontrivial \( H \)-stable nilpotent ideals. When \( A \) is \( H \)-semiprime, each \( H \)-stable ideal \( J \) gives rise to an element in \( \mathcal{F} \). Therefore, \( H \)-semiprimeness implies that \( \text{Ann}(J) \) is left annihilator of \( J \) and right annihilator of \( J \) in \( A \); and \( J \odot \text{Ann}(J) \in \mathcal{F} \).

In what follows we consider \( Q_H(A) \) for an \( H \)-module algebra \( A \) which is also an \( H \)-comodule. We first show that if \( \langle A, \cdot, \eta \rangle \) is an \( H \)-commutative algebra then \( Q_H(A) \) "commutes" in a sense with \( A \).

**Proposition 3.1.** Let \( \langle A, \cdot, \eta \rangle \) be an \( H \)-commutative algebra. If \( q \in Q_H(A) \) and \( a \in A \), then:

1. \( qa = \Sigma (a_{-1} \cdot q) a_0 \).
2. If \( H \) has a bijective antipode then \( qa = \Sigma a_0(S^{-1}(a_{-1}) \cdot q) \).

**Proof.**

1. Let \( I \in \mathcal{F} \) so that \( qI \subset A \) and \( x \in I \). Then,

\[
aqx = \sum (a_{-1} \cdot qx) a_0 \quad \text{by (2)}
= \sum (a_{-2} \cdot q)(a_{-1} \cdot x) a_0
= \sum (a_{-1} \cdot q) a_0 x \quad \text{by (2)}.
\]

Hence \( (aq - \Sigma (a_{-1} \cdot q) a_0)I = 0 \), and we are done.

2.

\[
qa = \Sigma e(a_{-1}) qa_0
= \Sigma (a_{-1} \cdot (S^{-1}a_{-2} \cdot q)) a_0
= \Sigma a_0(S^{-1}(a_{-1}) \cdot q) \quad \text{by part 1}.
\]
We consider next some consequences of Proposition 3.1. Recall that each \( \sigma \in G(H^0) \) determines an automorphism \( \hat{\sigma} \) of \( A \) defined by

\[
a^\hat{\sigma} = \sum \langle \sigma, a_{-1} \rangle a_0.
\]  

(4)

Note that \( \hat{\sigma} \) is a group homomorphism, in particular \( \hat{\sigma} \circ \hat{\tau} = \hat{\sigma} \circ \hat{\tau} \).

**Corollary 3.2.** Let \( \langle A, \cdot, \eta \rangle \) be an \( H \)-commutative algebra, and let \( \sigma \in G(H^0) \) and \( x \in (Q_H(A))_x \); then:

1. For all \( a \in A \), \( ax = xa^\hat{\sigma} \). In particular \( x \) is an \( A \)-normalizing element.
2. If \( I \in \mathcal{I} \) and \( xI \subset A \) then \( xI \) is a two sided \( H \)-stable ideal of \( A \).
3. \((Q_H(A))^H \subset Z(Q_H(A))\), the center of \( Q_H(A) \).

**Proof.**

1.

\[
ax = \sum (a_{-1} \cdot x) a_0 \quad \text{by } H\text{-commutativity}
\]

\[
= \sum \langle \sigma, a_{-1} \rangle xa_0 \quad \text{since } x \in (Q_H(A))_x
\]

\[
= \sum x\langle \sigma, a_{-1} \rangle a_0
\]

\[
= xa^\hat{\sigma}.
\]

2. By the first part, \( xI \) is a two sided ideal. Let \( h \in H \); then for all \( a \in I \),
\[
h \cdot (xa) = \sum (h_1 \cdot x)(h_2 \cdot a) = \sum \langle \sigma, h_1 \rangle h_2 \cdot a \in xI.
\]

Thus \( xI \) is \( H \)-stable.

3. By the first part, for each \( q \in (Q_H(A))^H = (Q_H(A))_x \), we have: \( aq = qa \) for all \( a \in A \). It is a routine argument in quotient rings to show that \( q \) is central in \( Q_H(A) \).  

Another useful fact is:

**Lemma 3.3.** Let \( x \in (Q_H(A))_x \), \( a \in A \), and \( h \in H \); then:

1. \((h \cdot a)x = \sum \langle \sigma^{-1}, h_2 \rangle h_1 \cdot (ax)\),
2. \(x(h \cdot a) = \sum \langle \sigma^{-1}, h_1 \rangle h_2 \cdot (xa)\).

**Proof.**

1.

\[
(h \cdot a)x = \sum h_2 \cdot (aSh_2 \cdot x) \quad \text{by (1)}
\]

\[
= \sum h_2 \cdot (a\langle \sigma, Sh_2 \rangle x)
\]

\[
= \sum \langle \sigma^{-1}, h_2 \rangle h_1 \cdot (ax).
\]
2.

\[ \sum \langle \sigma^{-1}, h_1 \rangle h_2 \cdot (xa) = \sum \langle \sigma^{-1}, h_1 \rangle (h_2 \cdot x)(h_3 \cdot a) = \sum \langle \sigma^{-1}, h_1 \rangle (\sigma, h_2 \cdot x)(h_3 \cdot a) = x(h \cdot a). \]

Let \( A \) be an \( H \)-module algebra. When there is no danger of ambiguity write \( Q = Q_H(A) \). Recall [BCM, 3.9] that \( Q_\sigma Q_\tau \subseteq Q_{\sigma \tau} \) for all \( \sigma, \tau \in G(H^0) \) and that \( S_Q = \sum_{\sigma \in G(H^0)} Q_\sigma \) is a direct sum. That is, \( S_Q \) is a \( G(H^0) \)-graded algebra. Corollary 3.2 indicates that if \( \langle A, \cdot\rangle \) is \( H \)-commutative then \( \hat{\cdot} \) is a so-called \((Kh2)\) \( X \)-inner automorphism. Following [Kh1, CM] we show that when \( A \) is \( H \)-semiprime, \( Q_\sigma \) has a nice form; it is a cyclic module over \( Q_H \).

