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Existence and uniqueness of positive solutions for higher order nonlocal fractional differential equations

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ABSTRACT

By means of a monotone iterative technique, we establish the existence and uniqueness of the positive solutions for a class of higher conjugate-type fractional differential equation with one nonlocal term. In addition, the iterative sequences of solution and error estimation are also given. In particular, this model comes from economics, financial mathematics and other applied sciences, since the initial value of the iterative sequence can begin from an known function, this is simpler and helpful for computation.

1. Introduction

In this paper, we are concerned with the existence and uniqueness of positive solutions for the following singular nonlinear $(n-1, 1)$ conjugate-type fractional differential equation with one nonlocal term

$$\begin{cases} D_{0+}^{\alpha}x(t) + f(t, x(t)) = 0, \quad 0 < t < 1, \ n-1 < \alpha \le n, \\ x^{(k)}(0) = 0, \quad 0 \le k \le n-2, \qquad x(1) = \displaystyle\int_0^1 x(s)dA(s), \end{cases} \tag{1.1}$$

where $\alpha \geq 2$, D_{0+}^{α} is the standard Riemann–Liouville derivative, A is a function of bounded variation and $\int_0^1 u(s)dA(s)$ denotes the Riemann–Stieltjes integral of u with respect to A, dA can be a signed measure.

Since $\int_0^1 u(s)dA(s)$ denotes the Riemann–Stieltjes integral in BCs (1.1), this implies the case of BCs (1.1) covers the multi-point BCs and also integral BCs in a single framework. For a comprehensive study of the case when there is a Riemann–Stieltjes integral boundary condition at both ends, see [1].

As the boundary value problem in economics, financial mathematics and other applied science has a wide range of applications, in recent years, there have been many papers investigating the existence and uniqueness of the positive solution for local or nonlocal boundary value problems of the second or higher order ordinary differential equations, we refer the readers to [2–9] and the references cited therein. For the case where α is an integer, Du and Zhao [9] investigated the following multi-point boundary problem

$$\begin{cases} -x''(t) = f(t, x(t)), \quad 0 < t < 1, \\ x(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i x(\eta_i), \qquad x(1) = 0. \end{cases} \tag{1.2}$$

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They assumed f is decreasing in u and obtained the existence of $C[0,1]$ positive solutions w for (1.2) with the property that $w(t) \geq m(1-t)$ for some $m > 0$. In a recent paper [10], Webb and Zima studied the problem

$$\begin{cases} x''(t) + k^2 x = f(t, x(t)), & 0 < t < 1, \\ x(0) = 0, \quad x(1) = \int_0^1 x(s)dA(s) \end{cases} \tag{1.3}$$

when dA is allowed to be a signed measure, and obtained the existence of multiple positive solutions under suitable conditions on $f(t,x)$. And then, by applying the monotone iterative technique, Mao et al. [8] established the existence and uniqueness of the positive solution for singular integral boundary value problem (1.3). When α is a fraction, Goodrich [11] dealt with a problem similar to (1.1) but with local conditions, by deriving properties of the Green's function and by using the well-known Guo–Krasnoselskii's fixed point theorem, the author established some nice existence results of at least one positive solution provided that $f(t,x)$ satisfies some growth conditions. Similarly, a significant work is developed by Goodrich [12] to study another fractional problem of nonlocal-type similar to (1.1) by utilizing different techniques from [11] and here. Recently, the same problem (1.1) is treated by Wang et al. [13] through cone theoretic techniques, where $f(t,x)$ can be singular at $x = 0$. Their techniques are also rather different from the ones presented here.

We have found that until now no result has been established for the existence and uniqueness of positive solutions for the problem (1.1) of a fractional differential equation when f has singularities at $t = 0$ and (or) 1. This paper thus aims to establish the existence and uniqueness of positive solutions for the problem (1.1), moreover we also obtain error estimates and the convergence rate of positive solutions with the property that there exist $M > m > 0$ such that $mt^{\alpha-1} \leq w^* \leq Mt^{\alpha-1}$.

2. Preliminaries and lemmas

For the convenience of the reader, we present here the Riemann–Liouville definitions for the fractional integral and derivative from fractional calculus which are to be used in the later sections.

