On Electric and Magnetic Problems for Vector Fields in Anisotropic Nonhomogeneous Media

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0. Introduction

In this paper we continue our study of the equations

\begin{align}
\text{curl } v u &= j & \text{in } \Omega, \\
\text{div } u &= \rho & \text{in } \Omega, \\
n \cdot u|_\Gamma &= \lambda,
\end{align}

\begin{align}
\text{and}
\text{curl } u &= j & \text{in } \Omega, \\
\text{div } \varepsilon u &= \rho & \text{in } \Omega, \\
n \wedge u|_\Gamma &= \sigma,
\end{align}

\(\Gamma = \partial \Omega\), initiated by Saranen [13]. In the above, \(n\) is the outward-drawn unit normal to the boundary and \(\wedge\) denotes the exterior product. According to the simple models for static magnetic fields (resp. electric fields) which are governed by (0.1) (resp. (0.2)), we call (0.1) the magnetic type problem and (0.2) the electric type problem. Considering bounded smooth domains \(\Omega \subset \mathbb{R}^3\), we discussed in [13], by means of an appropriate Hilbert space method, the solvability and the representation of the solutions for both problems (0.1) and (0.2). Such a new approach was necessary to cover the general nonhomogeneous cases where \(v\) and \(\varepsilon\) are matrix-valued functions.

Here our aim is twofold. First, we will now include exterior domains in our consideration. This will be achieved by using certain weighted spaces which slightly restrict the behavior of the fields at infinity. To find the scalar potentials, we have to solve the Dirichlet and Neumann problems in exterior domains. Consequently, in order to define the vector potentials, we solve appropriate auxiliary second-order boundary problems for vector fields.

The other aspect of this paper is to point out that the dimension of the space of solutions for (0.1) or (0.2) with homogeneous right side is
independent of the coefficient matrix \( v \) or \( \varepsilon \) describing the anisotropic phenomena. The importance of this result lies in the fact that for the identity matrices \( v = 1 \), \( \varepsilon = 1 \), the dimensions of these null spaces are known to be topological invariants which can be given by the Betti numbers for \( \Omega \) [6, 12].

Regarding the exterior domains, we also prove a substitute for the Friedrichs inequality giving a bound for all first derivatives of the fields satisfying a homogeneous boundary condition. Such an inequality had so far only been known for smooth bounded domains and for bounded convex domains in the case of the electric boundary condition (cf. Saranen [14]). To obtain our result, some assumptions on the behavior of the coefficient matrices \( v \) and \( \varepsilon \) at infinity are needed.

For the classical theory of problems (0.1) and (0.2) with \( \varepsilon = v = 1 \) using the method of integral equations, we refer to Kress [5, 6] as well as to Martensen [9]. For a constructive approach, see Hermann and Kersten [4] and for a finite element application, see Neittaanmäki and Saranen [10, 11].

1. Bounded Domains

In this section, we assume that the domain \( \Omega \subset \mathbb{R}^3 \) is smooth and bounded. First we introduce some notation which will be useful when problems (0.1) and (0.2) are discussed in the frame of Hilbert space methods. In the following, \( v \) and \( \varepsilon \) are symmetric positive definite measurable bounded matrices with real-valued entries \( v_{ij}(x) \) (resp. \( \varepsilon_{ij}(x) \)) in the domain \( \Omega \). Furthermore, the inequalities

\[
\begin{align*}
\nu_0 |\xi|^2 &\leq \langle \xi | v(x) \xi \rangle \leq \nu_1 |\xi|^2, \\
\varepsilon_0 |\xi|^2 &\leq \langle \xi | \varepsilon(x) \xi \rangle \leq \varepsilon_1 |\xi|^2
\end{align*}
\]

hold for certain numbers \( \nu_0, \nu_1 > 0 \) and \( \varepsilon_0, \varepsilon_1 > 0 \) for all \((\xi, x) \in \mathbb{C}^3 \times \Omega \). Here \( \langle \xi | \eta \rangle = \xi_l \bar{\eta}_l \) (summation convention) denotes the inner product in \( \mathbb{C}^3 \) with associated norm \( | \cdot | \).