**Proposition 3.4.** Let \( \langle A, \cdot, \eta \rangle \) be an \( H \)-commutative \( H \)-semiprime algebra. Let \( \sigma \in G(H^0) \); then:

1. For any \( x \in Q_\sigma \) there exists \( x' \in Q_{\sigma^{-1}} \) so that \( x = xx'x \) and \( x' = x'x'x' \).

2. \( x'x' = xx' = e_x \in Q_H \subseteq \mathbb{Z}(Q), \ x' e_x = x', \ x'e_x = x' \), and \( e_x \) is an idempotent.

3. If \( x \in Q_\sigma, \ y \in Q \) with \( x = ey \), and \( e \) is an idempotent then \( x = e_yx \) and \( e = ee \). If, moreover, \( y \in Q_\sigma \), then \( e = e_y e_x \).

4. The element \( x' \) is the unique element of \( Q_{\sigma^{-1}} \) so that \( xx'x = x \) and \( x'x'x' = x' \).

**Proof.** By Corollary 3.2.1 any \( x \in Q_\sigma \) is an \( A \)-normalizing element of \( \text{Mar}(A) \); hence [CM, Theorem 4] applies. So, there exists an \( x' \in \text{Mar}(A) \) satisfying \( xx'x = x \), \( x'x'x' = x' \) and \( xx' = x'x = e_x \in \mathbb{Z}(\text{Mar}(A)) \). We show first that \( x^2 \in Q \). Let \( I \in \mathcal{H} \) so that \( xI \subseteq A \). Then by Corollary 3.2.2 \( xI \) is an \( H \)-stable ideal of \( A \). Let \( K = \text{Ann}(xI) \), \( J = xI \oplus K \in \mathcal{H} \). Note that if \( a \in K \), then \( axI = 0 \) implies \( ax = 0 \); hence \( e_x a = ae_x = ax = 0 \). Therefore, \( e_x J = e_x (xI \oplus K) = xI \subseteq A \) which implies in turn that \( e_x \in Q \). Now, \( K = \text{Ann}(xI) = \text{Ann}(x(I \cap J)) \) and \( x' K = x' e_x K = 0 \). Hence \( J' = x(I \cap J) \oplus K \in \mathcal{H} \), and \( x' J' \subseteq A \). This implies that \( x' \in Q \).
We show now that $e_x \in Q^H$ and $x' \in Q_{\sigma^{-1}}$. Let $l \in I$, $a \in K$, $h \in H$; then for all $b = xl + a \in J$,

$$(h \cdot e_x)b$$

$$= \sum h_1 \cdot (e_xSh_2 \cdot b) \quad (\text{by (1)})$$

$$= \sum h_1 \cdot (e_xSh_2 \cdot (xl + a))$$

$$= \sum h_1 \cdot (e_x(Sh_3 \cdot x)(Sh_2 \cdot l)) \quad (\text{since } K \text{ is } H\text{-stable and } e_xK = 0)$$

$$= \sum h_1 \cdot (e_x(\sigma, Sh_2)xSh_2 \cdot l)$$

$$= \sum h_1 \cdot (x(\sigma, Sh_2)Sh_2 \cdot l) \quad (\text{since } e_x x = x)$$

$$= \sum \langle \sigma, h_1 \rangle \langle \sigma, Sh_2 \rangle xl$$

$$= \sum \langle \sigma, h_1 \rangle \langle \sigma, Sh_2 \rangle e_x(xl + a)$$

$$= e(h)e_xb.$$  

Hence $h \cdot e_x = e(h)e_x$. Now,

$$h \cdot x' = (h \cdot x')xx' = \sum h_1 \cdot (x' Sh_2 \cdot x) x' = \sum h_1 \cdot (\langle \sigma, Sh_2 \rangle x' x) x' = \langle \sigma^{-1}, h \rangle x'.$$

This concludes the proof of the first two parts.

3. If $x = ey$, and $e$ is an idempotent then $x = ex$. Thus $xx' = eex'$, so $e_x = ee_x = e_x e$. This in turn implies that $x = e_x x = e_x ey = e_x y$. If, moreover, $y \in Q_{\sigma}$, then $x = e_x y = e_x e_y y$. Setting $e = e_x e_y$, we have the above $e_x = e_x e_y$.

4. Assume $xx'' x = x$ and $x'' xx'' = x''$, where $x'' \in Q_{\sigma^{-1}}$. Then $e = xx'' \in Z(Q)$ and $e x = x$. By (3) this implies $e_x = ee_x = xx'' e_x = x x''$. Hence $x'' = x'' xx'' = x'' e_x = e_x x'' e_x = x'' x' x' x' = x' x' x' = x'$.

We call $x'$ a partial inverse of $x$.

Define a partial ordering on $Q_{\sigma}$ as follows: Let $x, y \in Q_{\sigma}$; then $x \leq y$ iff there exists an idempotent $e \in Z(Q)$ so that $x = ey$. By Proposition 3.4.3 this is equivalent to saying that $x = e_x y$. We are ready to prove:

**Theorem 3.5.** Let $\langle A, \cdot, \eta \rangle$ be an $H$-semiprime $H$-commutative algebra, $\sigma \in G(H^2)$, and $Q = Q_H(A)$. Then:

1. Every chain in $Q_{\sigma}$ has an upper bound. That is, if $x_1 \leq x_2 \leq \cdots$ is a chain in $Q_{\sigma}$ and $\{e_x\}$ is as in Proposition 3.4, then there exists $x \in Q_{\sigma}$ so that $x_j = e_x x$ for all $j$. 