Definition 2.1 (*See [14]*). Let $\alpha > 0$ with $\alpha \in \mathbb{R}$. Suppose that $x : [a, \infty) \to \mathbb{R}$. Then the αth Riemann–Liouville fractional integral is defined to be

$$D_{0+}^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)ds$$

whenever the right-hand side is defined. Similarly, with $\alpha > 0$ with $\alpha \in \mathbb{R}$, we define the αth Riemann–Liouville fractional derivative to be

$$D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_a^t (t-s)^{n-\alpha-1} x(s)ds,$$

where $n \in \mathbb{N}$ is the unique positive integer satisfying $n-1 \leq \alpha < n$ and $t > a$.

Proposition 2.1 (*See [15,14]*). *Let $\alpha > 0$, and $f(x)$ is integrable, then*

$$D_{0+}^{-\alpha} D_{0+}^{\alpha} f(x) = f(x) + c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \cdots + c_n x^{\alpha-n}$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, n is the smallest integer greater than or equal to α.

Proposition 2.2 (*See [15,14]*). *The equality*

$$D_{0+}^{\alpha} D_{0+}^{-\alpha} f(x) = f(x), \quad \alpha > 0$$

holds for $f \in L^1(a,b)$.

Lemma 2.1 (*See [16]*). *Given $y \in L^1(0,1)$, then the problem*

$$\begin{cases} D_{0+}^{\alpha} x(t) + y(t) = 0, & 0 < t < 1, \\ x(0) = x'(0) = \cdots = x^{(n-2)} = 0, & x(1) = 0, \end{cases} \tag{2.1}$$

has the unique solution

$$x(t) = \int_0^1 G(t,s) y(s)ds, \tag{2.2}$$

where $G(t,s)$ is the Green function of BCs (2.1) and is given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1}, & 0 \leq t \leq s \leq 1, \\ [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.3}$$

Lemma 2.2 (*See [16]*). *The function* $G(t,s)$ *has the following properties:*

(1)

$$G(t,s)=G(1-s,1-t),\quad \text{and}\quad G(t,s)>0,\quad \text{for } t,s\in(0,1).$$

(2)

$$t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}\le \Gamma(\alpha)G(t,s)\le(\alpha-1)s(1-s)^{\alpha-1},\quad \text{for } t,s\in[0,1],$$
$$\Gamma(\alpha)G(t,s)\le(\alpha-1)t^{\alpha-1}(1-t),\quad \text{for } t,s\in[0,1]. \tag{2.4}$$

By Lemma 2.1, the unique solution of the problem

$$\begin{cases} D^{\alpha}_{0+}x(t)=0, & 0<t<1,\\ x(0)=x'(0)=\cdots=x^{(n-2)}=0, & x(1)=1,\end{cases} \tag{2.5}$$

is $t^{\alpha-1}$. *Defining* $\mathcal{G}_A(s)=\int_0^1 G(t,s)dA(t)$, *as in* [17], *we see the Green function for the nonlocal BCs (1.1) is given by*

$$H(t,s)=\frac{t^{\alpha-1}}{1-\mathcal{C}}\mathcal{G}_A(s)+G(t,s),\qquad \mathcal{C}=\int_0^1 t^{\alpha-1}dA(t). \tag{2.6}$$

Lemma 2.3. *Let* $0\le\mathcal{C}<1$ *and* $\mathcal{G}_A(s)\ge 0$ *for* $s\in[0,1]$, *then the Green function defined by (2.6) satisfies:*

(1) $H(t,s)>0$, *for all* $t,s\in(0,1)$.
(2) *There exist two constants* c,d *such that*

$$ct^{\alpha-1}\mathcal{G}_A(s)\le H(t,s)\le dt^{\alpha-1}\le d,\quad t,s\in[0,1]. \tag{2.7}$$

Proof. (1) is obvious. For (2), by Lemma 2.2 and the definition of $H(t,s)$, we have

$$H(t,s)=\frac{t^{\alpha-1}}{1-\mathcal{C}}\mathcal{G}_A(s)+G(t,s)\ge\frac{t^{\alpha-1}}{1-\mathcal{C}}\mathcal{G}_A(s)=ct^{\alpha-1}\mathcal{G}_A(s).$$

On the other hand, notice A is a function of bounded variation and $\mathcal{G}_A(s)\ge 0$ for $s\in[0,1]$, $G(t,s)$ is continuous on $s,t\in[0,1]$ and

$$G(t,s)\le\frac{(\alpha-1)(1-t)t^{\alpha-1}}{\Gamma(\alpha)}\le\frac{t^{\alpha-1}}{\Gamma(\alpha-1)},$$

it is easy to know there exists a constant $\mathcal{F}>0$ such that $\mathcal{G}_A(s)\le\mathcal{F}$. Consequently, there exists a constant d such that

$$H(t,s)=\frac{t^{\alpha-1}}{1-\mathcal{C}}\mathcal{G}_A(s)+G(t,s)\le dt^{\alpha-1},\quad t,s\in[0,1],$$

where $d=\frac{\mathcal{F}}{1-\mathcal{C}}+\frac{1}{\Gamma(\alpha-1)}$. □

3. Main results

In this section, for convenience of presentation, we now present below assumptions to be used in the rest of the paper.