The spaces \( L^2(\Omega) \) and \( L^2(\Omega)^3 \) (square integrable fields) have the usual inner product \( \langle \cdot | \cdot \rangle_0 \) with norm \( \| \cdot \|_0 \). The norm in the Sobolev spaces \( H^k(\Omega) \) and \( H^k(\Omega)^3 \) is denoted by \( \| \cdot \|_k \). For the pair of boundary spaces \( H^{-1/2}(\Gamma), H^{1/2}(\Gamma) \), we denote by \( \langle \cdot | \cdot \rangle_0 \) the duality \( \langle u | \varphi \rangle_0 = u(\varphi) \) extending the \( L^2(\Gamma) \) inner product. The same notation is used for the pair \( H^{-1/2}(\Gamma)^3, H^{1/2}(\Gamma)^3 \).

We abbreviate

\[
\begin{align*}
H(\text{div } v) &= \{ u \in L^2(\Omega)^3 | \text{div } vu \in L^2(\Omega) \}, \\
H(\text{curl } v) &= \{ u \in L^2(\Omega)^3 | \text{curl } vu \in L^2(\Omega)^3 \},
\end{align*}
\]

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and write \( H(\text{div}) = H(\text{div} \varepsilon) \) for \( \varepsilon = 1 \) as well as \( H(\text{curl}) = H(\text{curl} \nu) \) for \( \nu = 1 \). The space \( H(\text{div}) \) is a Hilbert space with respect to the inner product belonging to the norm \( \| u \|_d = (\| u \|_0^2 + \| \text{div} u \|_0^2)^{1/2} \). Similarly \( H(\text{curl}) \) is a Hilbert space with norm \( \| u \|_c = (\| u \|_0^2 + \| \text{curl} u \|_0^2)^{1/2} \). Furthermore, we use the notation

\[
H(\text{div} \varepsilon | 0) = \{ u \in H(\text{div} \varepsilon) | \text{div} \varepsilon u = 0 \}, \tag{1.4}
\]

\[
H(\text{curl} \nu | 0) = \{ u \in H(\text{curl} \nu) | \text{curl} \nu u = 0 \}. \tag{1.5}
\]

The characteristic feature of the space \( H(\text{div}) \) is that associated with it there exists a trace operator \( u \mapsto n \cdot u : H(\text{div}) \mapsto H^{-1/2}(\Gamma) \) defined as the unique continuous extension of the normal component for smooth fields [2, p. 356]. Similarly, in the space \( H(\text{curl}) \) there exists a trace operator \( u \mapsto n \wedge u : H(\text{curl}) \mapsto H^{-1/2}(\Gamma)^3 \) defined as the unique continuous extension of the same mapping for smooth fields [2, p. 341].

The space \( H(\text{curl} \nu) \cap H(\text{div}) \), respectively, \( H(\text{curl}) \cap H(\text{div} \varepsilon) \), is endowed with the Hilbert norm

\[
\| u \|_{d, \text{curl} \nu} = (\| u \|_0^2 + \| \text{curl} \nu u \|_0^2 + \| \text{div} u \|_0^2)^{1/2}, \tag{1.6}
\]

respectively,

\[
\| u \|_{c, \text{div} \varepsilon} = (\| u \|_0^2 + \| \text{curl} u \|_0^2 + \| \text{div} \varepsilon u \|_0^2)^{1/2}. \tag{1.7}
\]

Then the mapping \( M(\nu) : H(\text{curl} \nu) \cap H(\text{div}) \mapsto L^2(\Omega)^3 \times L^2(\Omega) \times H^{-1/2}(\Gamma) \), respectively, the mapping \( E(\varepsilon) : H(\text{curl}) \cap H(\text{div} \varepsilon) \mapsto L^2(\Omega)^3 \times L^2(\Omega) \times (H^{-1/2}(\Gamma)^3) \), defined by

\[
M(\nu) u = (\text{curl} \nu u, \text{div} u, n \cdot u), \tag{1.8}
\]

respectively,

\[
E(\varepsilon) u = (\text{curl} u, \text{div} \varepsilon u, n \wedge u), \tag{1.9}
\]

is continuous, when the range space is endowed with the natural product topology.