There exists $x_\sigma \in Q$ so that $Q_\sigma = Q^H x_\sigma$. In particular $e_{x_\sigma} y = y$, all $y \in Q_\sigma$.

3. $Q_{\sigma^{-1}} = Q^H (x_\sigma)'$, where $(x_\sigma)'$ is the partial inverse of $x_\sigma$.

Proof. Let $e_j = e_{x_j}$ and let $I_j \in \mathcal{F}$ so that $x_j I_j$ and $e_j I_j$ are contained in $A$. Let $I = \sum e_{I_j} K = \text{Ann}(I)$ and $J = I \oplus K \in \mathcal{F}$. Define $f: J \to A$ by
\[ f(a + b) = x_j a, \quad a \in e_{I_j}, \quad b \in K. \]

This map is well defined for, say, $a \in e_k I_k$ with $k > j$; then $x_j < x_k$ implies that $x_j = e_j x_k$, which implies that $x_k a = x_k e_j a = x_j a$. Set $x = [f] \in Q^H$. Note that $xe_j = x_j$ and that $(h \cdot x)K = 0$, all $h \in H$. We wish to prove that $x \in Q_\sigma$. Well, for all $h \in H, a \in e_j I_j, b \in K$,
\[ (h \cdot x)(a + b) = (h \cdot x)e_j a = (h \cdot (xe_j))a = (h \cdot x_j)a = \langle \sigma, h \rangle x_j a = \langle \sigma, h \rangle x(a + b). \]

This implies that $h \cdot x = \langle \sigma, h \rangle x$ and hence that $x \in Q_\sigma$.

2. Let $x_1 \leq x_2 \leq \cdots$ be any chain in $Q_\sigma$; then by the first part this chain has an upper bound $x \in Q_\sigma$, so that $x_i = e_{x_i} x$ for all $i$. We may apply Zorn’s lemma to find a maximal element $x_\sigma \in Q_\sigma$. Let $(x_\sigma)' \in Q_{\sigma^{-1}}$ be defined as in Proposition 3.4; that is: $(x_\sigma)' x_\sigma = e_{x_\sigma} \in Q^H$. Let $z \in Q_\sigma$; then since $1 - e_{x_\sigma} \in Q^H$, $x_\sigma + (1 - e_{x_\sigma})z \in Q_\sigma$. But $x_\sigma \leq x_\sigma + (1 - e_{x_\sigma})z$, for $e_{x_\sigma}(x_\sigma + (1 - e_{x_\sigma})z) = e_{x_\sigma} x_\sigma = x_\sigma$. By maximality of $x_\sigma$, we have: $(1 - e_{x_\sigma})z = 0$. Hence $z = ze_{x_\sigma} = z(x_\sigma)' x_\sigma$. Since $z(x_\sigma)' \in Q_\sigma Q_{\sigma^{-1}} \subset Q^H$, we are done.

3. We wish to prove that $(x_\sigma)'$ is maximal in $Q_{\sigma^{-1}}$. Assume that there exists $y \in Q_{\sigma^{-1}}$ with $(x_\sigma)' < y$; then $(x_\sigma)' = e_{x_\sigma} y$. Hence $(x_\sigma)' y = e_{x_\sigma} y$ and by Proposition 3.4.3 we also have that $e_{x_\sigma} y = e_{x_\sigma}$. Thus $(x_\sigma)' y = e_{x_\sigma}$ and hence $e_{x_\sigma} y = x_\sigma(x_\sigma)' y = x_\sigma e_{x_\sigma} = x_\sigma$. By maximality of $x_\sigma$, $y = x$ and by uniqueness of the partial inverse $y' = x'$, which proves our claim.

When $A$ is $H$-prime we have:

**Corollary 3.6.** Let $H$ be a Hopf algebra, let $T$ be a pointed sub-Hopf algebra of $H^0$, and let $\langle A, \cdot, \eta \rangle$ be an $H$-commutative $H$-prime algebra. Then $Q_T$ is an $H$-simple algebra.

**Proof.** Let $0 \neq I$ be an $H$-stable ideal of $Q_T$. Since $T$ is pointed, $I$ contains a simple $T$-subcomodule, which is of the form $I_\sigma, \sigma \in G(T)$. Since $A$ is $H$-prime, so is $Q$, and hence $Q^H$ is a domain. If $0 \neq x \in I_\sigma$ then $xx' = e \in Q^H$ is an idempotent, and thus equals 1. We have shown that $I_\sigma$ contains an invertible element; hence $I = Q_T$. 

We first show that for certain $A$, $Q_n(A)$ satisfies the conditions of Theorem 2.7. First we show a general result, similar in flavor to results about the Martindale ring of quotients [Kh2].

**Theorem 3.7.** Assume $A$ is an $H$-semiprime $H$-module algebra and set $R = Z(Q)^H$, where $Q = Q_n(A)$. Then $Q$ is a nonsingular injective $R$-module.

