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# Ramanujan type identities and congruences for partition pairs

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ABSTRACT

Using elementary methods, we establish several new Ramanujan type identities and congruences for certain pairs of partition functions.



## 1. Introduction and statement of main results

Let $p(n)$ denote the number of unrestricted partitions of a non-negative integer $n$, then

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1a}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.1b}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.1c}$$

These are commonly known as the Ramanujan congruences. Ramanujan gave several proofs of (1.1). (A detailed account can be found in [6, Ch. 2].) In particular, he deduced (1.1a) and (1.1b) from the following exact identities [29]

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.2a}$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}, \tag{1.2b}$$

where we adopted the standard notation

$$(a;q)_\infty = \prod_{j=1}^{\infty}(1-aq^j) \quad \text{and} \quad (a_1,\ldots,a_m;q)_\infty = (a_1;q)_\infty \cdots (a_m;q)_\infty.$$

For brevity we shall also use $E(n)$ to denote $(q^n;q^n)_\infty$.

By imposing certain restrictions on the parts of the partitions, one can obtain variants of the partition function. For example, an overpartition of $n$ is a partition of $n$ where we may overline the first occurrence of a part. We define in the following, seven variants of the partition function. Let

- $po(n)$ denote the number of partitions of $n$ into odd parts;
- $pe(n)$ denote the number of partitions of $n$ into even parts;

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- $\overline{p}(n)$ denote the number of overpartitions of $n$;
- $\overline{po}(n)$ denote the number of overpartitions of $n$ into odd parts;
- $\overline{pe}(n)$ denote the number of overpartitions of $n$ into even parts;
- $pod(n)$ denote the number of partitions of $n$ where the odd parts are distinct;
- $ped(n)$ denote the number of partitions of $n$ where the even parts are distinct.

The eight corresponding generating functions are

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{1}{E(1)}, \tag{1.3a}$$

$$\sum_{n=0}^{\infty} po(n)q^n = \frac{1}{(q;q^2)_\infty} = \frac{E(2)}{E(1)}, \tag{1.3b}$$

$$\sum_{n=0}^{\infty} pe(n)q^n = \frac{1}{(q^2;q^2)_\infty} = \frac{1}{E(2)}, \tag{1.3c}$$

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{E(2)}{E(1)^2}, \tag{1.3d}$$

$$\sum_{n=0}^{\infty} \overline{po}(n)q^n = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} = \frac{E(2)^3}{E(1)^2E(4)}, \tag{1.3e}$$

$$\sum_{n=0}^{\infty} \overline{pe}(n)q^n = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} = \frac{E(4)}{E(2)^2}, \tag{1.3f}$$

$$\sum_{n=0}^{\infty} pod(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} = \frac{E(2)}{E(1)E(4)}, \tag{1.3g}$$

$$\sum_{n=0}^{\infty} ped(n)q^n = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{E(4)}{E(1)}. \tag{1.3h}$$

Properties of these partition functions, as well as identities and congruences satisfied by them, have been the subject of many recent studies [3,4,11,15,19–21,25,28]. For example, Hirschhorn and Sellers [19] proved that the overpartition function $\overline{p}(n)$ satisfies the following analogue of (1.2)

$$\sum_{n=0}^{\infty} \overline{p}(2n+1)q^n = 2\frac{E(2)^2E(8)^2}{E(1)^4E(4)}. \tag{1.4}$$

Apart from studying a single partition function, one may also consider a partition pair formed from partition functions $p_1(n)$ and $p_2(n)$. Let $Q_{(p_1,p_2)}(n)$ denote the number of partitions of $n$ into two colors (say, red and blue), where the parts colored red satisfy restrictions of partitions counted by $p_1(n)$, while the parts colored blue satisfy restrictions of partitions counted by $p_2(n)$. The generating function for $Q_{(p_1,p_2)}(n)$ is then the product of the generating functions of $p_1(n)$ and $p_2(n)$.