Now magnetic problem (0.1) for bounded domains is understood in the following sense:

\( (\text{Mb}) \) Given the data \( (j, \rho, \lambda) \in L^2(\Omega)^3 \times L^2(\Omega) \times H^{-1/2}(\Gamma) \), find the field \( u \in H(\text{curl} \nu) \cap H(\text{div}) \) such that

\[
M(\nu) u = (j, \rho, \lambda). \tag{1.10}
\]

Similarly, electric problem (0.2) takes the form:
(Eb) Given the data \((j, p, \sigma) \in L^2(\Omega) \times L^2(\Omega) \times (H^{-1/2}(\Gamma))^3\), find the field \(u \in H(\text{curl}) \cap H(\text{div} \, \varepsilon)\) such that

\[
E(\varepsilon) u = (j, p, \sigma). \tag{1.11}
\]

Let us describe the solvability of these problems. In order to find and to represent the solutions for (Mb), we need as an auxiliary problem the Neumann boundary value problem (Nb)

\[
\text{div}(\nu^{-1} \nabla p) = \rho, \quad n \cdot (\nu^{-1} \nabla p) = \lambda, \quad (p|\Gamma)_0 = 0. \tag{1.12}
\]

This problem is defined in the usual weak sense: find \(p \in H^1(\Omega)\) such that \((p|\Gamma)_0 = 0\) and that

\[
(\nu^{-1} \nabla p|\nabla p)_0 = - (p|\varphi)_0 + \langle \lambda \rangle_0 \varphi_0, \quad \varphi \in H^1(\Omega). \tag{1.13}
\]

Here \(\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)\) is the familiar trace operator extending the mapping \(\varphi \to \varphi|_\Gamma\), [8]. Equation (1.13) is uniquely solvable if the equation \((\rho|\Gamma)_0 - \langle \lambda \rangle_0 = 0\) holds and the solution \(p\) plays the role of a scalar potential in the representation of the solutions of the problem (Mb).

Before discussing the determination of the vector potential, we introduce some more notation. We write

\[
H_0(\text{div}) = \{ u \in H(\text{div}) | n \cdot u = 0 \}, \tag{1.14a}
\]

\[
H_0(\text{curl}) = \{ u \in H(\text{curl}) | n \wedge u = 0 \}. \tag{1.14b}
\]

These spaces have also other characterizations [13, Lemma 1.3])

\[
H_0(\text{div}) = \overline{\mathcal{D}(\Omega)}^\text{\#d}, \tag{1.15a}
\]

\[
H_0(\text{curl}) = \overline{\mathcal{D}(\Omega)}^\text{\#c}, \tag{1.15b}
\]

as well as

\[
H_0(\text{div}) = \{ u \in H(\text{div}) | (u|\nabla \varphi)_0 = - (\text{div} \, u|\varphi)_0 , \varphi \in H^1(\Omega) \}, \tag{1.16a}
\]

\[
H_0(\text{curl}) = \{ u \in H(\text{curl}) | (u|\text{curl} \, \varphi)_0 = (\text{curl} \, u|\varphi)_0 , \varphi \in H(\text{curl}) \}. \tag{1.16b}
\]

Finally, let \(\mathcal{H}(m|\nu)\), respectively \(\mathcal{H}(e|\varepsilon)\), stand for the space of \(\nu\)-harmonic fields of the magnetic type, respectively, \(\varepsilon\)-harmonic fields of the electric type, that is

\[
\mathcal{H}(m|\nu) = \{ u \in H(\text{curl} \, \nu) \cap H_0(\text{div}) | \text{curl} \, \nu u = 0, \text{div} \, u = 0 \}, \tag{1.17a}
\]

\[
\mathcal{H}(e|\varepsilon) = \{ u \in H_0(\text{curl}) \cap H(\text{div} \, \varepsilon) | \text{curl} \, u = 0, \text{div} \, \varepsilon u = 0 \}. \tag{1.17b}
\]
The vector potential $q$ is required to be solenoidal, $\text{div} \ q = 0$, and satisfy the equation
\[ \text{curl}(\nabla \text{curl} \ q) = j, \quad n \wedge q|_\Gamma = 0, \quad (1.18) \]
with an additional condition which is needed in order to fix a unique solution. With this condition, our weak formulation for (1.18) reads find $q \in H_0(\text{curl}) \cap H(\text{div} \mid 0)$ such that
\[ (\nabla \text{curl} \ q \mid \nabla \text{curl} \varphi)_0 = (j \mid \varphi)_0, \quad \varphi \in H_0(\text{curl}) \cap H(\text{div} \mid 0) \quad (1.19) \]
together with the orthogonality condition ($L^2(\Omega)^3$-orthogonality)
\[ q \in \mathcal{H}(e \mid 1)^\perp. \quad (1.20) \]

