Physics Letters B 761 (2016) 207–212

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Relativistic corrections to the algebra of position variables and spin-orbital interaction

Alexei A. Deriglazov [a,b,*], Andrey M. Pupasov-Maksimov [a]

[a] Departamento de Matemática, ICE, Universidade Federal de Juiz de Fora, MG, Brazil
[b] Laboratory of Mathematical Physics, Tomsk Polytechnic University, 634050 Tomsk, Lenin Ave. 30, Russian Federation

ARTICLE INFO

Article history:
Received 11 July 2016
Received in revised form 15 August 2016
Accepted 15 August 2016
Available online 18 August 2016
Editor: M. Cvetič

Keywords:
Vector model of relativistic spin
First relativistic corrections
Problem of covariant formalism
Hydrogen atom spectrum
Theories with constraints
Non commutative position

ABSTRACT

In the framework of vector model of spin, we discuss the problem of a covariant formalism [35] concerning the discrepancy between relativistic and Pauli Hamiltonians. We show how the spin-induced non-commutativity of a position accounts the discrepancy on the classical level, without appeal to the Dirac equation and Foldy–Wouthuysen transformation.

1. Introduction

In series of previous works [1–4] we developed a Poincare-invariant variational formulation describing particle with spin. This classical model provides a unified description of both Frenkel and BMT equations [5]. The latter are considered as a basic tool in the analysis of the polarization precession measurements [6]. In [7] we extend the variational formulation to the general relativity, where the classical models of a spinning particle are widely used to describe a rotating body in pole-dipole approximation [8–16]. Another possible application can be related with the kinetic theory of chiral medium, where, in the regime of weak external fields and weak interactions between spinning (quasi)-particles, each particle can be considered as moving along a classical trajectory [17].

For variational formulations provide a striating point to the canonical quantization [18], they have incredible theoretical importance connecting classical and quantum descriptions of nature. Canonical quantization of the free relativistic spinning particle (within our variational formulation, [19]) leads to the positive-energy part of the Dirac equation in the Foldy–Wouthuysen representation. It also identifies [19] the non-commutative Pryce's d-type center of mass operator[1] as the quantum observable which corresponds to the classical position variable. Non-commutativity of (physically meaningful) position operators for relativistic spinning particles was noticed already by Pryce [20]. He shown that coordinates of the relativistic center-of-mass have to obey non-trivial Poisson brackets. As a result, the corresponding quantum observables do not commute. Therefore a physically meaningful position operators of a spin-1/2 should be non-commutative.

Recent theoretical studies revive Snyder's attempts [21] to solve fundamental physical problems by introducing non-commutativity of the space [22]. It is believed that this fundamental non-commutativity may be important at Plank length scale λ_P. Extensive studies of non-commutativity cover both classical and quantum theories, as well relativistic and non-relativistic situations. Postulating non-commutative deformation of position operators [31] one can study physical consequences and estimate possible effects. Calculations of the hydrogen spectrum corrections strongly limit possible non-commutativity of coordinate parameters in the Dirac equation [26–30].

* Corresponding author at: Departamento de Matemática, ICE, Universidade Federal de Juiz de Fora, MG, Brazil.
E-mail addresses: alexei.deriglazov@ufjf.edu.br (A.A. Deriglazov), pupasov.maksimov@ufjf.edu.br (A.M. Pupasov-Maksimov).

[1] See also [32], where the same result was obtained for the classical particle with anticommuting spin variables.

http://dx.doi.org/10.1016/j.physletb.2016.08.034

In the present work we will study effects of a natural non-commutativity of Pryce's d-type center of mass (at both classical and quantum levels) in the description of electron interacting with an electromagnetic background. Our considerations extend results of [19] towards a quantization of interacting spinning particle.

In the free theory, different candidates for the position operator are almost indistinguishable. All these operators obey the same Heisenberg equations (uniform rectilinear motion), and the difference in their expectation values is of Compton wave length order, λ_C. In the interacting case, the problem of the identification of quantum position observables becomes more complicated.[2]

Fleming [25] noted:

"The simplest form of interaction is that due to a static potential which may be expressed in terms of the position operator of the particle. For a relativistic particle, however, the important question arises of which position operator should be used. The conventional approach, in which the position operator is assumed to be local, forces the choice of the center of spin."[3]

He also observed, that a formal substitution of Pryce d-type operator into the potential leads to some reasonable corrections:

"The first correction term to a spherically symmetric local potential will be recognized as the spin-orbit coupling that Thomas derived many years ago as a consequence of classical relativity and which appears in the nonrelativistic limit of the Dirac equation for spin particles."

Analogous situation was observed in general relativity, [35, 37–39] where a formal substitution of a non-local position variable into potential results in correct equations of motion for the spinning particle. Restricting ourselves to the case of special relativity, in the present work we provide some theoretical grounds for such substitution.

The paper is organized as follows. In the first section we present general considerations of the structure of classical and quantum Hamiltonians for a spinning particle. In the second section we give a brief description of the vector model for the classical description of a relativistic spinning particle. In the third section we will realize classical algebra of Dirac brackets by quantum operators in the case of a stationary electro-magnetic background. This realization will deform free Foldy–Wouthusen Hamiltonian and at low energies will give Pauli Hamiltonian with correct spin-orbital interaction. In the conclusion we discuss obtained results.