**Proof.** We first prove nonsingularity. Let $E$ be an essential ideal of the commutative ring $R$, and $L = \text{Ann}_Q(E)$. Then $L$ is an $H$-stable ideal of $Q$. Set $L' = \text{Ann}_Q(L)$; then $L'$ is $H$-stable by (1) and $L \cap L' = \{0\}$ since $Q$ is also $H$-semiprime. Set $B = L \oplus L'$ and define $\phi: B \to B$ by $\phi(a + a') = a$. Then $\phi = \phi^2$ is a left and right $R$-module map. Now, $\phi^{-1}(A) = A \cap L \neq 0$ and it is easy to see that $A \cap L' = \text{Ann}_Q(A \cap L)$. Take $T = (A \cap L) \oplus (A \cap L')$; then $T \in \mathcal{F}$ and $\phi^{-1}(A) \cap T = A \cap L$ is an $H$-stable ideal of $A$. Let $e = [\phi_T]$. Then $e \in Q$ is an idempotent, $e \in Z(Q)$ because $\phi_T$ is also left $A$-linear, and $e \in L$. Moreover, $ex = \phi(x)$ for all $x \in B$. We show that $e \in R$. Since $Le = eL$ and $L$ is $H$-stable we have $h \cdot e \in Le$ for all $h \in H$. This implies by (1) that $h \cdot e = (h \cdot e)e = \Sigma h_1 \cdot (eSh_2 \cdot e)$. Since $h_2 \cdot e \in eL$, the latter expression equals $e(h)e$. We have shown that $e \in R$. But $L'e = eL' = \phi(L') = 0$; hence this is true for $E \subseteq L'$. This is a contradiction since $E$ is an essential ideal of $R$ and $e$ is an idempotent. Hence $Q$ is a nonsingular $R$-module.

We prove now that $Q$ is an injective $R$-module. Let $I = I(Q)$ be the injective hull of the $R$-module $Q$. We show that $I = Q$. Let $m \in I$ and set $E = m^{-1}Q$. By Proposition 2.1.7, $E$ is an essential ideal of $R$. For each $x \in E$ there exists $F_x \in \mathcal{F}$ such that $xF_x \subset A$. Similarly, since $mx \in Q$, there exists $F_{mx} \in \mathcal{F}$ such that $mxF_{mx} \subset A$. Then

$$J = \sum_{x \in E} x(F_x \cap F_{mx})$$

is an $H$-stable ideal of $R$ since $x \in Z(Q)^H$ and $F_x$ and $F_{mx}$ are $H$-stable ideals of $A$. If $yJ = 0$ for some $y \in A$ then $yx = 0$ for all $x \in E$. But by the previous part, $Q$ is a nonsingular $R$-module. Therefore $y = 0$. We have shown that $J \in \mathcal{F}$. Now, $mJ \subset A$, so we define $\beta: J \to A$ by $\beta(j) = mj$ for $j \in J$. Since $\beta$ is a right $A$-module map, there exists $q \in Q$ such that $q = [\beta]$, and hence $(q - m)J' = 0$, where $J' \in \mathcal{F}$ and $J' \subset J$. But, $(q - m)E \subset Q$ and $(q - m)J^2E = 0$. Since $E \subset Z(Q)$, this implies that $(q - m)E = 0$. Since $Q$ is a nonsingular $R$-module and $Q \subset_{ess} I$, so is $I$ (by Proposition 2.1.8). Thus $q = m$ and so $I = Q$. 

As a corollary we have:

**Theorem 3.8.** Let \( \langle A, \cdot, \eta \rangle \) be an \( H \)-semiprime, \( H \)-commutative, \( H \)-module algebra. Set \( Q = Q_H(A) \); then

1. \( Q \) is a nonsingular injective \( Q_H \)-module.
2. Let \( C \) be a pointed subcoalgebra of \( H^0 \) and let \( W \subset V \) be finite-dimensional left coideals of \( C \). Then \( Q_V/Q_W \) is an injective and projective \( Q_H \)-module generated over \( Q_H \) by \( \dim V/W \) elements.

**Proof.**

1. By [CW2, Corollary 2.4.2], \( Q_H \subset Z(Q) \); thus \( R = Q_H \) and the rest follows from Theorem 3.7.

2. By Theorem 3.5, \( Q_{\sigma} \) is a cyclic \( Q_H \) module for each \( \sigma \in G(H^0) \). By this fact and the first part Theorem 2.7 can be applied to \( Q \) and \( Q_H \) with \( t = 1 \) and the result follows.

**4. APPLICATIONS AND EXAMPLES**

We start by applications to \( H \)-commutative algebras in \( \text{YD}^H \), the "Yetter–Drinfeld" category, and in particular to quasitriangular Hopf algebras.

**Definition 4.1.** The "Yetter–Drinfeld" category \( \text{YD}^H \) is the category of objects which are left \( H \)-modules, left \( H \)-comodules, and satisfy the compatibility condition

\[
\sum h_{1''} \otimes h_2 \cdot v = \sum (h_1 \cdot v)_{-1} h_2 \otimes (h_1 \cdot v)_0. \tag{5}
\]

A particular example of an object in \( \text{YD}^H \) is \( H \) as in Example 1.3.

The subalgebra \( \sum_{\sigma \in G(H^0)} H_{\sigma} \) is then termed the semicenter of \( H \), for it is a well known notion for \( H = U(L) \), the enveloping algebra of a Lie algebra. The semicenter was explored in [BCM] for general \( H \) and in [MP] for \( H = kG \), when \( H \) was assumed to be prime. For both \( H = U(L) \) and prime \( H = kG \), the semicenter is a commutative algebra, which is both graded and acted upon by \( G(H^0) \). In the following we explore the existence of similar results for any \( H \)-commutative algebra, \( A \), in \( \text{YD}^H \). It is easy to see that \( A_{kG(H^0)} = \sum_{\sigma \in G(H^0)} A_{\sigma} \) is graded by \( G(H^0) \). We show that \( G(H^0) \) acts on \( A_{kG(H^0)} \) by automorphisms.