For example, partitions into two colors with no restrictions on the red parts, but where the blue parts must be even, are counted by $Q_{(p,pe)}(n)$. Chan [8] recently showed that such partitions satisfy the following remarkable identity

$$\sum_{n=0}^{\infty} Q_{(p,pe)}(3n+2)q^n = 3\frac{E(3)^3E(6)^3}{E(1)^4E(2)^4}. \tag{1.5}$$

Pairs of partition functions have been studied recently in [7,9,12,13,17,22–24,35].

We shall use elementary methods to prove several new identities and congruences that are analogous to (1.2) and (1.1). In Section 3, we prove the following two theorems

**Theorem 1.1.**

$$\sum_{n=0}^{\infty} Q_{(\overline{po},ped)}(3n+1)q^n = 3\frac{E(2)^4E(3)^5}{E(1)^8E(6)}, \tag{1.6a}$$

$$\sum_{n=0}^{\infty} Q_{(\overline{po},ped)}(3n+2)q^n = 6\frac{E(2)^3E(3)^2E(6)^2}{E(1)^7}. \tag{1.6b}$$

**Theorem 1.2.**

$$\sum_{n=0}^{\infty} Q_{(\overline{p},pod)}(3n+1)q^n = 3\frac{E(2)^6E(3)^6}{E(1)^{11}E(4)^3}. \tag{1.7}$$

In Section 4, we shall establish the following congruences

**Theorem 1.3.**

$$\sum_{n=0}^{\infty} Q_{(p,\overline{po})}(5n+4)q^n \equiv 0 \pmod 5, \tag{1.8a}$$

$$\sum_{n=0}^{\infty} Q_{(p,ped)}(5n+2)q^n \equiv 0 \pmod 5, \tag{1.8b}$$

$$\sum_{n=0}^{\infty} Q_{(\overline{po},pod)}(5n+2)q^n \equiv 0 \pmod 5, \tag{1.8c}$$

$$\sum_{n=0}^{\infty} Q_{(\overline{pe},ped)}(5n+3)q^n \equiv 0 \pmod 5 \tag{1.8d}$$

*and*

$$\sum_{n=0}^{\infty} Q_{(po,\overline{p})}(7n+2)q^n \equiv 0 \pmod 7. \tag{1.8e}$$

We end the introduction with a brief discussion on modular forms. All of the partition functions that we consider have generating functions that are (up to a factor of $q$) either modular forms, reciprocals of modular forms or are congruent to some modular forms. If we let $f(n)$ denote their Fourier coefficients, then it has been shown [1,2,16,27,30,33] that there exists infinitely many congruences of the form

$$f(An+B) \equiv 0 \pmod M,$$

but for certain types of partition functions, there exists only finitely many *Ramanujan type congruences*, which are of the form

$$f(\ell n+b) \equiv 0 \pmod \ell,$$

where $\ell$ is a prime. For instance, (1.1) are the only Ramanujan type congruences for $p(n)$ and the only Ramanujan type congruence for $\overline{p}(n)$ is for the prime 2.

For this reason, we concentrate on Ramanujan type congruences. A computer search was carried out on all pairs of partition functions formed from the list in (1.3) for congruences modulo primes up to 101. The only ones found are for primes 2, 3, 5 and 7. Our aim is to explain all of these Ramanujan type congruences and identities in an elementary manner. In the next section we will prove some preliminary results. Ramanujan type congruences modulo 3 are discussed in Section 3 while Section 4 is devoted to the remaining congruences.