2. Model independent discussion of the quantum and classical Hamiltonians of a spinning particle

From quantum point of view, at low energies an electron interacting with a background electromagnetic field is described by the two-component Schrödinder equation. Pauli Hamiltonian[4] includes spin-orbital and Zeeman interactions

$$\hat{H}_{ph} = \frac{1}{2m}(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A})^2 - eA_0 + \frac{e(g-1)}{2m^2c^2}\hat{\mathbf{S}}[\hat{\mathbf{p}}\times\mathbf{E}] - \frac{eg}{2mc}\mathbf{B}\hat{\mathbf{S}} = \hat{H}_{charge} + \hat{H}_{spin-em}\,. \quad (1)$$

Gyromagnetic ratio g is a coupling constant of spin with an electromagnetic field. In principle, in non-relativistic theory one can expect different coupling constants for the third and the fourth terms of the Hamiltonian. Experimental observations of the hydrogen spectrum lead to the factor $g-1$ in the third term and to the factor g in the last term. Thus, Hamiltonian explains Zeeman effect and reproduces fine structure of the energy levels of the hydrogen atom. This Hamiltonian follows also from the non-relativistic limit of the Dirac equation in the Foldy–Wouthuysen representation [18, 36].

From classical point of view, models of spinning particles are based on a Lagrangian or Hamiltonian mechanics, both in the relativistic and non-relativistic regime [23]. In a covariant formulation, the spin part of the Hamiltonian describing an interaction between spin S and electromagnetic field reads

$$H_{spin-em-cov} \sim \frac{eg}{2m^2c^2}\mathbf{S}[\mathbf{p}\times\mathbf{E}] - \frac{eg}{2mc}\mathbf{BS}\,. \quad (2)$$

We emphasize that the expression (2) follows from the analysis of all possible terms in covariant equations of motion and thus is a model-independent [35]. It can also be predicted from symmetry considerations on the level of a Hamiltonian. For instance, if we take the Frenkel spin-tensor $S^{\mu\nu}$, the only Lorentz-invariant combination that could give the desired terms is $F_{\mu\nu}S^{\mu\nu} = 2E^iS^{i0} + \epsilon^{ijk}S^{ij}B^k$ (see our notations in Appendix).

For the classical gyromagnetic ratio $g = 2$, the classical spin-orbital interaction in (2) differs by the famous and troublesome factor[5] of $\frac{1}{2}$ from its quantum counterpart in (1). It seems quantization of $H_{spin-em-cov}$ will not reproduce quantum behavior given by $\hat{H}_{spin-em}$. The issue about this difference was raised already in 1926 [34] and still remains under discussion [35].

In principle, Hamiltonian $H_{spin-em}$ can be obtained, if one impose a non-covariant supplementary condition on spin, $2S^{i0}p_0 + S^{ij}p_j = 0$, where $p_0 = -mc$ in the non-relativistic limit. On a first glance, any covariant spin-supplementary condition [8,34,42–44] would give $H_{spin-em-cov}$ and the discrepancy factor of $\frac{1}{2}$.

In the next section we study this issue in the framework of vector model of a spinning particle [4]. We show that the vector model provides an answer on a pure classical ground, without appeal to the Dirac equation. In a few words, it can be described as follows. The relativistic vector model involves a second-class constraints, which should be taken into account by passing from the Poisson to Dirac bracket. The emergence of a higher non-linear classical brackets that accompany the relativistic Hamiltonian (2) is a novel point, which apparently has not been taken into account in literature. If we pretend to quantize the model, it is desirable to find a set of variables with the canonical brackets. The relativistic Hamiltonian (2), when written in the canonical variables, just gives (1).

[2] Another related problem is in the identification of spin operator, since a change of the center of mass definition leads to the modification if the spin definition. [24] compares Pauli, Foldy–Wouthuysen, Czachor, Frenkel, Chakrabarti, Pryce, and Fradkin–Good spin operators in different physical situations and concluded that interaction with electromagnetic potentials allows to distinguish between various spin operators experimentally.

[3] Fleming calls the Newton–Wigner position operator as the center of spin, while Pryce d-type operator is called as the center of mass.

[4] We will write quantum Hamiltonians and other operators using the hat, the same observables without the hat correspond to the classical theory. Thus (1) defines also classical Pauli-like Hamiltonian.

[5] This factor is often referred to Thomas precession [33]. We will not touch this delicate and controversial issue [34,40] since the covariant formalism automatically accounts the Thomas precession [41].

3. Vector model of spinning particle in the parametrization of physical time

To find the classical brackets that accompany H_{cov} we need a systematically developed model of a spinning particle. Here we consider the vector model and briefly describe the construction of the Hamiltonian and the brackets in a stationary electromagnetic field. For a detailed discussion of the model, see [4].