If the right-hand side $j$ is also solenoidal, then one concludes by applying the orthogonal decomposition
\[ L^2(\Omega)^3 = H(\text{div} \mid 0) \oplus \nabla H_0(\Omega), \quad (1.21) \]
that the problem (1.19), (1.20) is equivalent to the problem: find $q \in H_0(\text{curl}) \cap H(\text{div}) \cap (\mathcal{H}(e \mid 1)^\perp)$ such that
\[ (\nabla \text{curl} \ q \mid \nabla \text{curl} \varphi)_0 + (\text{div} \ q \mid \text{div} \varphi)_0 = (j \mid \varphi)_0, \quad \varphi \in H_0(\text{curl}) \cap H(\text{div}). \quad (1.22) \]

This last formulation was employed in [13, Lemma 2.3] with the result that (1.22), hence (1.19) with (1.20), is uniquely solvable.

The following theorem describes the solvability of the magnetic problem (Mb):

**THEOREM 1.1.** For the mapping $M(v): H(\text{curl} \ v) \cap H(\text{div}) \mapsto L^2(\Omega)^3 \times L^2(\Omega) \times H^{-1/2}(\Gamma) =: Y$ defined by (1.8), the following assertions hold:

(a) The range $R(M(v))$ of $M(v)$ is given by
\[ R(M(v)) = \{(j, \rho, \lambda) \in Y \mid j \in \mathcal{H}(e \mid 1)^\perp \cap H(\text{div} \mid 0), \quad (\rho \mid 1)_0 - \langle \lambda \mid 1 \rangle_0 = 0 \}. \quad (1.23) \]

(b) The dimension of the null space $N(M(v)) = \mathcal{H}(m \mid v)$ is finite and independent of $v$.

(c) The field $u$ is a solution of the problem (Mb) if and only if it has the representation
\[ u = u_0 + v^{-1} \nabla p + \text{curl} \ q, \quad (1.24) \]
where $u_0 \in \mathcal{H}(m \mid v)$, the scalar potential $p \in H^1(\Omega)$, $(p \mid 1)_0 = 0$ is defined by
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(1.13), and the vector potential \( q \in H_0(\text{curl}) \cap H(\text{div} \mid 0) \) satisfies (1.19) with (1.20).

Proof. All the assertions except for the equality of the dimension of the space \( \mathcal{H}(m \mid v) \) for all \( v \) were proved in [13, Theorem 2.4]. Note, however, that the general assumption in [13] that \( \epsilon \) and \( v \) are smooth fields in \( \Omega \) can be replaced by our assumptions here. For the following essentially shorter argument than our original proof of the rest we thank K. J. Witsch.

Recalling the definition of the space \( H(\text{curl} \mid v \mid 0) \), we obtain the orthogonal decomposition

\[
L^2(\Omega)^3 = H(\text{curl} \mid v \mid 0) \oplus \text{curl} H_0(\text{curl}),
\]

where \( \oplus^r \) denotes the orthogonality with respect to the inner product \( (u \mid v)_\Theta := (u \mid v)_0 \), and where \( \text{curl} H_0(\text{curl}) = \{ \text{curl} \phi \mid \phi \in H_0(\text{curl}) \} \). On the other hand, one can verify the inclusion

\[
\text{curl} H_0(\text{curl}) \subset H_0(\text{div}) \cap H(\text{div} \mid 0).
\]

Relations (1.25) and (1.26) yield the decomposition

\[
H_0(\text{div}) \cap H(\text{div} \mid 0) = \mathcal{H}(m \mid v) \oplus \text{curl} H_0(\text{curl}).
\]

Since the space \( \mathcal{H}(m \mid v) \) is finite dimensional, the assertion follows from the fact that the codimension of the space \( \text{curl} H_0(\text{curl}) \) in \( H_0(\text{div}) \cap H(\text{div} \mid 0) \) is unique.