(B0) A is a function of bounded variation such that $\mathcal{G}_A(s)\ge 0$ for $s\in[0,1]$ and $0\le\mathcal{C}<1$, where $\mathcal{C}$ is defined by (2.6).
(B1) $f\in C((0,1)\times[0,+\infty),[0,+\infty))$; $f(t,x)$ is nondecreasing in x and for any $r\in(0,1)$, there exists a constant $0<\lambda<1$ such that, for any $(t,x)\in(0,1)\times[0,+\infty)$,

$$f(t,rx)\ge r^{\lambda}f(t,x). \tag{3.1}$$

Remark 3.1. If (B1) holds, then we easily prove, for any $r\ge 1$, and for any $(t,x)\in(0,1)\times[0,+\infty)$, (3.1) becomes

$$f(t,rx)\le r^{\lambda}f(t,x). \tag{3.2}$$

Remark 3.2. There are many kinds of functions satisfying condition (B1). In fact, let

$$\Omega=\{f\mid \text{The function } f(t,x) \text{ satisfies condition (B1)}\},$$

$a_i(t)$ $(i=0,1,2,\ldots,m)$ are nonnegative continuous on $(0,1)$ which can be singular at $t=0$ and (or) 1. Then it is easily verified directly that the following facts hold:

(1) $\sum_{i=1}^{m} a_i(t)x^{b_i} \in \Omega$, where $0 < b_i < 1$ $(i = 1, 2, \ldots, m)$ are constants.
(2) If $0 < \mu_i < +\infty (i = 1, 2, \ldots, m)$ and $\mu > \max_{1\le i\le m}\{\mu_i\}$, then

$$\left[a_0(t) + \sum_{i=1}^{m} a_i(t)x^{\mu_i}\right]^{\frac{1}{\mu}} \in \Omega.$$

(3) If $f(t, x) \in \Omega$, then $a_i(t)f(t, x) \in \Omega,\ i = 0, 1, 2, \ldots, m$.
(4) If $f_i(t, x) \in \Omega$ $(i = 1, 2, \ldots, m)$, then $\max_{1\le i\le m}\{f_i(t, x)\} \in \Omega, \min_{1\le i\le m}\{f_i(t, x)\} \in \Omega, \max_{1\le i\le m}\{f_i(t, x)\} + \min_{1\le i\le m}\{f_i(t, x)\} \in \Omega$.

Our discussion is in the space $E = C[0, 1]$, it is a Banach space if it is endowed with the form $\|x\| = \max_{t\in[0,1]} |x(t)|$ for any $x \in E$. Let

$$P = \{x \in C[0, 1] : x(t) \ge 0, t \in [0, 1]\}.$$

Clearly P is a normal cone in the Banach space E. Now let us define a sub-cone of P

$$D = \{x(t) \in P :\ \text{there exist two positive numbers } L_x \ge l_x \text{ such that } l_x t^{\alpha-1} \le x(t) \le L_x t^{\alpha-1},\ t \in [0, 1]\}.$$

Theorem 3.1. *Suppose* (B0) , (B1) *hold. If*

$$0 < \int_0^1 f(t, t^{\alpha-1})dt < +\infty. \tag{3.3}$$

Then the BCs (1.1) have a unique positive solution $w^*(t)$ in D. Moreover for any initial value $w_0 \in D$, the sequence of functions defined by

$$w_n = \int_0^1 H(t, s)f(s, w_{n-1}(s))ds, \quad n = 1, 2, \ldots$$

converges uniformly to the unique positive solution $w^*(t)$ on $[0, 1]$ as $n \to +\infty$, furthermore we have error estimation

$$\|w_n(t) - w^*(t)\| \le 2(1 - \kappa^{\lambda^n})\|v_0\|,$$

and with the rate of convergence

$$\|w_n - w^*\| = o(1 - \kappa^{\lambda^n}),$$

where $0 < \kappa < 1$ is a positive constant which is determined by $w_0(t)$.