Configuration space of the vector model of spinning particle is parameterized by a point $x^\mu(\tau)$ of a world-line and a vector $\omega^\mu(\tau)$ attached to that point. The configuration-space variables are taken in an arbitrary parametrization τ of the world-line. The conjugate momenta of the variables are denoted by p^μ and π^μ, correspondingly. Frenkel spin-tensor in the vector model is a composite quantity, $S^{\mu\nu}=2(\omega^\mu\pi^\nu-\omega^\nu\pi^\mu)$. The free Lagrangian can be written in a number of equivalent forms [11,19]. To describe the spin-field interaction through the gyromagnetic ratio g, we use the Lagrangian with an auxiliary variable $\lambda(\tau)$

$$S=\int d\tau\frac{1}{4\lambda}\Big[\dot{x}N\dot{x}+D\omega ND\omega - \sqrt{[\dot{x}N\dot{x}+D\omega ND\omega]^2-4(\dot{x}ND\omega)^2}\Big] - \frac{\lambda}{2}(m^2c^2-\frac{\alpha}{\omega^2})+\frac{e}{c}A\dot{x}, \tag{3}$$

where $D\omega^\mu\equiv\dot{\omega}^\mu-\lambda\frac{eg}{2c}F^{\mu\nu}\omega_\nu$. The auxiliary variable provides a homogeneous transformation law of $D\omega$ under the reparametrizations, $D_{\tau'}\omega=\frac{d\tau}{d\tau'}D\omega$. The matrix $N_{\mu\nu}$ is the projector on the plane orthogonal to ω^ν, $N_{\mu\nu}=\eta_{\mu\nu}-\frac{\omega_\mu\omega_\nu}{\omega^2}$. The parameter m is mass, while α determines the value of spin. The value $\alpha=\frac{3\hbar^2}{4}$ is fixed by quantization conditions and corresponds to an elementary spin one-half particle. In the spinless limit, $\alpha=0$ and $\omega^\mu=0$, the functional (3) reduces to the well known Lagrangian of the relativistic particle, $\frac{1}{2\lambda}\dot{x}^2-\frac{\lambda}{2}m^2c^2+\frac{e}{c}A\dot{x}$.

Frenkel considered the case $g=2$ and found approximate equations of motion neglecting quadratic and higher terms in spin, fields and field gradients. Equations of motion obtained from (3) coincide with those of Frenkel in this approximation [34].

To find relativistic Hamiltonian in the physical-time parametrization,[6] we use the Hamiltonian action associated with (3). This reads [4], $\int d\tau\ p\dot{x}+\pi\dot{\omega}-\lambda_i T_i$, where λ_i are Lagrangian multipliers associated with the primary constraints T_i. The variational problem provides both equations of motion and constraints of the vector model in an arbitrary parametrization. Using the reparametrization invariance of the functional, we take physical time as the evolution parameter, $\tau=\frac{x^0}{c}=t$, then the functional reads

$$S_H=\int dt\ c\tilde{\mathcal{P}}_0-eA^0+p_i\dot{x}^i+\pi_\mu\omega^\mu - \Big[\frac{\lambda}{2}\Big(-\tilde{\mathcal{P}}_0^2+\mathcal{P}_i^2-\frac{eg}{4c}(FS)+m^2c^2+\pi^2-\frac{\alpha}{\omega^2}\Big) + \lambda_2\omega\pi+\lambda_3\mathcal{P}\omega+\lambda_4\mathcal{P}\pi\Big], \tag{4}$$

where $\tilde{\mathcal{P}}_0=p_0-\frac{e}{c}A_0$ and $\mathcal{P}^i=p^i-\frac{e}{c}A^i$ is $U(1)$-invariant canonical momentum.

We can treat the term associated with λ as a kinematic constraint of the variational problem. Following the known prescription of classical mechanics, we can solve the constraint,

$$\tilde{\mathcal{P}}_0=-\tilde{\mathcal{P}}^0=-\sqrt{\mathcal{P}_i^2-\frac{eg}{4c}(FS)+m^2c^2+\pi^2-\frac{\alpha}{\omega^2}}, \tag{5}$$

and substitute the result back into Eq. (4), this gives an equivalent form of the functional

$$S_H=\int dt\ p_i\dot{x}^i+\pi_\mu\dot{\omega}^\mu - \Big[c\sqrt{\mathcal{P}_i^2-\frac{eg}{4c}(FS)+m^2c^2+\pi^2-\frac{\alpha}{\omega^2}}+eA^0 + \lambda_2\omega_\mu\pi^\mu+\lambda_3\mathcal{P}_\mu\omega^\mu+\lambda_4\mathcal{P}_\mu\pi^\mu\Big], \tag{6}$$

where the substitution (5) is implied in the last two terms as well. The sign in front of the square root (5) was chosen according to the right spinless limit, $L=-mc\sqrt{-\dot{x}^\mu\dot{x}_\mu}$. The expression in square brackets is the Hamiltonian.

The variational problem implies the first-class constraints $T_2\equiv\omega\pi=0$, $T_5\equiv\pi^2-\frac{\alpha}{\omega^2}=0$. They determine gauge symmetries and physical observables of the theory. The quantities $x^i(t)$, $\mathcal{P}^i(t)$ and $S^{\mu\nu}(t)$ have vanishing Poisson brackets with the constraints and hence are candidates for observables. The set

$$T_3=-\mathcal{P}^0\omega^0+\mathcal{P}^i\omega^i=0, \qquad T_4=-\mathcal{P}^0\pi^0+\mathcal{P}^i\pi^i=0, \tag{7}$$

where

$$\mathcal{P}^0\equiv\sqrt{\mathcal{P}_i^2-\frac{eg}{4c}(FS)+m^2c^2} \tag{8}$$

represents a pair of second class constraints. In all expressions below the symbol $\mathcal{P}^0$ represents the function (8). The constraints imply the spin-supplementation condition

$$S^{\mu\nu}\mathcal{P}_\nu=0, \tag{9}$$

as well as the value-of-spin condition $S^{\mu\nu}S_{\mu\nu}=8\alpha$.