For the electric problem (Eb), the scalar potential \( p \) can be uniquely defined as a solution of the Dirichlet problem (Db)

\[
\text{div}(\epsilon \nabla p) = \rho, \quad p \in H^1_0(\Omega).
\]

It turns out that the conditions for the solvability of (Eb) read

\[
j \in H(\text{div} \mid 0),
\]

There exists \( v \in H(\text{curl}) \) such that \( n \cdot (j - \text{curl} v) = 0 \) and that \( \sigma = n \wedge v \),

\[
(j \mid h)_0 - \langle \sigma \mid h \rangle_0 = 0, \quad h \in \mathcal{H}(m \mid 1).
\]

Condition (1.29b) appears in [13] in an inadequate form for the solvability corrected in an errata to [13]. Let us briefly argue that (1.29) is sufficient.

If we can find the vector potential \( q \in H(\text{curl}) \cap H_0(\text{div}) \) such that

\[
\text{curl}(\epsilon^{-1} \text{curl} q) = j, \quad n \wedge (\epsilon^{-1} \text{curl} q) = \sigma,
\]

then \( w = \nabla p + \epsilon^{-1} \text{curl} q \) is a particular solution of (Eb). We may
additionally require that the field $q$ is solenoidal and that the orthogonality condition
\[ q \in H(m|1) \]
holds.

Denote $V := H(\text{curl}) \cap H_0(\text{div})$. A suitable weak formulation for (1.30) with (1.31) reads: find $q \in V$, $\text{div} q = 0$, $q \in H(m|1)$ such that
\[ (e^{-1} \text{curl } q | \text{curl } \varphi)_0 = (j|\varphi)_0 - (n \wedge \sigma | n \wedge \varphi)_0 \]
for all $\varphi \in V$, $\text{div} \varphi = 0$, $\varphi \in H(m|1)$. This problem is uniquely solvable by [13, Lemma 2.2]. By applying the decomposition
\[ L^2(\Omega)^3 = \overline{\nabla H^1(\Omega)} \oplus H(m|1) \oplus \text{curl}(H_0(\text{curl})) \]
to $\varphi \in V$ we conclude that by (1.29) Eq. (1.32) is valid for all $\varphi \in V$.

This implies the first relation in (1.30) and the second relation (1.31) follows then from (1.32) by testing with all field $\varphi \in V$, cf. [13] the proof of Lemma 3.1.

Theorem 1.2 describes the solvability of the electric type problem (Eb).

**Theorem 1.2.** For the mapping $E(\varepsilon): H(\text{curl}) \cap H(\text{div} \varepsilon) \mapsto L^2(\Omega)^3 \times L^2(\Omega) \times (H^{-1/2}(\Gamma)^3) =: Z$ defined by (1.9), the following assertions are true:

(a) The range $R(E(\varepsilon))$ of $E(\varepsilon)$ is given by
\[ R(E(\varepsilon)) = \{(j, p, \sigma) \in Z | (1.29) \text{ valid}\}. \]

(b) The dimension of the null space $N(E(\varepsilon)) = \mathcal{H}(\varepsilon|e)$ is finite and independent of $\varepsilon$.

(c) The field $u$ is a solution of the problem (Eb) if and only if it has the representation
\[ u = u_0 + \nabla p + e^{-1} \text{curl } q, \]
where $u_0 \in \mathcal{H}(\varepsilon|e)$, the scalar potential $p \in H_0^1(\Omega)$ is defined by (1.28), and the vector potential $q \in H(\text{curl}) \cap H_0(\text{div})$ is the unique solenoidal solution of (1.32) together with (1.31).

2. Exterior Domains

From now on, we assume that $\Omega$ is an exterior domain in the sense that the complement $\Omega^c = \mathbb{R}^3 \setminus \Omega$ is a compact set with a nonvoid interior. The boundary $\Gamma = \partial \Omega$ is assumed to be smooth. The assumptions on the matrices $\nu$ and $\varepsilon$ are same as before. Regarding the Friedrichs inequality, we shall
make additional assumptions on the regularity as well as on the behavior of the coefficient matrices $v$ and $\varepsilon$ at infinity.

