3-D elastodynamic contact problem for an interface crack under harmonic loading

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A B S T R A C T
The paper concerns 3-D fracture dynamic problems for elastic bimaterials with cracks located at the bonding interface under the time-harmonic loading. The problem for a penny-shaped interface crack under normally incident tension–compression wave is solved taking the crack’s closure into account. The dynamic stress intensity factors are computed as functions of the loading frequency and compared with those obtained neglecting the crack’s closure.

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1. Introduction

The modern engineering standards demand that the components’ design should incorporate an awareness of various safety factors. The great attention is paid to the failure analysis as an essential tool to improve materials’ reliability, which helps prevent accidents and disasters causing by the unpredicted fracture. Thus the dynamic response of intra- and inter-component cracks to elastic waves is a topic of long-standing interest in fracture mechanics [1–6]. The growth of interfacial cracks has a significant influence on the integrity of layered composite materials. Consequently, the problems of bimaterials and layered materials with interlayer cracks are of particular importance [7–13]. Today, due to the great improvement in computer technology and computational methods, it is possible to solve many complex interface crack problems accurately and efficiently [14–19].

It should also be emphasised that the opposite faces of existing cracks interact with each other under deformation, drastically affecting the distributions of the stress and strain fields. In general, when bodies are brought together they will either be in contact at a point, along a line, or over a surface. In complex loading and geometrical configurations, a combination of the above mentioned contacts is possible. Upon application of a load on these bodies in contact, the initial contact region will change. The extent of changes will depend on the magnitude of the external load, the elastic properties of the bodies and frictional behaviour at the contact interfaces, which are unknown beforehand. Due to the non-linearity of the dynamic cracks’ faces contact interaction the researchers almost always neglect the effects of the cracks’ closure in spite of its evident significance [20–29].

One of the first ever attempts to solve the dynamic interface cracks problem taking the crack’s closure into account was recently undertaken by the authors of the present study [28], who considered 2-D bimaterials with linear interface cracks under harmonic loading. It was shown that the crack’s closure changes the solution both quantitatively and qualitatively, as the difference between comparable quantities (in particular, stress intensity factors) obtained taking the crack closure into account and neglecting it can reach 30–50%. The solution was obtained based on the system of boundary integral equations

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for the general case of an interface crack between two dissimilar elastic materials under dynamic loading derived in [30–36].

The reviews of the results obtained for cracked bimaterials neglecting the cracks’ closure are given in [5,6,17–19,35,36].

The current work is devoted to the study of the linearly elastic bimaterial with a penny-shaped interface crack under the normally incident time-harmonic tension–compression wave. It must be noted that in contrast to problems for homogeneous materials with cracks under the normally incident longitudinal loading an oscillating singularity appears at the vicinity of the crack front; and the normal displacements and tangential tractions arise at the bonding interface, which results in non-zero shear stress intensity factors even for the case considered [1,8,15,21,29,34,36]. The dynamic stress intensity factors (opening and transverse shear modes) are computed as functions of the frequency of the incident wave. For the very first time, the results for the 3-D interface crack problem are obtained taking the crack’s closure into account.

2. Methodology

Let us consider an unbounded bimaterial which consists of two dissimilar linearly elastic homogeneous isotropic half-spaces \( \Omega^{(1)} \) and \( \Omega^{(2)} \) with plane boundaries \( \Gamma^{(1)} \) and \( \Gamma^{(2)} \). Henceforth, the superscript \((1)\) refers to the upper half-space and the superscript \((2)\) refers to the lower half-space. The crack is located at the bimaterial interface. The boundary \( \Gamma^{(m)} \) (\( m = 1, 2 \)) consist of the infinite part \( \Gamma^{(m)\infty} \) and the finite part \( \Gamma^{(m)cr} \), and the crack surface \( \Gamma^{cr} \) is formed by two faces, \( \Gamma^{(1)cr} \) and \( \Gamma^{(2)cr} \) (see Fig. 1).