To represent the Hamiltonian in a more familiar form, we take into account the second-class constraints by passing from Poisson to Dirac bracket. As the constraints involve conjugate momenta of the position $\mathbf{x}$, this leads to nonvanishing brackets for the position variables. In the result, the position space is endowed, in a natural way, with a noncommutative structure which originates from accounting of spin degrees of freedom. For the convenience, an exact form of Dirac brackets of our observables is presented in the Appendix. Since the Dirac bracket of any quantity with second-class constraints vanishes, we can omit them from the Hamiltonian. The first-class constraints can be omitted as well, as they do not contribute into equations of motion for physical variables. In the result we obtain the relativistic Hamiltonian

$$H_{cov}=c\sqrt{\vec{\mathcal{P}}^2-\frac{eg}{4c}F_{\mu\nu}S^{\mu\nu}+m^2c^2}+eA^0. \tag{10}$$

Equations of motion follow from this Hamiltonian with use of the Dirac bracket[7]: $\frac{dz}{dt}=\{z,H_{cov}\}_D$.

4. First relativistic corrections and fine structure of hydrogen spectrum

To quantize our relativistic theory we need to find quantum realization of highly non-linear classical brackets (21)–(21). They remain non-canonical even in absence of interaction. For instance,

[6] Which is necessary for the canonical quantization.

[7] We emphasize that the use of canonical brackets will lead to different equations. In our opinion, this turns out to be the reason for debates around the controversial results obtained by different groups, see the discussion in [35].

Eq. (21) in a free theory reads $\{x^i, x^j\} = \frac{1}{2mcp^0} S^{ij}$. It is worth noting that non-relativistic spinning particle [3,23] implies the canonical brackets, so the deformation arises as a relativistic correction induced by spin of a particle. Technically, the deformation arises from the fact that the constraints, used to construct the Dirac bracket, mixes up the space-time and inner-spin coordinates. Quantum realization of the brackets in a free theory has been obtained in [19], while in an interacting theory its explicit form is unknown. Therefore we quantize the interacting theory perturbatively, considering c^{-1} as a small parameter and expanding all quantities in power series. Let us consider the approximation $o(c^{-2})$ neglecting c^{-3} and higher order terms. For the Hamiltonian (11) we have $H_{ph} \approx mc^2 + \frac{\mathcal{P}^2}{2m} - \frac{\mathcal{P}^4}{8m^3c^2} - \frac{eg}{8mc}(FS)$. Since the last term is of order $(mc)^{-1}$, resolving the constraint $S^{\mu\nu}\mathcal{P}_\nu = 0$ with respect to S^{i0} we can approximate $\mathcal{P}^0 = mc$, then $S^{i0} = \frac{1}{mc} S^{ij}\mathcal{P}^j$. Using this expression we obtain

$$\begin{aligned} H_{ph} &= mc^2 + \frac{\mathcal{P}^2}{2m} - \frac{\mathcal{P}^4}{8m^3c^2} + eA^0 \\ &\quad + \frac{eg}{2mc}\left[\frac{1}{mc}\mathbf{S}[\mathcal{P}\times\mathbf{E}] - \mathbf{BS}\right] + o\left(\frac{1}{c^2}\right) \\ &= H_{charge} + H_{spin-em-cov} + o\left(\frac{1}{c^2}\right). \end{aligned} \tag{11}$$

Due to the second and fourth terms, we need to know the operators $\hat{\mathcal{P}}^i$ and $\hat{x}^i$ up to order c^{-2}, while $\hat{S}^{ij} \sim \hat{\mathbf{S}}$ should be found up to order c^{-1}. With this approximation, the commutators $[\hat{x}, \hat{x}]$, $[\hat{x}, \hat{\mathcal{P}}]$, and $[\hat{\mathcal{P}}, \hat{\mathcal{P}}]$ can be computed up to order c^{-2}, while the remaining commutators can be written only up to c^{-1}. Therefore, we expand the right hand sides of Dirac brackets (21) in this approximation

$$\begin{aligned} &\{x^i, x^j\} = \frac{1}{2m^2c^2} S^{ij} + o\left(\frac{1}{c^2}\right), \\ &\{x^i, \mathcal{P}^j\} = \delta^{ij} + o\left(\frac{1}{c^2}\right), \\ &\{x^i, S^{jk}\} = 0 + o\left(\frac{1}{c}\right), \\ &\{\mathcal{P}^i, \mathcal{P}^j\} = \frac{e}{c} F^{ij} + o\left(\frac{1}{c^3}\right), \\ &\{\mathcal{P}^i, S^{jk}\} = o\left(\frac{1}{c^2}\right), \\ &\{S^{ij}, S^{kl}\} = 2(\delta^{ik}S^{jl} - \delta^{il}S^{jk} - \delta^{jk}S^{il} + \delta^{jl}S^{ik}) + o\left(\frac{1}{c}\right). \end{aligned} \tag{12}$$