Proof. Firstly, let us define an operator $T : D \to E$ by

$$(Tx)(t) = \int_0^1 H(t, s)f(s, x(s))ds, \tag{3.4}$$

then a fixed point of the operator T is a solution of the BCs (1.1), moreover T is well defined and $T : D \to D$.

In fact, for any $x \in D$, there exist two positive numbers $L_x > 1 > l_x$ such that

$$l_x t^{\alpha-1} \le x(t) \le L_x t^{\alpha-1}, \quad t \in [0, 1]. \tag{3.5}$$

Thus it follow from (2.7) and (3.2)–(3.5) that

$$\int_0^1 H(t, s)f(s, x(s))ds \le d(L_x)^{\lambda} \int_0^1 f(s, s^{\alpha-1})ds < +\infty.$$

On the other hand, by (2.7), (3.1), (3.2) and (3.5), we have

$$\int_0^1 H(t, s)f(s, x(s))ds \le d(L_x)^{\lambda} \int_0^1 f(s, s^{\alpha-1})ds t^{\alpha-1},$$

and

$$\int_0^1 H(t, s)f(s, x(s))ds \ge c(l_x)^{\lambda} \int_0^1 \mathcal{g}_A(s)f(s, s^{\alpha-1})ds t^{\alpha-1},$$

which implies that T is well defined and $T(D) \subset D$.

Now let $w_0 \in D$ be given, then there exist four positive constants $l_{w_0}, L_{w_0}, \widetilde{l}_{w_0}, \widetilde{L}_{w_0}$ such that

$$l_{w_0} t^{\alpha-1} \le w_0 \le L_{w_0} t^{\alpha-1}, \qquad \widetilde{l}_{w_0} t^{\alpha-1} \le T w_0 \le \widetilde{L}_{w_0} t^{\alpha-1},$$

consequently,

$$\frac{\widetilde{l}_{w_0}}{L_{w_0}} w_0 \leq T w_0 \leq \frac{\widetilde{L}_{w_0}}{l_{w_0}} w_0.$$

Take

$$t_0 \leq \min \left\{ \left(\frac{\widetilde{l}_{w_0}}{L_{w_0}} \right)^{\frac{1}{1-\lambda}}, \left(\frac{l_{w_0}}{\widetilde{L}_{w_0}} \right)^{\frac{1}{1-\lambda}} \right\},$$

then $t_0 \in (0, 1)$ and

$$t_0^{1-\lambda} w_0 \leq T w_0 \leq \left(\frac{1}{t_0} \right)^{1-\lambda} w_0. \tag{3.6}$$

Let

$$u_0 = t_0 w_0, \qquad v_0 = \frac{1}{t_0} w_0, \quad t_0 \in (0, 1). \tag{3.7}$$

Then $u_0 \leq v_0$. Now we define

$$u_n = Tu_{n-1}, \qquad v_n = T v_{n-1}, \quad (n = 1, 2, \ldots),$$

then notice that T is a increasing operator by (B1) and

$$T(rx) = \int_0^1 H(t, s) f(s, rx(s)) ds \geq r^\lambda \int_0^1 H(t, s) f(s, x(s)) ds = r^\lambda Tx, \quad 0 < r < 1, \tag{3.8}$$

$$T(rx) = \int_0^1 H(t, s) f(s, rx(s)) ds \leq r^\lambda \int_0^1 H(t, s) f(s, x(s)) ds = r^\lambda Tx, \quad r \geq 1. \tag{3.9}$$

One can obtain by (3.6)–(3.8)

$$Tu_0 \geq t_0^\lambda T w_0 \geq t_0 w_0 = u_0, \qquad T v_0 \leq \left(\frac{1}{t_0} \right)^\lambda T w_0 \leq \frac{1}{t_0} w_0 = v_0. \tag{3.10}$$

It follows from induction and (3.10) that

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{3.11}$$

Notice that $u_0 = t_0^2 v_0$, by induction it is easy to obtain

$$u_n \geq (t_0^2)^{\lambda^n} v_n, \quad (n = 0, 1, 2, \ldots). \tag{3.12}$$

Since P is a normal cone with normality constant 1, and $u_{n+p} - u_n \leq v_n - u_n$ for any $p \in N$, we have

$$\|u_{n+p} - u_n\| \leq \|v_n - u_n\| \leq (1 - (t_0^2)^{\lambda^n}) \|v_0\| \to 0, \quad \text{as } n \to +\infty. \tag{3.13}$$