**Lemma 4.2.** Let \( A \in \text{YD}^H \), \( \sigma, \tau \in G(H^0) \). Then

\[
(A_{\sigma})^{\hat{\tau}} = A_{\tau^{-1} \sigma \tau}
\]

where \( \hat{\tau} \) is defined in (4).
Proof. Let \( a \in A_\sigma, h \in H \). Then

\[
h \cdot a^\tau = h \cdot \sum \langle \tau, a_{-1} \rangle a_0
\]

\[
= \sum \langle \tau^{-1}, h_1 \rangle \langle \tau, h_2 \rangle \langle \tau, a_{-1} \rangle h_3 \cdot a_0
\]

\[
= \sum \langle \tau^{-1}, h_1 \rangle \langle \tau, h_2 a_{-1} \rangle (h_3 \cdot a_0)
\]

\[
= \sum \langle \tau^{-1}, h_1 \rangle \langle \tau, (h_2 \cdot a)_{-1} h_3 \rangle (h_2 \cdot a)_0 \quad \text{(by (5))}
\]

\[
= \sum \langle \tau^{-1}, h_1 \rangle \langle \tau, h_3 \rangle \langle \sigma, h_2 \rangle \langle \tau, a_{-1} \rangle a_0 \quad \text{(since } a \in A_\sigma)\]

\[
= \langle \tau^{-1} \sigma \tau, h \rangle a^\tau.
\]

That is, \( a^\tau \in A_{\tau^{-1} \sigma \tau} \). Conversely, let \( b \in A_{\tau^{-1} \sigma \tau} \). Then by the above: \( a = b^{\tau^{-1}} \in A_{\tau^{-1} \sigma \tau}^{-1} = A_\sigma \), and so \( b = a^\tau \in (A_\sigma)^\tau \). \qed

Moreover,

**Theorem 4.3.** Let \( A \in H \mathcal{H} \mathcal{D} \) be an \( H \)-semiprime \( H \)-commutative algebra, and \( \sigma \in G(H^0) \); then:

1. For any \( x \in A_\sigma \), \( x^\tau = x \).
2. For any \( i, j \), if \( x \in A_{\sigma^i} \) and \( y \in A_{\sigma^j} \), then \( xy = yx \).

Proof. 1. For simplicity of notation let us write \( a^\sigma \) for \( a^\tau \). Let \( x \in A_\sigma \) and \( a \in A \); then by Corollary 3.2 \( ax = xa^\sigma \). In particular, \( x^2 = xx \). This implies that \( x(x^\sigma - x) = 0 \). On the other hand, by Lemma 4.2 \( x^\sigma \in A_\sigma \) as well, as thus \( xx^\sigma = x^2x^\sigma \). This implies that \( (x^\sigma - x)x^\sigma = 0 \). Hence \( (x^\sigma - x)^3 = 0 \). Since \( x^\sigma \) and \( x \) are in \( A_\sigma \), so is \( x^\sigma - x \). Thus by Corollary 3.2, \( I = A(x^\sigma - x) = (x^\sigma - x)A \) is an \( H \)-stable ideal of \( A \). By the above, \( I^3 = 0 \). Since \( A \) is \( H \)-semiprime, \( I \) must be 0. That is \( x^\sigma = x \) as claimed.

2. Let \( x \in A_{\sigma^i} \) and \( y \in A_{\sigma^j} \). By Lemma 4.2 \( x^{\sigma^n} \in A_{\sigma^n} \) and \( y^{\sigma^n} \in A_{\sigma^n} \) for all \( m \in \mathbb{Z} \). By part 1: \( x = x^{\sigma^i} = x^{\sigma^{-i}} \) and \( y = y^{\sigma^{-j}} \). Let \( d = \text{g.c.d}(i, j) \); then

\[
1 = \frac{i}{d} t + \frac{j}{d} t \quad \text{some } s, t \in \mathbb{Z}.
\]

We may choose \( s < 0 \). Now by Corollary 3.2 applied repeatedly we have

\[
xy = y^{\sigma^i} x = \cdots = x^{\sigma^{-s} \sigma^{i-s}} y^{\sigma^{i-s}} = y^{\sigma^{-s} \sigma^{i-s}} x^{\sigma^{-s}}.
\]

Set \( n = \frac{i}{d} (-s) \); then \( nj = \frac{i}{d} (-s) \) and so \( i \mid nj \). Since \( x^{\sigma^i} = x \), we have \( x^{\sigma^{i-s}} = x \). While by (6)

\[
n + 1 = \frac{i}{d} (-s) + 1 = \frac{j}{d} t.
\]
This implies that \((n + 1)i = jn + 1\). Hence \(j(n + 1)i\), and so \(y^{y^{(n+1)i}} = y\). By (7) we have \(xy = yx\) as claimed.

If \(G(H^0)\) is abelian (for example if \(H\) is quasitriangular), more can be said. The next theorem generalizes [C] which was proved for the semicenter of a semiprime cocommutative \(H\).

**Theorem 4.4.** Assume \(\sigma, \tau \in G(H^0)\) commute and \(A \in \mathcal{H}^0\mathcal{Y}\mathcal{D}\) is an \(H\)-semiprime \(H\)-commutative algebra; then:

1. If \(x \in Q_\sigma\) and \(y \in Q_\tau\), then \(xy = yx\), where \(z \in Q^H \subset Z(Q)\).
2. If \(x \in Q_\sigma, y \in Q_\tau, \sigma^i = 1 = \tau^j\), and \(g.c.d(i, j) = 1\) then \(xy = yx\).