In order to consider problems (0.1) and (0.2) in the case of exterior domains, it is appropriate to introduce some weighted spaces. As the weighting factor, we use the function $(1 + |x|)^{\alpha}$ for various values $\alpha \in \mathbb{R}$. Let us fix the following notation:

\begin{align*}
L^{2,\alpha}(\Omega) &= \{ w \in L^{2}_{loc}(\Omega) \mid (1 + |x|)^{\alpha} w \in L^{2}(\Omega) \}, \\
H^{\alpha}(\text{curl } v) &= \{ u \in L^{2}_{loc}(\Omega)^{3} \mid (1 + |x|)^{\alpha - 1} u \in L^{2}(\Omega)^{3}, \\
&\quad (1 + |x|)^{\alpha} \text{curl } vu \in L^{2}(\Omega)^{3} \}, \\
H^{\alpha}(\text{div } \varepsilon) &= \{ u \in L^{2}_{loc}(\Omega)^{3} \mid (1 + |x|)^{\alpha - 1} u \in L^{2}(\Omega)^{3}, \\
&\quad (1 + |x|)^{\alpha} \text{div } \varepsilon u \in L^{2}(\Omega)^{3} \}.
\end{align*}

Especially in the spaces $H^{\alpha}(\text{div}) = H^{\alpha}(\text{div } \varepsilon)$, $\varepsilon = 1$, there exists a continuous trace operator $u \mapsto n \cdot u: H^{\alpha}(\text{div}) \to H^{-1/2}(\Gamma)$ defined in an obvious way such that $n \cdot u = n \cdot (\varphi u)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^{3})$ which satisfy $\varphi(x) \equiv 1$ in some neighbourhood of $\Gamma$. The space $H^{\alpha}(\text{div})$ is thereby endowed with the norm

\[ \| u \|_{\alpha,\Gamma} = (\| (1 + |x|)^{\alpha - 1} u \|_{0}^{2} + \| (1 + |x|)^{\alpha} \text{div } u \|_{0}^{2})^{1/2}. \]

Similarly, if $H^{\alpha}(\text{curl}) = H^{\alpha}(\text{curl } v)$, $v = 1$, is endowed with the norm

\[ \| u \|_{\alpha,\Gamma} = (\| (1 + |x|)^{\alpha - 1} u \|_{0}^{2} + \| (1 + |x|)^{\alpha} \text{curl } u \|_{0}^{2})^{1/2}. \]

then there exists continuous trace operator $u \mapsto n \wedge u: H^{\alpha}(\text{curl}) \to H^{-1/2}(\Gamma)^{3}$ satisfying $n \wedge u = n \wedge (\varphi u)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^{3})$, $\varphi(x) \equiv 1$ in a neighbourhood of $\Gamma$.

As in the case of the bounded domains, we define the operators $M(v)$ and $E(\varepsilon)$ corresponding to the magnetic, respectively electric, type problem as follows

\begin{align*}
M(v): H^{1}(\text{curl } v) \cap H^{1}(\text{div}) \hookrightarrow L^{2,1}(\Omega)^{3} \times L^{2,1}(\Omega) \times H^{-1/2}(\Gamma), \\
E(\varepsilon): H^{1}(\text{curl}) \cap H^{1}(\text{div } \varepsilon) \hookrightarrow L^{2,1}(\Omega)^{3} \times L^{2,1}(\Omega) \times (H^{-1/2}(\Gamma)^{3}),
\end{align*}

where

\begin{align*}
M(v) u &= (\text{curl } vu, \text{div } u, n \cdot u), \\
E(\varepsilon) u &= (\text{curl } u, \text{div } \varepsilon u, n \wedge u).
\end{align*}

The magnetic type boundary value problem (0.1) for exterior domains reads
(Me) Given \((j, \rho, \lambda) \in L^{2,1}(\Omega)^3 \times L^{2,1}(\Omega) \times H^{-1/2}(\Gamma)\), find the field \(u \in H^1(\text{curl } v) \cap H^1(\text{div } v)\) such that

\[
M(v) u = (j, \rho, \lambda).
\] (2.5)