In the absence of body forces, the stress–strain state of both half-spaces is defined by the Lamé dynamic equations of the linear elasticity for the displacement vector \( u^{(m)}(x, t) \)

\[
(\ddot{u}^{(m)} + \mu^{(m)} \Delta u^{(m)}(x, t)) = \rho^{(m)} \partial^2 \partial^2 u^{(m)}(x, t), \quad x \in \Omega^{(m)}, \quad t \in [0, \infty),
\]

where \( \Delta \) is the Laplace operator, \( \dot{u}^{(m)} \) and \( \mu^{(m)} \) are the Lamé elastic constants, \( \rho^{(m)} \) is the specific material density.

![Fig. 1. Interface crack under loading.](image)
The standard conditions of continuity for displacements, \( \mathbf{u}^{(1)}(x, t) = \mathbf{u}^{(2)}(x, t) \), and stresses, \( \mathbf{p}^{(1)}(x, t) = -\mathbf{p}^{(2)}(x, t) \), where \( \mathbf{p}^{(m)}(x, t) \) is the traction vector, are satisfied at the bonding interface \( \Gamma^* = \Gamma^{(1)} \cap \Gamma^{(2)} \). At the crack surface the traction vectors \( \mathbf{g}^{(m)}(x, t) \), caused by the external loading, are given.

It is also assumed that there are no initial displacements of the points of the body, and the Sommerfeld radiation-type condition, which provides a finite elastic energy of an infinite body, was imposed at infinity on the vector of displacements.

In order to include the contact interaction of the opposite crack’s faces into consideration, the Signorini unilateral constraints must be imposed for the normal components of the contact force and the displacement vectors

\[
[u_n(x, t)] = 0, \quad [q_n(x, t)] = 0, \quad [u_n(x, t)]q_n(x, t) = 0, \quad x \in \Omega, \quad t \in [0; T],
\]

where \( u(x, t) = \mathbf{u}^{(1)}(x, t) - \mathbf{u}^{(2)}(x, t) \) is the displacement discontinuity vector; and \( q(x, t) \) is the contact force that arises in the contact region, which is unknown beforehand, changes in time under deformation of the material and must be determined as a part of solution. The contact region also depends on the frequency, magnitude and direction of the external loading complicating the problem even more and making it highly non-linear [21,23–26,28,29].

It must also be noted that in reality the friction of the opposite crack faces would also affect the solution of the problem. In order to simplify the analysis, the friction was not taken into account in the case presented here. Hence, considering the friction will be the natural next stage of the research (see also [20,22,23,25,27]).

The constraints (2) ensure that there is no interpenetration of the opposite crack faces; the normal component of the contact force is unilateral and it is absent for any non-zero opening of the crack. Note that, due to the contact interaction the traction vector at the crack faces, \( \mathbf{p}^{(m)}(x, t) \), is the superposition of the initial traction caused by the incident wave, \( \mathbf{g}^{(m)}(x, t) \), and the contact force, \( q(x, t) \).

The allowance for crack faces contact interaction makes the resulting process a steady-state periodic, but not a harmonic one. Hence, the components of the stress–strain state cannot be represented as a function of coordinates multiplied by an exponential function [21,23–26,28,29]. Here all components of the solution are expanded into the Fourier series

\[
f(\bullet, t) = \text{Re} \left\{ \sum_{k=-\infty}^{+\infty} f^k(\bullet)e^{ik\omega t} \right\},
\]

where \( \omega_k = 2\pi k/T \), and the appropriate Fourier coefficients are given as

\[
f^k(\bullet) = \frac{\omega}{2\pi} \int_0^T f(\bullet, t)e^{-ik\omega t}dt.
\]

For the general case of the dynamic loading of the bimaterial with interface crack the Somigliana dynamic representations for the components of the displacement and traction fields in the upper and lower half-spaces can be used in order to obtain the necessary system of boundary integral equations corresponding to the considered problem [30,32], see also [31,33–36]. Using the results presented by the authors of the current study for the 2-D case in [28], the following system of boundary integral equations for the Fourier coefficients of displacements and tractions at the interface and the crack faces can be obtained in the considered 3-D case of the harmonic loading