**Proof.** First assume \(Q_\sigma = Q^H x\) and \(Q_\tau = Q^H y\). Since \(xy \in Q_{\sigma\tau}, yx \in Q_{\sigma\tau}\), and \(\sigma\tau = \tau\sigma\), we have \(xy, yx \in Q_{\sigma\tau} \subset Q_{\sigma\tau}\). By Theorem 3.5 \(Q_{\sigma\tau} = Q^H t\). Thus there exists \(z \in Q^H\) so that \(xy = zt\). But then \(xy = xye = zte = tzx\). Since \(tx' \in Q_{\sigma}, Q_{\tau-1}\), we have \(tx' \in Q_{\tau}\). Hence \(tx' = wy\), some \(w \in Q^H\). Thus \(xy = zwxy\), where \(zw \in Q^H\). Now, for any \(a \in Q_\sigma, b \in Q_\tau\), the result follows from the above since \(a = mx\), \(b = py\), where \(m, p \in Q^H\).

2. Since \(\sigma\tau = \tau\sigma\), Lemma 4.2 implies that \((A_\sigma)^{-1} = A_{\sigma^{-1}}\), and \((A_\tau)^{-1} = A_{\tau^{-1}}\). That is \(x^{-1} \in A_\sigma\) and \(y^{-1} \in A_{\tau}\). Applying Corollary 3.2 repeatedly we have:

\[
xy = y^{\sigma^{-1}}x = x^{\tau^{-1}}y^{\sigma^{-1}} = \cdots = y^{\sigma^{-(n + 1)}}x^{\tau^{-n}}.
\]

Since \(g.c.d(i, j) = 1, 1 = is + jt\), where \(t < 0\). Set \(n = j(-t)\); then \(x^{\tau^{-n}} = x^{\tau^{n}} = x\), since \(\tau^1 = 1\), while \(n + 1 = is\) implies that \(y^{\sigma^{-(n+1)}} = y^{\sigma^n}\). Since \(\sigma^i = 1, y^{\sigma^{(n+1)}} = y\). We have shown that \(y^{\sigma^{-(n+1)}}x^{\tau^{-(n+1)}} = xy\). Hence \(xy = yx\).

Examples in which \(z\) in the above theorem is not 1 exist.

More can be said when \((H, R)\) is quasitriangular. Note that for such \(H\), if \(M\) is a left \(H\)-module then \(M \in \mathcal{H}^0\mathcal{Y}\mathcal{D}\) by \(\eta(m) = \sum R^2 \otimes R^1 \cdot m\), for all \(m \in M\), where \(R = \sum R^2 \otimes R^1\). Moreover, as is well known \(R\) induces a braiding on \(H^*\) by: \(\langle a|b\rangle = \sum \langle a, R^2 \rangle \langle b, R^1 \rangle\), all \(a, b \in H^*\). In this case there is an interesting constraint on the possible \(\sigma\) so that \(A_\sigma \neq 0\).

**Lemma 4.5.** Let \((H, R)\) be a quasitriangular Hopf algebra and \(A\) an \(H\)-module algebra. Let \(x \in A_\sigma\); then

1. \(x^{\sigma} = \langle\sigma|\sigma\rangle x\) and
2. if \(u = \Sigma(SR^2)R^1\) then \(u \cdot x = \langle\sigma|\sigma\rangle^{-1}x\).
Proof. 1. Let \( x \in A_\sigma \); then
\[
x^{\hat{\sigma}} = \sum \langle \sigma, x \rangle x_0
= \sum \langle \sigma, R^2 \rangle R^1 \cdot x
= \sum \langle \sigma, R^2 \rangle \langle \sigma, R^1 \rangle x
= \langle \sigma | \sigma \rangle x.
\]

2. Observe that
\[
\langle \sigma, u \rangle = \langle \sigma, \sum (SR^2)R^1 \rangle = \sum \langle \sigma, SR^2 \rangle \langle \sigma, R^1 \rangle = \langle \sigma^{-1}, R^2 \rangle \langle \sigma, R^1 \rangle.
\]
But this equals \( \langle \sigma^{-1} | \sigma \rangle \) which equals \( \langle \sigma | \sigma \rangle^{-1} \). Thus
\[
\langle \sigma, u \rangle = \langle \sigma | \sigma \rangle^{-1};
\] (8)
hence if \( x \in A_\sigma \), then \( u \cdot x = \langle \sigma, u \rangle x = \langle \sigma | \sigma \rangle^{-1} x \).

**Corollary 4.6.** Let \((H, R)\) be a quasitriangular Hopf algebra, and \( A \) an \( H \)-semiprime \( H \)-commutative algebra (with \( \eta \) defined as above). If \( A_\sigma \neq 0 \) then \( \langle \sigma | \sigma \rangle = 1 \), and \( u \) acts as the identity on \( A_\sigma \).

**Proof.** By Theorem 4.3 \( x^{\hat{\sigma}} = x \) for any \( x \in A_\sigma \), which by Lemma 4.5 implies that if \( x \neq 0 \) then \( \langle \sigma | \sigma \rangle = 1 \).

The \( H \)-semiprimeness condition in Corollary 4.6 cannot be omitted as seen in the following example:

**Example 4.7 [CW1, Ra].** Let \( H = kZ_2 \) where \( Z_2 = \{1, g\} \) and \( \text{Char} \ k \neq 2 \). Set \( R = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) \); then \( u = g \). Let \( A = A_1 \oplus A_g \) be a \( Z_2 \)-graded algebra, where \( a \in A_1 \) has degree 0 and \( a \in A_g \) has degree 1. Assume that \( A \) is a commutative superalgebra: \( ab = (-1)^{\deg a \deg b} ba \). This is equivalent to \( A \) being \( H \)-commutative. It is not \( H \)-semiprime since \( a^2 = 0 \) for each \( a \in A_g \), so \( Aa \) is a nilpotent \( H \)-stable ideal of \( A \). Now, let \( H^* = Sp(p_1, p_g) \), where \( \{p_1, p_g\} \) is a dual basis of \( \{1, g\} \); then \( \sigma = p_1 - p_g \in G(H^*) \) and \( A_\sigma = A_g \). But then \( \langle \sigma, u \rangle = -1 \). This, by (8), implies that \( \langle \sigma | \sigma \rangle = -1 \). Also note that for each \( x \in A_\sigma \), \( x^{\hat{\sigma}} = -x \), exhibiting the necessity of \( H \)-semiprimeness in Theorem 4.3.