An operator realization of these brackets reads

$$\hat{\mathcal{P}}_i = -i\hbar\frac{\partial}{\partial x^i} - \frac{e}{c}A_i(\mathbf{x}), \tag{13}$$

$$\hat{x}_i = x_i - \frac{\hbar}{4m^2c^2}\epsilon_{ijk}\hat{P}^j\sigma^k, \tag{14}$$

$$\hat{S}^{ij} = \hbar\epsilon_{ijk}\sigma^k, \tag{15}$$

then

$$\hat{S}^i = \frac{1}{4}\epsilon_{ijk}S^{jk} = \frac{\hbar}{2}\sigma^i, \tag{16}$$

$$\hat{S}^{i0} = \frac{\hbar}{mc}\epsilon_{ijk}\hat{\mathcal{P}}^j\sigma^k. \tag{17}$$

By construction of a Dirac bracket, the operator $\hat{S}^{i0}$ automatically obeys the desired commutators up to order c^{-1}.

We substitute these operators into the classical Hamiltonian (11). Expanding $A^0(\hat{\mathbf{x}})$ in a power series, we obtain an additional contribution of order c^{-2} to the potential due to non-commutativity of the position operator

$$eA^0\left(x_i - (2mc)^{-2}\epsilon_{ijk}\hat{\mathcal{P}}^j\hat{S}^k\right) \approx eA^0(\mathbf{x}) - \frac{e}{2m^2c^2}\hat{\mathbf{S}}[\hat{\mathcal{P}}\times\mathbf{E}]. \tag{18}$$

The contribution has the same structure as fifth term in the Hamiltonian (11). In the result, the quantum Hamiltonian up to order c^{-2} reads

$$\hat{H}_{ph} = mc^2 + \frac{\hat{\mathcal{P}}^2}{2m} - \frac{\hat{\mathcal{P}}^4}{8m^3c^2} + eA^0 + \frac{e(g-1)}{2m^2c^2}\hat{\mathbf{S}}[\hat{\mathcal{P}}\times\mathbf{E}] - \frac{eg}{2mc}\mathbf{B}\hat{\mathbf{S}}. \tag{19}$$

The first three terms corresponds to an increase of relativistic mass. The last two terms coincides with those in Eq. (1). We could carry out the same reasoning in the classical theory, by asking on the new variables z' that obey the canonical brackets as a consequence of Eq. (12). In the desired approximation they are $\mathcal{P}^i = \mathcal{P}'^i - \frac{e}{c}A^i(x'^j)$, $x^i = x'^i - \frac{1}{4m^2c^2}S'^{ij}\mathcal{P}'^j$ and $S^{ij} = S'^{ij}$. In the result, we have shown that non-commutativity of electron's position at the Compton-scale is responsible for the fine structure of hydrogen atom.

5. Conclusions

Relativistic spinning particles give an example of noncommutative system, with noncommutative geometry of position space induced by spin of the particle. The "parameter of noncommutativity" is being proportional to spin-tensor. As a consequence, canonical quantization of the variational model of electron gives (in the leading approximation) the Pauli Hamiltonian. Our calculations show that

1) classical interaction of spin with electromagnetic field is given by manifestly covariant term $S^{\mu\nu}J_{\mu\nu}$ [4] and all constraints are covariant as well;
2) phase space is endowed with a non-trivial symplectic structure (Dirac brackets), in particular, position variables become noncommutative due to non-vanishing Dirac brackets;
3) the Thomas precession automatically appears in the equations of motion [1] due to non-trivial Dirac bracket, without modification of the Hamiltonian;
4) quantization of classical model for free electron leads to the positive energy part of Dirac equation in the Foldy–Wouthuysen representation, the free Hamiltonian acts in the space of two-component spinors and reads $\hat{H}_{phys}(F=0) = \sqrt{\hat{\mathbf{p}}^2 + m^2c^2}$, position operator of free electron is the Pryce's d-type [19,20];
5) quantization of classical model in the case of a stationary electromagnetic background formally leads to the Hamiltonian

$$\hat{H}_{phys}(F) = c\sqrt{(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}}))^2 - \frac{eg}{4c}\hat{S}^{\mu\nu}F_{\mu\nu}(\hat{\mathbf{x}}) + m^2c^2} + eA^0(\hat{\mathbf{x}}),$$

which, up to $o(c^{-2})$ order, coincides with the positive energy part of Dirac Hamiltonian in the Foldy–Wouthuysen representation. It would be interesting to compare high-order terms;
6) non-commutativity of position operator results in the Thomas 1/2-correction of spin-orbital interaction coming from $eA^0(\hat{\mathbf{x}})$ term.[8]

[8] For instance, similar corrections were obtained in [29]. However, they appear from the non-commutativity introduced in the Dirac representation, therefore they give additional contribution to the correct spectrum as if non-commutativity acts twice.

Table 1
Auxiliary Poisson brackets.