This implies that $\{u_n\}$ is a Cauchy sequence, so u_n converges to some $w^* \in D$, from (3.13) and

$$\|v_n - w^*\| \leq \|v_n - u_n\| + \|u_n - w^*\|,$$

we also have $v_n \to w^*$. Thus $w^* \in D$ is a fixed point of T, and $w^* \in [u_0, v_0]$. For any initial value $w_0 \in D$, it follows from $u_0 \leq w_0 \leq v_0$ that $u_n \leq w_n \leq v_n$, $(n = 1, 2, \ldots)$. So

$$\|w_n - w^*\| \leq \|w_n - u_n\| + \|u_n - w^*\| \leq 2\|v_n - u_n\| \leq 2(1 - (t_0^2)^{\lambda^n}) \|v_0\|,$$

which implies the sequence of functions defined by

$$w_n = \int_0^1 H(t, s) f(s, w_{n-1}(s)) ds, \quad n = 1, 2, \ldots$$

converges uniformly to the positive solution $w^*(t)$ on [0, 1] as $n \to +\infty$. Furthermore we have error estimation

$$\|w_n - w^*\| \leq 2(1 - \kappa^{\lambda^n}) \|v_0\|,$$

and with the rate of convergence

$$\|w_n - w^*\| = o(1 - \kappa^{\lambda^n}),$$

where $0 < \kappa = t_0^2 < 1$ is a positive constant which is determined by $w_0(t)$.

Next we shall prove the uniqueness of positive solution for BCs (1.1). In fact, for any fixed point $\overline{w} \in D$ of T, since $\overline{w}, w^* \in D$ and the definition of D, let

$$t_1 = \sup\{t > 0 : \overline{w} \geq t w^*\},$$

then $0 < t_1 < +\infty$. Now we prove $t_1 \geq 1$. Otherwise, if $0 < t_1 < 1$, then

$$\overline{w} = T\overline{w} \geq T(t_1 w^*) \geq t_1^{\lambda} T w^* = t_1^{\lambda} w^*,$$

which contradicts the definition of t_1 since $t_1^{\lambda} > t_1$. Thus $t_1 \geq 1$ and $\overline{w} \geq w^*$, in the same way we also have $\overline{w} \leq w^*$, thus $\overline{w} = w^*$, i.e., w^* is a unique fixed point of T in D. Of course, it also is a unique positive solution of BCs (1.1). □

Remark 3.3. In Theorem 3.1, we not only give the condition of the existence of a unique positive solution, but also establish an iterative sequence of solution and error estimation. In particular, since $t^{\alpha-1} \in D$, and the initial value of the iterative sequence can begin from $w_0 = t^{\alpha-1}$, this is simpler and helpful for computation.

Remark 3.4. Theorem 3.1 still holds if (B1) is replaced by the following condition:

(B^*1) $f \in C((0, 1) \times [0, +\infty), [0, +\infty))$; there exist constants $0 < \lambda_1 \leq \lambda_2 < 1$ such that, for any $t \in (0, 1)$, $x \in [0, +\infty)$,

$$c^{\lambda_2} f(t, x) \leq f(t, cx) \leq c^{\lambda_1} f(t, x), \quad 0 < c < 1.$$

(B^*1) is adopted by Wei [2–5] to prove the necessary and sufficient condition for the existence of positive solutions for some nonlinear integer order differential equation with two-point or multi-point boundary conditions. It follows from (B^*1) that $f(t, x)$ is nondecreasing in x, then (B1) is weaker than (B^*1). In fact, for any $t \in (0, 1)$, $u, v \in [0, \infty)$, without loss of the generality, let $0 \leq u \leq v$. If $v = 0$, obviously $f(t, u) \leq f(t, v)$ holds. If $v \neq 0$, let $c_0 = u/v$, then $0 \leq c_0 \leq 1$. It follows from (B^*1) that

$$f(t, u) = f(t, c_0 v) \leq c_0^{\lambda_1} f(t, v) \leq f(t, v).$$

Thus $f(t, x)$ is increasing on x in $[0, \infty)$. So our main result extends some recent works of Wei [2–5]. In addition, we also establish the iterative sequence of the solution and error estimation; these are not done in [2–5], this implies our result is better than those of [2–5].

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