In what follows we consider applications to finite-dimensional Hopf algebras. In this case, if \( A \) is a left \( H \)-module algebra then \( A^{\text{fin}} = A \). Our first application is to \( A = H \), with the adjoint action of \( H \) on itself.
Theorem 4.8. Let $H$ be a finite-dimensional semisimple Hopf algebra acting on itself by the adjoint action. Let $\sigma \in \text{G}(H^*)$; then:

1. $H_\sigma = H^\sigma = Z(H)$ [BCM, 3.12].
2. $H_\sigma = Z(H)x_\sigma = \{x \in H | hx = xh^\sigma \text{ for all } h \in H\}$, with $h^\sigma$ as defined in (4).
3. For each $x \in H_\sigma$ there exists a unique $x' \in H_{\sigma^{-1}}$ so that $xx' = x'x = e_x \in Z(H)$ is an idempotent, and $xe_x = x$, $x'e_x = x'$.
4. For all $x \in H_\sigma$, $x^\sigma = x$.
5. Let $m$ be the order of $\sigma$; then $S_\sigma = \sum_{i=1}^m H\sigma^i$ is a commutative algebra with $1$.
6. If $\sigma \neq e$ and $x \in H_\sigma$, then $\langle e, x \rangle = 0 = \langle \sigma, x \rangle$.
7. If $G(H^*)$ is abelian then for $x \in H_\sigma$, $y \in H_\sigma$, there exists $c \in Z(H)$ so that $xy = cyx$. If the orders of $\sigma$ and $\tau$ are relatively prime then $xy = yx$.

Proof. 1. Since $H$ is finite dimensional and semisimple, the only ideal of $H$ with trivial annihilator is $H$; hence $\mathcal{F} = \{H\}$, and so $Q_{H}(H) = H$. Since $\langle H, \text{adj}, \Delta \rangle$ is $H$-commutative as discussed in Example 1.3, we can use Theorem 3.5. This proves the first equality. As for the second equality, if $x$ satisfies the right hand side condition, then for all $h \in H$

$$h \cdot x = \sum h_1 xSh_2 = \sum xh_1^\sigma Sh_2 = \langle \sigma, h \rangle x;$$

hence $x \in Q_\sigma$. The converse follows from Corollary 3.2.

2. Is a consequence of Proposition 3.4.
3. Is a consequence of Theorem 4.3.1.
4. Is a consequence of Theorem 4.3.2.
5. Since $H$ is semisimple, the integral $0 \neq t \in Z(H) = H_\epsilon$. Hence for $x \in H_\sigma$, $tx \in H_\sigma$. But $tx = \langle e, x \rangle t \in H_\sigma$. Since $H_\epsilon \cap H_\sigma = \{0\}$, it follows that if $\sigma \neq e$ then $\langle e, x \rangle = 0$. Now, by (4) $x = x \leftarrow \sigma$; thus $x = x \leftarrow \sigma^{-1}$ as well. But then $\langle \sigma, x \rangle = \langle \sigma, x \leftarrow \sigma^{-1} \rangle = \langle e, x \rangle = 0$.
6. Follows from Theorem 4.4.

If $H$ is finite dimensional and $H^*$ is pointed then as a corollary of Theorem 3.8 we have:

Theorem 4.9. Let $H$ be an $n$-dimensional Hopf algebra so that $H^*$ is pointed. Let $\langle A, \cdot, \eta \rangle$ be an $H$-semiprime $H$-commutative algebra; then $Q = Q_H(A)$ is generated over $Q^H$ by $n$ elements.

Remark 4.10. Under the hypothesis of Theorem 4.9 $Q$ is a (finite) centralizing extension of $Q^H$; hence there is a strong connection between these two rings. In particular, their prime ideals, radicals, chain conditions, and module structures are strongly related [BR1, BR2, RS].
If $A$ is $H$-simple then $Q_H(A) = A$; hence as a corollary of this theorem we have:

**Corollary 4.11.** Let $H$ be and $H^*$ be as in Theorem 4.9 and let $\langle A, \cdot, \eta \rangle$ be an $H$-commutative $H$-simple algebra. Then $\dim A$ over the field $A^H$ is at most $n$.

If $A$ is a commutative $H$-module algebra then $\langle A, \cdot, \eta \rangle$ as in Example 1.2 is $H$-commutative. By the above we have:

**Corollary 4.12.** Let $H$ and $H^*$ be as in Theorem 4.9, and let $A$ be a commutative $H$-module algebra. If $A$ is $H$-semiprime then $Q = Q_H(A)$ is generated over $Q^H$ by $n$ elements. If $A$ is $H$-simple then $\dim A$ over $A^H$ is at most $n$.