	$\{\mathcal{P}^0, *\}$	$\{T_3, *\}$	$\{T_4, *\}$
x^i	$-\frac{\mathcal{P}^i}{\mathcal{P}^0}$	$-\omega^i + \frac{\omega^0\mathcal{P}^i}{\mathcal{P}^0}$	$-\pi^i + \frac{\pi^0\mathcal{P}^i}{\mathcal{P}^0}$
$\mathcal{P}^i$	$-\frac{e}{\mathcal{P}^0c}[(F\vec{\mathcal{P}})^i + \frac{g}{8}\partial^i(SF)]$	$\frac{e\omega^0}{\mathcal{P}^0c}[(F\vec{\mathcal{P}})^i + \frac{}{8}\partial^i(SF)] - \frac{e}{c}(F\vec{\omega})^i$	$\frac{e\pi^0}{\mathcal{P}^0c}[(F\vec{\mathcal{P}})^i + \frac{g}{8}\partial^i(SF)] - \frac{e}{c}(F\vec{\pi})^i$
$\mathcal{P}^0$	0	$\frac{e}{2\mathcal{P}^0c}[(g-2)(\vec{\mathcal{P}}F\vec{\omega}) + \frac{g}{8}\omega^i\partial^i(SF) - \mu F^{0i}\mathcal{P}^{[0}\omega^{i]}]$	$\frac{e}{2\mathcal{P}^0c}[(g-2)(\vec{\mathcal{P}}F\vec{\pi}) + \frac{g}{8}\pi^i\partial^i(SF) - \frac{g}{2}F^{0i}\mathcal{P}^{[0}\pi^{i]}]$
ω^μ	$-\frac{eg}{2\mathcal{P}^0c}(F\omega)^\mu$	$\frac{\omega^0 eg}{2\mathcal{P}^0c}(F\omega)^\mu$	$-\mathcal{P}^\mu + \frac{\pi^0 eg}{2\mathcal{P}^0c}(F\omega)^\mu$
π^μ	$-\frac{eg}{2\mathcal{P}^0c}(F\pi)^\mu$	$\mathcal{P}^\mu + \frac{\omega^0 eg}{2\mathcal{P}^0c}(F\pi)^\mu$	$\frac{\pi^0 eg}{2\mathcal{P}^0c}(F\pi)^\mu$
$J^{\mu\nu}$	$-\frac{eg}{2\mathcal{P}^0c}(FS)^{[\mu\nu]}$	$\frac{\omega^0 eg}{2\mathcal{P}^0c}(FS)^{[\mu\nu]} - 2\mathcal{P}^{[\mu}\omega^{\nu]}$	$\frac{\pi^0 eg}{2\mathcal{P}^0c}(FS)^{[\mu\nu]} - 2\mathcal{P}^{[\mu}\pi^{\nu]}$

In the considered approximation our Hamiltonian $\hat{H}_{phys}(F)$ coincides with the Pauli Hamiltonian for the case of stationary fields. Therefore, within this approximation there is no any difference between standard and non-commutative approach to the spin-orbital interaction except a conceptual one. However, in the case of non-stationary fields the classical Hamiltonian changes form. Further studies of time-dependent electromagnetic fields and next order corrections may give suggestions for the experimental searches of effects produced by non-commutativity.

Acknowledgements

The work of AAD has been supported by the Brazilian foundations CNPq (Conselho Nacional de Desenvolvimento Cientifico e Tecnológico—Brasil) and FAPEMIG (Fundação de Amparo à Pesquisa do Estado de Minas Gerais—Brasil). The work of AMPM is supported by FAPEMIG (Demanda Universal 2015).

Appendix A

Notation Our variables are taken in arbitrary parametrization τ, then $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. The square brackets mean antisymmetrization, $\omega^{[\mu}\pi^{\nu]} = \omega^\mu\pi^\nu - \omega^\nu\pi^\mu$. For the four-dimensional quantities we suppress the contracted indexes and use the notation $\dot{x}^\mu N_{\mu\nu}\dot{x}^\nu = \dot{x}N\dot{x}$, $N^\mu{}_\nu\dot{x}^\nu = (N\dot{x})^\mu$, $\omega^2 = \eta_{\mu\nu}\omega^\mu\omega^\nu$, $\eta_{\mu\nu} = (-,+,+,+)$, $\mu = (0, i)$, $i = 1, 2, 3$, Notation for the scalar functions constructed from second-rank tensors are $FS = F_{\mu\nu}S^{\mu\nu}$, $S^2 = S^{\mu\nu}S_{\mu\nu}$.

Electromagnetic field:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = (F_{0i} = -E_i,\ F_{ij} = \epsilon_{ijk}B_k),$$
$$E_i = -\frac{1}{c}\partial_t A_i + \partial_i A_0, \quad B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} = \epsilon_{ijk}\partial_j A_k.$$

Spin-tensor:

$$S^{\mu\nu} = 2(\omega^\mu\pi^\nu - \omega^\nu\pi^\mu) = (S^{i0} = D^i,\ S_{ij} = 2\epsilon_{ijk}S_k),$$

then $S_i = \epsilon_{ijk}\omega_j\pi_k = \frac{1}{4}\epsilon_{ijk}S_{jk}$. Here S_i is three-dimensional spin-vector of Frenkel and D_i is dipole electric moment.