## 2. Preliminaries

In this section, we will state several key results needed in our proofs. We begin with three crucial 3-dissection results for Ramanujan's theta function $\psi$, and the generating functions for $\overline{p}(n)$ and $pod(n)$

**Lemma 2.1.**

$$\frac{E(2)^2}{E(1)} = \frac{E(6)E(9)^2}{E(3)E(18)} + q\frac{E(18)^2}{E(9)}, \tag{2.1a}$$

$$\frac{E(2)}{E(1)^2} = \frac{E(6)^4E(9)^6}{E(3)^8E(18)^3} + 2q\frac{E(6)^3E(9)^3}{E(3)^7} + 4q^2\frac{E(6)^2E(18)^3}{E(3)^6}, \tag{2.1b}$$

$$\frac{E(2)}{E(1)E(4)} = \frac{E(18)^9}{E(3)^2E(9)^3E(12)^2E(36)^3} + q\frac{E(6)^2E(18)^3}{E(3)^3E(12)^3} + q^2\frac{E(6)^4E(9)^3E(36)^3}{E(3)^4E(12)^4E(18)^3}. \tag{2.1c}$$

**Proof.** Eq. (2.1a) is Corollary (ii) on page 49 of [5]. Eq. (2.1b) is the 3-dissection of the overpartition generating function and an elementary proof of which can be found in [19, Th. 1]. Finally to obtain (2.1c), the 3-dissection of the generating function for partitions with odd parts distinct, we replace $q$ by $-q$ in [21, Lem. 2] and use Jacobi's triple product identity [6, Th. 1.3.3] to write the various sums as products. □

Our second lemma comes from [31, Eq. 5.1]

**Lemma 2.2.**

$$\frac{E(2)^4E(3)^8}{E(1)^8E(6)^4} = 1 + 8q\frac{E(2)E(6)^5}{E(1)^5E(3)}. \tag{2.2}$$

The next result is named after Winquist [34] who used it to obtain an elementary proof of (1.1c)

**Theorem 2.3** (*Winquist*).

$$F_1(x)F_2(y) - F_1(y)F_2(x) = -\frac{2}{x}\left(xq, \frac{q}{x}, yq, \frac{q}{y}, xy, \frac{q^2}{xy}, \frac{x}{y}, \frac{yq^2}{x}, q^2, q^2; q^2\right)_\infty,$$

*where*

$$F_1(x) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2}(x^{3n} + x^{-3n}),$$

$$F_2(x) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+2n}(x^{3n+1} + x^{-3n-1}).$$

We remark that the above result is not exactly the identity found in [34] but one of the four variants of it [32, Eq. 3.1]. In his landmark paper [26], Macdonald introduced and classified affine root systems and associated each root system to an identity that equates infinite series to products. Winquist's identity corresponds to the $B_2$ root system as can be seen from the factors $x, y, xy, x/y$ in the infinite product. An easy consequence of Winquist's identity is

**Corollary 2.4.**

$$\frac{E(2)^{14}}{E(1)^4E(4)^4} = \frac{E(4)^{10}}{E(8)^4} + 4q\frac{E(2)^4E(8)^4}{E(4)^2}. \tag{2.3}$$

**Proof.** Apply the differential operator $x\frac{\mathrm{d}}{\mathrm{d}x}$ twice to Theorem 2.3 and substitute the value of $-1$ for both $x$ and $y$ to get

$$\sum_{m,n=-\infty}^{\infty} \left((3m+1)^2 - (3n)^2\right) q^{3m^2+2m+3n^2} = \frac{E(2)^{14}}{E(1)^4E(4)^4}.$$

Now replace $(m, n)$ by $\left(\frac{i+j}{2}, \frac{i-j}{2}\right)$ to obtain

$$\sum_{i\equiv j \pmod 2} (3i+1)(3j+1)q^{\frac{3i^2+2i}{2}+\frac{3j^2+2j}{2}} = \left(\sum_{i=-\infty}^{\infty}(6i+1)q^{\frac{4(3i^2+i)}{2}}\right)^2 + \left(\sum_{i=-\infty}^{\infty}(6i+2)q^{6i^2+4i+\frac{1}{2}}\right)^2$$

and use Entries 8(ix) and 8(x) in page 114 of [5] to convert the last line into a sum of two products. Incidentally these two entries correspond to the $BC_1$ identities in the Macdonald system. □