In the last part of this section we consider infinite-dimensional $H$. Let $T$ be a pointed sub-Hopf algebra of $H^0$, and let $\{C_i\}$ be the associated coradical filtration of $T[TW]$, where $C_i = C_{i-1} \oplus K_i$ for all $i \geq 0$, $C_{-1} = 0$. Fix $\sigma \in G(T)$; set

$$K_\sigma,i = \{ t \in K_i | \Delta(t) = \tau \otimes t + t \otimes \sigma + \sum t_j \otimes s_j, \tau \in G(T), t_j, s_j \in C_{i-1} - G(T) \}$$

and

$$C_{\sigma,k} = \sum_{i \leq k} K_{\sigma,i}.$$

Using coassociativity, it is straightforward to check that $C_{\sigma,k}$ is a left coideal of $T$. Let $A$ be an $H$-module algebra; set

$$A_{\sigma,k} = \rho^{-1}(A^\text{fin} \otimes C_{\sigma,k}),$$

and

$$A_{\sigma,=} = \bigcup_k A_{\sigma,k}.$$

By the comodule axioms, $Q_T = \bigoplus_{\sigma \in G(T)} Q_{\sigma,=}$, and $Q_T$ is graded by $G(T)$. Note also that for $\sigma \in G(T)$, $A_{\sigma,0} = A_{\sigma,}$. We take now again $A = H$ as in Example 1.3. Since the action is the adjoint action, every two-sided ideal of $H$ is $H$-stable and hence $H$-semiprimeness is equivalent to usual semiprimeness and $Q_H(H)$ is just the usual Martindale ring of quotients $Q(H)$. Thus, in order to apply results of the previous sections we require that $H$ is semiprime. We first prove:

**Proposition 4.13.** Let $H$ be a semiprime Hopf algebra, let $Q$ be its Martindale ring of quotients, and let $T$ be a pointed sub-Hopf algebra of $H^0$. 
Then

\[ Q_{\sigma,k} = \left\{ q \in Q \mid hq = qh^\sigma + \sum_{m \in c_{\sigma,k}} q_m (h \leftrightarrow t_m), \text{ all } h \in H \right\}, \]

where \( \rho(q) = q \otimes \sigma + \sum_{m \in c_{\sigma,k}} q_m \otimes t_m \) and \( h \leftrightarrow t_m = \sum (t_m, h) h_2 \). In particular \( Q^H = Z(Q) \).

**Proof.** If \( q \) satisfies the right hand side condition, then

\[ h \cdot q = \sum h_1 q S h_2 = \left( q h_1^\sigma + \sum_{m \in c_{\sigma,k}} q_m (h \leftrightarrow t_m) \right) S h_2 = \langle \sigma, h \rangle q + \sum_{m \in c_{\sigma,k}} \langle t_m, h \rangle q_m. \]

By the definition of \( \rho \) on \( Q^{\text{fin}} \), it follows that \( \rho(q) = q \otimes \sigma + \sum_{m \in c_{\sigma,k}} q_m \otimes t_m \), hence \( q \in Q_{\sigma,k} \). Conversely if \( q \in Q_{\sigma,k} \) then for all \( h \in H \),

\[ h \cdot q = \langle \sigma, h \rangle q + \sum_{m \in c_{\sigma,k}} \langle t_m, h \rangle q_m. \]

Hence

\[ hq = \sum (h_1 \cdot q) h_2 = \sum \left( \langle \sigma, h_1 \rangle q + \sum_{m \in c_{\sigma,k}} \langle t_m, h_1 \rangle q_m \right) h_2 \]

which equals the expression on the right. \( \square \)

A nice example is \( O_q(SL_n) \) (e.g., [D]). It is known that when \( q \) is not a root of unity then \( U_q(sl_n) \subset (O_q(SL_n))^0 \). Recall that \( U_q(sl_n) \) is generated as an algebra by \( a_i, e_i, \) and \( f_i \) for \( 1 \leq j \leq n - 1 \), where \( \{a_i\} \) are grouplike elements and

\[ \Delta(e_i) = a_i \otimes e_i + e_i \otimes 1, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes a_i^{-1}. \]

Choose \( T = U_q(sl_n) \) which is pointed; then it can be deduced from [ChM, Theorems A and B] that for all \( \sigma \in G(U_q(sl_n)) \),

\[ K_{\sigma,i} = \sigma \prod_{j \neq i} t_j^i, \quad \text{where } t_j = a_j^{-1} e_j \text{ or } t_j = f_j. \]

This implies that \( K_{\sigma,i} \) is finite dimensional for all \( \sigma \in G(T), i \geq 0 \). Thus we have:

**Theorem 4.14.** Let \( H = O_q(SL_n) \) and let \( Q \) be the Martindale ring of quotients of \( H \). Let \( T = U_q(sl_n) \subset (O_q(SL_n))^0 \). Then

1. \( Q^H = Z(Q) \) is a field.
2. For each \( \sigma \in G(T), Q_{\sigma} = Z(Q) x_{\sigma}, \) where \( x_{\sigma} \) is 0 or an invertible element which normalizes \( A \).
3. If $\sigma, \tau \in G(T)$, then $x_\sigma x_\tau = z x_\tau x_\sigma$, where $z \in Z(Q)$. 
4. $Q_{\sigma, k}$ is finite dimensional over $Z(Q)$ and moreover, $HQ_{\sigma, k} = Q_{\sigma, k} H$ for all $\sigma \in G(T)$, $k \geq 0$. 
5. $Q_T = \rho^{-1}(Q^{1n} \otimes T) = \bigoplus_{\sigma \in G(T)} Q_{\sigma, k}$ is a simple algebra graded by $G(T)$.

**Proof.**
1. Follows directly from the fact that $H = Q_q(SL_n)$ is prime.
2. Is a consequence of Theorem 3.5. The fact that $x_\sigma$ is invertible follows since $x_\sigma x_\sigma'$ is an idempotent in a field.
3. Is a consequence of Theorem 4.4 since $G(T)$ is abelian.
4. The first part follows by Theorem 3.8 and the fact that $K_{i, i}$ is finite dimensional for all $i > 0$ and $\sigma \in U_q(sl_n)$. The second part follows by Proposition 4.13.
5. Is a consequence of Corollary 3.6. □

REFERENCES