Dirac bracket Dirac bracket for the constraints (7) reads

$$\{A, B\}_D = \{A, B\} - \{A, T_3\}\{T_4, T_3\}^{-1}\{T_4, B\}$$
$$- \{A, T_4\}\{T_3, T_4\}^{-1}\{T_3, B\}.$$

Complete list of brackets computed in an arbitrary parametrization can be found in [2]. Here we present the brackets of the observables $x^i(t)$, $\mathcal{P}^i(t)$ and $S^{\mu\nu}(t)$. To compute them, we use the auxiliary Poisson brackets shown in the Table 1. We will use the notation

$$\begin{aligned}
u^0 &= \mathcal{P}^0 - \frac{(g-2)a}{2}(SF\mathcal{P})^0 + \frac{ga}{8}S^{0\mu}\partial_\mu(FS), \\
a &= \frac{-2e}{4m^2c^3 - e(g+1)(SF)}, \\
\triangle^{\mu\nu} &= -\frac{2ca}{eu^0}\mathcal{P}^{(0}S^{\mu\nu)}, \\
\mathcal{P}^{(0}S^{\mu\nu)} &= \mathcal{P}^0 J^{\mu\nu} + \mathcal{P}^\mu S^{\nu 0} + \mathcal{P}^\nu S^{0\mu}, \\
K^{\mu\nu} &= -\frac{gca}{4eu^0}S^{0\mu}\partial^\nu(SF), \qquad L^{\mu\nu\alpha} = -\frac{ga}{u^0}(FS)^{[\mu\nu]}S^{0\alpha}, \\
g^{\mu\nu} &= \eta^{\mu\nu} - \frac{2ca\mathcal{P}^0}{eu^0}\mathcal{P}^\mu\mathcal{P}^\nu.
\end{aligned} \tag{20}$$

Using the table, we obtain $\{T_3, T_4\} = \frac{eu^0}{2ca\mathcal{P}^0}$. Then

$$\begin{aligned}
&\{x^i, x^j\}_D = \frac{1}{2}\triangle^{ij}, \qquad \{x^i, \mathcal{P}^j\}_D = \delta^{ij} - \frac{e}{2c}\left[\triangle^{ik}F^{kj} - K^{ij}\right], \\
&\{\mathcal{P}^i, \mathcal{P}^j\}_D = \frac{e}{c}F^{ij} - \frac{e^2}{2c^2}\left[F^{ik}\triangle^{kn}F^{nj} - F^{[ik}K^{kj]}\right], \\
&\{S^{\mu\nu}, S^{\alpha\beta}\}_D = 2(g^{\mu\alpha}S^{\nu\beta} - g^{\mu\beta}S^{\nu\alpha} - g^{\nu\alpha}S^{\mu\beta} + g^{\nu\beta}S^{\mu\alpha}) \\
&\qquad + L^{\mu\nu[\alpha}\mathcal{P}^{\beta]}, \\
&\{S^{\mu\nu}, x^j\}_D = \mathcal{P}^{[\mu}\triangle^{\nu]j} + \frac{1}{2}L^{\mu\nu j}, \\
&\{S^{\mu\nu}, \mathcal{P}^j\}_D = \frac{e}{c}\left[-\mathcal{P}^\mu(\triangle^{\nu k}F^{kj} - K^{\nu j}) - (\mu \leftrightarrow \nu)\right. \\
&\qquad \left. + \frac{1}{2}L^{\mu\nu k}F^{kj}\right].
\end{aligned} \tag{21}$$

References

[1] A.A. Deriglazov, Variational problem for the Frenkel and the Bargmann–Michel–Telegdi (BMT) equations, Mod. Phys. Lett. A 28 (01) (2013) 1250234.

[2] A.A. Deriglazov, A.M. Pupasov-Maksimov, Frenkel electron on an arbitrary electromagnetic background and magnetic Zitterbewegung, Nucl. Phys. B 885 (2014) 1, arXiv:1401.7641 [hep-th].

[3] A.A. Deriglazov, A.M. Pupasov-Maksimov, Geometric constructions underlying relativistic description of spin on the base of non-Grassmann vector-like variable, SIGMA 10 (2014) 012, arXiv:1311.7005 [math-ph].

[4] A.A. Deriglazov, Lagrangian for the Frenkel electron, Phys. Lett. B 736 (2014) 278, arXiv:1406.6715 [physics.gen-ph].

[5] V. Bargmann, L. Michel, V.L. Telegdi, Phys. Rev. Lett. 2 (1959) 435.

[6] J.P. Miller, E. de Rafael, B.L. Roberts, Muon (g2): experiment and theory, Rep. Prog. Phys. 70 (2007) 795.

[7] W. Guzmán Ramírez, A.A. Deriglazov, A.M. Pupasov-Maksimov, Frenkel electron and a spinning body in a curved background, J. High Energy Phys. 1403 (2014) 109, arXiv:1311.5743 [hep-th].

[8] W.G. Dixon, Nuovo Cimento 34 (1964) 317.