\[
\begin{align*}
- \int_{\Gamma^{(1)}} \bar{p}^{(k)}(y)U^{(1)}(x, y, \omega_k)dy & = -\frac{1}{2} u^{(1)}(x) - \int_{\Gamma^{(1)}} u^{(1)}(y)W^{(1)}(x, y, \omega_k)dy + \int_{\Gamma^{(1)}} u^{(2)}(y)W^{(1)}(x, y, \omega_k)dy \\
& - \int_{\Gamma^{(1)}} \bar{p}^{(k)}(y)U^{(1)}(x, y, \omega_k)dy, \quad x \in \Gamma^{(1)}\setminus\Gamma^*,
\end{align*}
\]

\[
\begin{align*}
- \int_{\Gamma^{(2)}} \bar{p}^{(k)}(y)U^{(2)}(x, y, \omega_k)dy & = -\frac{1}{2} u^{(2)}(x) - \int_{\Gamma^{(2)}} u^{(2)}(y)W^{(2)}(x, y, \omega_k)dy + \int_{\Gamma^{(2)}} u^{(1)}(y)W^{(2)}(x, y, \omega_k)dy \\
& + \int_{\Gamma^{(2)}} \bar{p}^{(k)}(y)U^{(2)}(x, y, \omega_k)dy, \quad x \in \Gamma^{(2)}\setminus\Gamma^*,
\end{align*}
\]

\[
\begin{align*}
- \int_{\Gamma^{(1)}} \bar{p}^{(k)}(y)U^{(1)}(x, y, \omega_k)dy & = -\frac{1}{2} u^{(1)}(x) - \int_{\Gamma^{(1)}} u^{(1)}(y)W^{(1)}(x, y, \omega_k)dy + \int_{\Gamma^{(1)}} u^{(2)}(y)W^{(1)}(x, y, \omega_k)dy \\
& - \int_{\Gamma^{(1)}} \bar{p}^{(k)}(y)U^{(1)}(x, y, \omega_k)dy, \quad x \in \Gamma^*,
\end{align*}
\]

\[
\begin{align*}
- \int_{\Gamma^{(2)}} \bar{p}^{(k)}(y)U^{(2)}(x, y, \omega_k)dy & = -\frac{1}{2} u^{(2)}(x) - \int_{\Gamma^{(2)}} u^{(2)}(y)W^{(2)}(x, y, \omega_k)dy + \int_{\Gamma^{(2)}} u^{(1)}(y)W^{(2)}(x, y, \omega_k)dy \\
& + \int_{\Gamma^{(2)}} \bar{p}^{(k)}(y)U^{(2)}(x, y, \omega_k)dy, \quad x \in \Gamma^*,
\end{align*}
\]

where \( \bar{p}^{(k)}(x), p^{(k)}(x), u^{(k)}(x) \) and \( u^{(k)}(x) \), are the Fourier complex-valued coefficients of the total traction at the crack surface, unknown traction and displacements at the bonding interface and the opposite crack faces. The integration surface \( \Gamma^* \) coincides with the surface \( \Gamma^{(2)}\setminus\Gamma^* \); and variables \( u^{(k)}(x) = u^{(1)}(x) = u^{(2)}(x), p^{(k)}(x) = -p^{(1)}(x) = p^{(2)}(x), x \in \Gamma^* \) in introduced...
Eqs. (5)–(8) correspond to the conditions of continuity. The physical values of tractions and displacements are obtained according the Eq. (2).

The Green’s fundamental displacement tensor $U_{ij}^{(m)}(x, y, \omega)$ present in Eqs. (5)–(8) has the following form:

$$U_{ij}^{(m)}(x, y, \omega_k) = \frac{1}{4\pi\mu^{(m)}} \left( \psi^{(m)} \delta_{ij} - \chi^{(m)} \left( \frac{y_i - y_k}{r} \right) \left( \frac{y_j - y_k}{r} \right) \right),$$

where $\delta_{ij}$ is the Kronecker delta, $r$ is the distance between the observation point and the load point. The appropriate expressions for functions $\psi^{(m)}$ and $\chi^{(m)}$ in the considered 3-D case of the harmonic loading are [3,4,21]:

$$\psi^{(m)} = \left( \frac{1}{l_1^{(m)}} \right)^2 + \frac{1}{l_2^{(m)}} + 1 \cdot \frac{e^{\gamma x}}{r} - \left( \frac{c_2^{(m)}}{c_1^{(m)}} \right)^2 \frac{1}{l_1^{(m)}} + \frac{1}{l_1^{(m)}} \cdot \frac{e^{\gamma y}}{r},$$

$$\chi^{(m)} = \left( \frac{3}{l_1^{(m)}} \right)^2 + \frac{3}{l_2^{(m)}} + 1 \cdot \frac{e^{\gamma x}}{r} - \left( \frac{c_2^{(m)}}{c_1^{(m)}} \right)^2 \frac{3}{l_1^{(m)}} + 1 \cdot \frac{e^{\gamma y}}{r},$$

where $l_i^{(m)} = io\kappa_1/c_1^{(m)}$, $l_2^{(m)} = io\kappa_2/c_2^{(m)}$, $c_1^{(m)} = \sqrt{(\lambda^{(m)} + 2\mu^{(m)})/\rho^{(m)}}$ and $c_2^{(m)} = \sqrt{\mu^{(m)}/\rho^{(m)}}$ are the velocities of the longitudinal and the transversal waves in the material.

Consequently the integral kernel $W_{ij}^{(m)}(x, y, \omega_k)$ is obtained from $U_{ij}^{(m)}(x, y, \omega_k)$ and has the form [21,31,33,34]:

$$W_{ij}^{(m)}(x, y, \omega_k) = \chi^{(m)} n_i^{(m)}(y) \frac{\partial}{\partial y_j} U_{ij}^{(m)}(x, y, \omega_k) + \mu^{(m)} n_j^{(m)}(y) \frac{\partial}{\partial y_i} U_{ij}^{(m)}(x, y, \omega_k) + \frac{\partial}{\partial y_j} U_{ij}^{(m)}(x, y, \omega_k).$$

The exact expressions for integral kernels $U_{ij}^{(m)}(x, y, \omega_k)$ and $W_{ij}^{(m)}(x, y, \omega_k)$ in terms of the Taylor power series are given in Appendix.

The considered non-linear contact problem requires an iterative solution procedure, which consistently solves the Neumann and Dirichlet problems in the cracked solid in combination with projections onto subsets of admissible displacements and admissible contact forces [21]. During the iterative process, the solution changes until the distribution of physical values satisfying the contact constraints (2) is found. Here the algorithm initially proposed for dynamic problems in homogeneous cracked solids [21,37] was adapted and used to solve the considered 3-D interface problem. The piecewise-constant approximation of the known and unknown functions was used to solve the problem numerically.

3. Numerical results

A penny-shaped crack with the radius of $R$ at the bimaterial interface is considered as a numerical example. The materials of the upper and lower half-spaces have the typical properties of steel and epoxy: $E^{(1)} = 207$ GPa, $E^{(2)} = 4.6$ GPa; $\mu^{(1)} = 0.28$, $\mu^{(2)} = 0.36$; $\rho^{(1)} = 7800$ kg/m$^3$, $\rho^{(2)} = 1380$ kg/m$^3$.

A time-harmonic tension-compression wave propagates normally to the interface. It is assumed that the incident wave can be described by the following potential function.

![Fig. 2](image-url) Stress intensity factor (opening mode) plotted against the number of iterations for different iterative coefficients, the wave number $k_1 R = 0.25$. 
\[ U(x, t) = U_0 e^{i(k_1^{(1)}x - \omega t)}. \]

where \( U_0 \) and \( \omega \) are the amplitude and the frequency of the incident wave; \( k_1^{(1)} \) is the generalized wave number. This potential generates the displacement field \( u(x, t) = \text{grad}_x \Phi(x, t) \) and the traction \( g(x, t) \) caused by the incident wave is

\[ g(x, t) = (0, 0, -k_1^{(1)})^2 \Phi_0 \cos(\omega t), \quad x \in \Omega, \quad t \in [0, T]. \]

The dynamic stress intensity factors (the opening and the transverse shear modes) were computed in the vicinity of the crack front using the following asymptotic expressions [3,4,21]:

\[ K_I = \max_{R \to 0} \lim_{r \to 0} p_n^r(R + r, t) \sqrt{2\pi r}, \quad K_{II} = \max_{R \to 0} \lim_{r \to 0} p_s^r(R + r, t) \sqrt{2\pi r}, \]

where \( p_n^r(R + r, t) \) and \( p_s^r(R + r, t) \) are normal and tangential components of the traction vector at the bonding interface; \( r \) is the distance from the crack front.