**Corollary 2.5.**

$$1 - q\frac{E(1)E(12)^3}{E(3)^3E(4)} = \frac{E(1)E(4)^3E(6)^2}{E(2)^2E(3)^3E(12)}, \tag{2.4a}$$

$$1 + 2q\frac{E(1)E(12)^3}{E(3)^3E(4)} = \frac{E(2)^7E(12)}{E(1)^2E(3)^2E(4)^3E(6)}, \tag{2.4b}$$

$$1 - 2q\frac{E(1)E(12)^3}{E(3)^3E(4)} = \frac{E(1)^2E(4)E(6)^9}{E(2)^3E(3)^6E(12)^3}. \tag{2.4c}$$

**Proof.** Specialize $(x, y)$ in Theorem 2.3 to $(e^{2\pi i/3}, -q)$ and use Jacobi's triple product to get

$$-4q^{-1}\frac{E(3)^2E(4)E(6)}{E(2)E(12)} + 4\frac{E(1)E(6)E(12)^2}{E(2)E(3)} = -4q^{-1}\frac{E(1)E(4)^4E(6)^3}{E(2)^3E(3)E(12)^2}.$$

Simplifying the result yields (2.4a). Similarly, set $(x, y)$ to $(1, -q)$ for (2.4b) and set $(x, y)$ to $(1, -1)$ for (2.4c). □

## 3. Identities and congruences modulo 3

In this section we prove Theorems 1.1 and 1.2 and survey other results in the literature.

**Proof of Theorem 1.1.** It is easy to check that the generating function for $Q_{(\overline{po},ped)}(n)$ is

$$\frac{E(2)^3}{E(1)^3}.$$

Hence the theorem is a simple consequence of the following identity

$$\frac{E(2)^3}{E(1)^3} = \frac{E(6)}{E(3)} + 3q\frac{E(6)^4E(9)^5}{E(3)^8E(18)} + 6q^2\frac{E(6)^3E(9)^2E(18)^2}{E(3)^7} + 12q^3\frac{E(6)^2E(18)^5}{E(3)^6E(9)}. \tag{3.1}$$

Multiply (2.1a) by (2.1b) to obtain

$$\frac{E(2)^3}{E(1)^3} = \frac{E(6)^5E(9)^8}{E(3)^9E(18)^4} + 4q^3\frac{E(6)^2E(18)^5}{E(3)^6E(9)} + 3q\frac{E(6)^4E(9)^5}{E(3)^8E(18)} + 6q^2\frac{E(6)^3E(9)^2E(18)^2}{E(3)^7}.$$

It suffices to prove

$$\frac{E(6)^5E(9)^8}{E(3)^9E(18)^4} = \frac{E(6)}{E(3)} + 8q^3\frac{E(6)^2E(18)^5}{E(3)^6E(9)}.$$

By replacing $q^3$ by $q$ and after some rearrangement, we can see that the above result is equivalent to (2.2). □

As a corollary of (3.1), we record the following attractive identity, although it does not lead to a Ramanujan type congruence

$$\sum_{n=0}^{\infty} Q_{(\overline{po},ped)}(3n)q^n = \frac{E(2)}{E(1)} + 12q\frac{E(2)^2E(6)^5}{E(1)^6E(3)}. \tag{3.2}$$

**Proof of Theorem 1.2.** The generating function for $Q_{(\overline{p},pod)}(n)$ is

$$\frac{E(2)^2}{E(1)^3E(4)}.$$

Multiply (2.1b) by (2.1c) and collect the terms with exponent congruent to 1 modulo 3. Now replace $q^{3n+1}$ with $q^n$ to get

$$\frac{E(2)^6E(3)^6}{E(1)^{11}E(4)^3} + 2\frac{E(2)^3E(6)^9}{E(1)^9E(4)^2E(12)^3} + 4q\frac{E(2)^6E(3)^3E(12)^3}{E(1)^{10}E(4)^4}.$$