[9] I.B. Khriplovich, Particle with internal angular momentum in a gravitational field, Sov. Phys. JETP 69 (1989) 217, Zh. Eksp. Teor. Fiz. 96 (1989) 385.
[10] I.B. Khriplovich, Spinning relativistic particles in external fields, Acta Phys. Pol. B 1 (2008) 197, Proceedings supplement.
[11] A.A. Deriglazov, W.G. Ramírez, Ultra-relativistic spinning particle and a rotating body in external fields, arXiv:1511.00645.
[12] A.A. Deriglazov, W.G. Ramírez, Mathisson–Papapetrou–Tulczyjew–Dixon (MPTD) equations in ultra-relativistic regime and gravimagnetic moment, arXiv:1509.05357.
[13] W.G. Ramírez, A.A. Deriglazov, Lagrangian formulation for Mathisson–Papapetrou–Tulczyjew–Dixon (MPTD) equations, Phys. Rev. D 92 (2015) 124017.
[14] C. Armaza, M. Banados, Benjamin Koch, Can Schwarzschild black holes be accelerators of spinning massive particles?, arXiv:1510.01223 [gr-qc], 2015.
[15] A.B. Balakin, V.A. Popov, Spin-axion coupling, arXiv:1509.06058 [gr-qc], 2015.
[16] M.B. Frob, Quantum gravitational corrections for spinning particles, arXiv:1607.03129, 2015.
[17] J.Y. Chen, D.T. Son, M.A. Stephanov, H.U. Yee, Y. Yin, Lorentz invariance in chiral kinetic theory, Phys. Rev. Lett. 113 (18) (2014) 182302.
[18] P.A.M. Dirac, Quantum Mechanics, 4th ed., Oxford Univ. Press, London, 1958.
[19] A.A. Deriglazov, A.M. Pupasov-Maksimov, Lagrangian for Frenkel electron and position's non-commutativity due to spin, Eur. Phys. J. C 74 (2014) 3101, arXiv:1312.6247 [hep-th].
[20] M.H.L. Pryce, The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles, Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. 195 (1040) (1948) 62–81.
[21] H.S. Snyder, Quantized space-time, Phys. Rev. 71 (1947) 38–41.
[22] A.A. Deriglazov, Poincare covariant mechanics on noncommutative space, J. High Energy Phys. 03 (2003) 021.
[23] A.A. Deriglazov, Nonrelativistic spin: a la Berezin–Marinov quantization on a sphere, Mod. Phys. Lett. A 25 (32) (2010) 2769–2777.
[24] H. Bauke, S. Ahrens, C.H. Keitel, R. Grobe, Relativistic spin operators in various electromagnetic environments, Phys. Rev. A 89 (2014) 052101.
[25] G.N. Fleming, Nonlocal properties of stable particles, Phys. Rev. B 139 (1965) 963.
[26] M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Hydrogen atom spectrum and the Lamb shift in noncommutative QED, Phys. Rev. Lett. 86 (2001) 2716; Comments on the hydrogen atom spectrum in the noncommutative space, Eur. Phys. J. C 36 (2004) 251–252.
[27] L. Khodja, S. Zaim, New treatment of the noncommutative Dirac equation with a Coulomb potential, Int. J. Mod. Phys. A 27 (19) (2012) 1250100.
[28] W.O. Santos, A.M.C. Souza, The anomalous zeeman effect for the hydrogen atom in noncommutative space, Int. J. Theor. Phys. 51 (12) (2012) 3882–3890.
[29] V.G. Kupriyanov, A hydrogen atom on curved noncommutative space, J. Phys. A, Math. Theor. 46 (24) (2013) 245303.
[30] M. Gomes, V.G. Kupriyanov, A.J. da Šilva, Noncommutativity due to spin, Phys. Rev. D 81 (Apr 2010) 085024.
[31] A.F. Ferrari, M. Gomes, V.G. Kupriyanov, C.A. Stechhahn, Dynamics of a Dirac fermion in the presence of spin noncommutativity, Phys. Lett. B 718 (2013) 1475–1480.
[32] D.M. Gitman, I.V. Tyutin, Quantization of Fields with Constraints, Springer-Verlag, Berlin, 1990.
[33] L.H. Thomas, The motion of the spinning electron, Nature 117 (1926) 514.
[34] J. Frenkel, Die elektrodynamik des rotierenden elektrons, Z. Phys. 37 (4–5) (1926) 243–262.
[35] I.B. Khriplovich, A.A. Pomeransky, Equations of motion of spinning relativistic particle in external fields, J. Exp. Theor. Phys. 86 (1998) 839, Zh. Eksp. Teor. Fiz. 113 (1998) 1537, arXiv:gr-qc/9710098.
[36] L.L. Foldy, S.A. Wouthuysen, On the Dirac theory of spin 1/2 particles and its non-relativistic limit, Phys. Rev. 78 (1950) 29.
[37] R.P. Feynman, Quantum Electrodynamics, W A Benjamin, 1961.
[38] G.N. Fleming, Covariant position operators, spin, and locality, Phys. Rev. B 137 (1965) B188.
[39] H.C. Corben, Classical and Quantum Theories of Spinning Particles, Holden-Day, San Francisco, 1968.
[40] S.S. Stepanov, Thomas precession for spin and for a rod, Phys. Part. Nucl. 43 (2012) 128.
[41] S. Weinberg, Gravitation and Cosmology, Willey, New York, NY, 1972.
[42] A. Papapetrou, Proc. R. Soc. Lond. A209 (1951) 248.
[43] F.A.E. Pirani, Acta Phys. Pol. 15 (1956) 389.
[44] W. Tulczyjew, Acta Phys. Pol. 18 (1959) 393.