Fig. 3. Stress intensity factor (opening mode) plotted against the number of iterations for different iterative coefficients, the wave number \( k_1^{(1)} R = 0.50 \).

Fig. 4. Stress intensity factor for opening mode plotted against the wave number.
The convergence of the algorithm with respect to the iterative coefficient for different wave numbers is given in Figs. 2 and 3. The results are normalised by the corresponding static values. Note that within the considered range of iterative coefficients, the use of different coefficients affects only the rate of convergence but not the obtained results.

In Figs. 4 and 5 the dynamic stress intensity factors for the opening mode and the transverse shear mode for two combinations of the properties of the upper and the lower half-spaces (steel–epoxy and steel–steel) are given as functions of the dimensionless wave number $k_1^2R = \omega R/c_1^2$.

The presented numerical results for the opening and shear modes were obtained for the iterative coefficient $K_n = 1000$, the approximate area of the boundary element nearest to the crack front $h = R^2/214$, the number of Fourier coefficients $N_f = 5$, and the number of time steps within the period of oscillations $N_t = 50$. The results obtained for the crack between two identical half-spaces (steel–steel) are in very good agreement with the results for the homogeneous cracked body [26].

It is evident from Figs. 4 and 5, that the maximal stress intensity factors evaluated with allowance for the effect of crack closure are much smaller than the ones obtained when neglecting it. Furthermore, the maximums are achieved at different wave frequencies. Note that the dissimilarity of the upper and the lower half-spaces results in mixed mode stress distributions for any type of the loading (e.g. there are tangential displacements’ jumps and tangential tractions for the considered case of the normally incident tension–compression wave, and vice versa). Hence, due to this mutual dependence of the normal and tangential components of the solution the transverse shear mode of the stress intensity factor is affected by the crack closure even for the frictionless case considered in this paper.

4. Conclusions

The current paper is devoted to the study of the linearly elastic bimaterial with a penny-shaped interface crack under normally incident time-harmonic tension–compression wave. The problem is solved by using the boundary integral equations method and the Somigliana dynamic identity for the displacement field.

For the very first time the 3-D interface problem is solved taking the crack closure into account. The dynamic stress intensity factors (opening and transverse shear modes) are computed as functions of the frequency of the incident wave and compared with those obtained neglecting the crack’s closure. Analysis of the results shows that the crack closure significantly changes the solution.

It must also be noted that in reality the friction of the opposite crack faces would significantly affect the solution of the problem. Thus, considering the friction will be the natural next stage of the research.

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Appendix A

After the differentiation and some manipulations the integral kernels $U^{(m)}_{ij}(\mathbf{x}, \omega_k)$ and $W^{(m)}_{ij}(\mathbf{x}, \omega_k)$ present in systems of boundary integral Eqs. (5)–(8) in terms of the Taylor power series are as shown [21,31,33,34]:

\begin{align}
U^{(m)}_{13}(\mathbf{x}, \omega_k) &= U^{(m)}_{23}(\mathbf{x}, \omega_k) = U^{(m)}_{31}(\mathbf{x}, \omega_k) = U^{(m)}_{32}(\mathbf{x}, \omega_k) = 0, \\
U^{(m)}_{11}(\mathbf{x}, \omega_k) &= \frac{1}{8\pi\mu(\lambda + 2\mu)r^2} \left[ (\lambda + 3\mu) \frac{(\lambda + \mu)(y_1 - x_1)^2}{r^2} + \frac{1}{4\pi\mu r} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+2)} \right] \\
&\quad \times \left[ \left( l_{11}^{(m)} \right)^2 \left( c_{11}^{(m)} \right)^2 + \left( l_{12}^{(m)} \right)^2 \left( c_{12}^{(m)} \right)^2 \right], \\
U^{(m)}_{22}(\mathbf{x}, \omega_k) &= \frac{1}{8\pi\mu(\lambda + 2\mu)r^2} \left[ (\lambda + 3\mu) \frac{(\lambda + \mu)(y_2 - x_2)^2}{r^2} + \frac{1}{4\pi\mu r} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+2)} \right] \\
&\quad \times \left[ \left( l_{11}^{(m)} \right)^2 \left( c_{11}^{(m)} \right)^2 + \left( l_{12}^{(m)} \right)^2 \left( c_{12}^{(m)} \right)^2 \right], \\
U^{(m)}_{12}(\mathbf{x}, \omega_k) &= U^{(m)}_{21}(\mathbf{x}, \omega_k) = \frac{(\lambda + \mu)(y_1 - x_1)(y_2 - x_2)}{8\pi\mu(\lambda + 2\mu)r^3} + \frac{1}{4\pi\mu r} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+2)} \left( l_{11}^{(m)} \right)^2 \left( c_{11}^{(m)} \right)^2 + \left( l_{12}^{(m)} \right)^2 \left( c_{12}^{(m)} \right)^2 + \left( l_{21}^{(m)} \right)^2 \left( c_{21}^{(m)} \right)^2 + \left( l_{22}^{(m)} \right)^2 \left( c_{22}^{(m)} \right)^2 \right), \\
U^{(m)}_{33}(\mathbf{x}, \omega_k) &= \frac{1}{8\pi\mu(\lambda + 2\mu)r} \left[ (\lambda + 3\mu) + \frac{1}{4\pi\mu r} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+2)} \right] \\
&\quad \times \left[ \left( l_{11}^{(m)} \right)^2 \left( c_{11}^{(m)} \right)^2 + \left( l_{12}^{(m)} \right)^2 \left( c_{12}^{(m)} \right)^2 \right]. \\
W^{(m)}_{11}(\mathbf{x}, \omega_k) &= W^{(m)}_{22}(\mathbf{x}, \omega_k) = W^{(m)}_{33}(\mathbf{x}, \omega_k) = 0, \\
W_{13}(\mathbf{x}, \omega_k) &= -\frac{\mu(y_1 - x_1)}{4\pi(\lambda + 2\mu)r^3} + \frac{\mu(y_1 - x_1)}{4\pi r^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+2)} \left( l_{11}^{(m)} \right)^2 \left( c_{11}^{(m)} \right)^2 + \left( l_{12}^{(m)} \right)^2 \left( c_{12}^{(m)} \right)^2 + \left( l_{21}^{(m)} \right)^2 \left( c_{21}^{(m)} \right)^2 + \left( l_{22}^{(m)} \right)^2 \left( c_{22}^{(m)} \right)^2 \right), \\
W_{31}(\mathbf{x}, \omega_k) &= \frac{\mu(y_1 - x_1)}{4\pi(\lambda + 2\mu)r^3} + \frac{\mu(y_1 - x_1)}{4\pi r^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+2)} \left( l_{11}^{(m)} \right)^2 \left( c_{11}^{(m)} \right)^2 + \left( l_{12}^{(m)} \right)^2 \left( c_{12}^{(m)} \right)^2 + \left( l_{21}^{(m)} \right)^2 \left( c_{21}^{(m)} \right)^2 + \left( l_{22}^{(m)} \right)^2 \left( c_{22}^{(m)} \right)^2 \right], \\
W_{32}(\mathbf{x}, \omega_k) &= \frac{\mu(y_1 - x_1)}{4\pi(\lambda + 2\mu)r^3} + \frac{\mu(y_1 - x_1)}{4\pi r^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+2)} \left( l_{11}^{(m)} \right)^2 \left( c_{11}^{(m)} \right)^2 + \left( l_{12}^{(m)} \right)^2 \left( c_{12}^{(m)} \right)^2 + \left( l_{21}^{(m)} \right)^2 \left( c_{21}^{(m)} \right)^2 + \left( l_{22}^{(m)} \right)^2 \left( c_{22}^{(m)} \right)^2 \right].
\end{align}

References