It remains to prove

$$2\frac{E(2)^6E(3)^6}{E(1)^{11}E(4)^3} = 2\frac{E(2)^3E(6)^9}{E(1)^9E(4)^2E(12)^3} + 4q\frac{E(2)^6E(3)^3E(12)^3}{E(1)^{10}E(4)^4}.$$

Some simplification shows that the above is equivalent to (2.4c). □

We now consider other modulo 3 Ramanujan type identities that have recently been discovered. In [22], Kim proved the following identity using the theory of modular forms

$$\sum_{n=0}^{\infty} Q_{(\overline{p},\overline{pe})}(3n+2)q^n = 6\frac{E(3)^6E(4)^3}{E(1)^8E(2)^3}. \tag{3.3}$$

One can obtain an elementary proof of Kim's result with (2.1b) and (2.4c).

By squaring (2.1c) and (2.1b) respectively, Chen and Lin [12,13] showed that

$$\sum_{n=0}^{\infty} Q_{(pod,pod)}(3n+2)q^n = 3\frac{E(2)^4E(6)^6}{E(1)^6E(4)^6}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} Q_{(\overline{p},\overline{p})}(3n+2)q^n = 12\frac{E(2)^6E(3)^6}{E(1)^{14}}. \tag{3.5}$$

A computer search yielded no other modulo 3 Ramanujan type identities for pairs of partition functions arising from the list (1.3). However, several congruences of the type

$$f(3^\alpha n + b_\alpha) \equiv 0 \pmod 3,$$

for infinitely many values of $\alpha$, $\alpha \neq 1$, have been found in [4,12,13,18,20,21,25]. Chan [8] has also found that

$$Q_{(p,pe)}(3^j n + c_j) \equiv 0 \pmod{3^{2\lfloor j/2\rfloor+1}},$$

where $c_j = 1/8 \pmod{3^j}$ for all positive integers $j$.

We end this section with some new exact identities for $Q_{(ped,ped)}(n)$, that do not lead to Ramanujan type congruences

**Theorem 3.1.**

$$\sum_{n=0}^{\infty} Q_{(ped,ped)}(3n)q^n = \frac{E(2)^{10}E(3)^4}{E(1)^{10}E(4)^2E(6)^2}, \tag{3.6a}$$

$$\sum_{n=0}^{\infty} Q_{(ped,ped)}(3n+1)q^n = 2\frac{E(2)^9E(3)E(6)}{E(1)^9E(4)^2}, \tag{3.6b}$$

$$\sum_{n=0}^{\infty} Q_{(ped,ped)}(3n+2)q^n = 2\frac{E(2)^2E(3)^3E(4)^3E(12)}{E(1)^7E(6)^2} + 3\frac{E(2)E(4)E(6)^5}{E(1)^6E(12)}. \tag{3.6c}$$

**Proof.** Multiply (2.1b) by (2.1a) with $q$ replaced by $q^2$. Collect terms with the same exponents modulo 3 and use Corollary 2.5 to simplify the resulting expressions. □

## 4. Identities and congruences modulo other primes

We first discuss Ramanujan type congruences modulo 2. All of the results that we have found are due to one of the following 2-dissections

$$\frac{E(2)}{E(1)^2} = \frac{E(8)^5}{E(2)^4E(16)^2} + 2q\frac{E(4)^2E(16)^2}{E(2)^4E(8)}, \tag{4.1}$$

$$\frac{E(2)^2}{E(1)^4} = \frac{E(4)^{14}}{E(2)^{12}E(8)^4} + 4q\frac{E(4)^2E(8)^4}{E(2)^8}. \tag{4.2}$$

An elementary proof of (4.1) can be found in [19, Th. 1]. If we square the identity, we obtain

$$\frac{E(2)^2}{E(1)^4} = 4q\frac{E(4)^2E(8)^4}{E(2)^8} + \frac{E(8)^{10}}{E(2)^8E(16)^4} + 4q^2\frac{E(4)^4E(16)^4}{E(2)^8E(8)^2}.$$

Thus (4.2) follows from establishing

$$\frac{E(4)^{14}}{E(2)^{12}E(8)^4} = \frac{E(8)^{10}}{E(2)^8E(16)^4} + 4q^2\frac{E(4)^4E(16)^4}{E(2)^8E(8)^2},$$

which is equivalent to (2.3).

Using (4.2) and some straightforward manipulation, it follows that we have the following Ramanujan type identity

$$\sum_{n=0}^{\infty} Q_{(\overline{p},\overline{po})}(2n+1)q^n = 4\frac{E(2)E(4)^4}{E(1)^6}. \tag{4.3}$$

Similar results can be obtained for another 12 partition pairs.

We now turn to congruences modulo 5 and 7. We first note that for any prime $p$, we have

$$E(n)^p \equiv E(pn) \pmod{p}.$$

Using this fact, one explanation for all five congruences stated in Theorem 1.3 is that the respective generating functions are congruent to some modular forms modulo 5 or 7. All we require are known series representations for these modular forms which can be found in [14].

**Proof of Theorem 1.3.** We shall only prove (1.8a)

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{(p,\overline{po})}(n)q^n &= \frac{E(2)^3}{E(1)^3E(4)}\\ &\equiv \frac{E(1)^2E(4)^4}{E(2)^2} \times \frac{E(10)}{E(5)E(20)} \pmod{5}\\ &= \frac{E(1)^2E(4)^2}{E(2)} \times \frac{E(4)^2}{E(2)} \times \frac{E(10)}{E(5)E(20)}.\end{aligned}$$

Using Jacobi's triple product and Entry 8(x) of [5, Pg. 114], the first two infinite products have the following series representation

$$\left(\sum_{\alpha\equiv 1 \pmod 3} \alpha q^{\frac{\alpha^2-1}{3}}\right)\left(\sum_{\beta\equiv 1 \pmod 4} q^{\frac{\beta^2-1}{4}}\right) = \sum_{\substack{\alpha\equiv 1 \pmod 3\\ \beta\equiv 1 \pmod 4}} \alpha q^{\frac{4\alpha^2+3\beta^2-7}{12}}.$$

If the exponent of $q$ is congruent to 4 modulo 5, we have

$$(2\alpha)^2 + 3\beta^2 \equiv 12(4) + 7 \equiv 0 \pmod 5.$$

Since $-3$ is a quadratic nonresidue modulo 5, 5 must divide both $\alpha$ and $\beta$. Thus we have shown that

$$\sum_{n=0}^{\infty} Q_{(p,\overline{po})}(5n+4)q^n \equiv 0 \pmod 5.$$

The proofs for congruences (1.8b) to (1.8e) are similar. □

A computer calculation for all possible pairs of partitions arising from the list (1.3) has shown that there are no other Ramanujan type congruences for primes between 5 and 101, besides those given in Theorem 1.3, trivial congruences arising from (1.1) and the well known

$$\sum_{n=0}^{\infty} Q_{(p,p)}(5n+b)q^n \equiv 0 \pmod 5 \text{ for } b = 2, 3, 4.$$

Analogous to the modulo 3 case, Chen and Lin [12,13] have proved congruences of the type

$$f(5^\alpha n + b_\alpha) \equiv 0 \pmod 5,$$

for infinitely many values of $\alpha$, $\alpha \neq 1$, while it was shown in [10] that

$$Q_{(p,pe)}(5^j n + d_j) \equiv 0 \pmod{5^{\left\lfloor \frac{j}{2} \right\rfloor}},$$

where $d_j = 1/8 \pmod{5^j}$ for all positive integers $j$.

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