Respectively, the electric type problem (0.2) for exterior domains is defined as

(Ee) Given \((j, \rho, \sigma) \in L^{2,1}(\Omega)^3 \times L^{2,1}(\Omega) \times (H^{-1/2}(\Gamma)^3)\), find the field \(u \in H^1(\text{curl }) \cap H^1(\text{div } v)\) such that

\[
E(\varepsilon) u = (j, \rho, \sigma).
\] (2.6)

As for bounded domains, the solutions will be represented in the form

\[
u = u_0 + v^{-1} \nabla p + \text{curl } q \quad \text{for (Me),}
\] (2.7)

\[
u = u_0 + \nabla p + \varepsilon^{-1} \text{curl } q \quad \text{for (Ee).}
\] (2.8)

To define the scalar potential \(p\) and the vector potential \(q\), we have to solve certain auxiliary problems in the exterior domain \(\Omega\). The theory of solutions for these problems is different from that in the case of the bounded domains and is not known except for the Poisson equation with the Dirichlet boundary condition.

For convenience, we first recall the Dirichlet exterior problem

\[
\text{div}(\varepsilon \text{ div } p) = \rho, \quad p|_{\Gamma} = 0,
\] (2.9)

which is needed for solving the electric problem (Ee). We abbreviate

\[
H^{\alpha}(\nabla) = \{w \in L^{2,\alpha}_{\text{loc}}(\Omega)|(1 + |x|)^{\alpha-1} w \in L^2(\Omega), \quad (1 + |x|)^\alpha \nabla w \in L^2(\Omega)^3\},
\] (2.10)

\[
H^{\alpha}_0(\nabla) = \{w \in H^{\alpha}(\nabla)|\varphi w \in H^{1}_0(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^3)\}.
\] (2.11)

We remark that for the spaces \(H^{\alpha}_0(\nabla)\), other characterizations are also possible. For example,

\[
H^{\alpha}_0(\nabla) = \overline{\mathcal{D}(\Omega)}^{1,1},
\] (2.12)

holds, where \(|w|_1 = \|\nabla w\|_0\), Witsch [17, Satz 7.6]. This property essentially follows from the weighted Poincaré inequality

\[
\| (1 + |x|)^{-1} w\|_0 \leq c \|\nabla w\|_0,
\] (2.13)

which holds for all \(w \in \mathcal{D}(\Omega)\) [1, p. 388]. Consequently, one obtains in \(H^{\alpha}_0(\nabla)\) the coerciveness result

\[
\|w\|_{1,0} \leq c \|\nabla w\|_0,
\] (2.14)
where

\[ \| w \|_{1,\alpha} = (\| (1 + |x|)^{\alpha-1} w \|_{0}^2 + \| (1 + |x|)^{\alpha} \nabla w \|_{0}^2)^{1/2}. \] (2.15)

Using the standard argument relying on the Lax–Milgram theorem, we get for the exterior Dirichlet problem (De)

**Theorem 2.1.** Let \( p \in L^{2,1}(\Omega) \) be given. Then there exists a unique solution \( p \in H_0^0(\nabla) \) for the problem (2.9) defined in the sense that

\[ (\varepsilon \nabla p | \nabla \phi)_0 = -(p| \phi)_0 \quad \text{for all} \quad \phi \in H_0^0(\nabla). \] (2.16)

In order to discuss the nonhomogeneous exterior Neumann type problem which will be needed to find the scalar potential for the problem (Me), we introduce some additional lemmas. In the following, we will use the abbreviations \( B(R) = \{x \in \mathbb{R}^3 | |x| < R \} \), \( E(R) = \{x \in \mathbb{R}^3 | |x| > R \} \), and \( \Omega(R) = \Omega \cap B(R) \).

**Lemma 2.2.** Let \( \beta > 0 \) be given. Then the embedding \( H^0(\nabla) \subset L^{2,-1-\beta}(\Omega) \) is compact.

**Proof:** Let \( w_\nu \in H^0(\nabla) \) be a sequence with \( \| w_\nu \|_{0,1} = 1 \). Consequently, we have

\[ \| (1 + |x|)^{-1-\beta} w_\nu \|_{0, E(R)} \leq (1 + R)^{-\beta} \] (2.17)

for all \( \nu \in \mathbb{N} \), \( R > 0 \). The desired converging subsequence of \( w_\nu \) in \( L^{2,-1-\beta}(\Omega) \) can therefore be found by the familiar diagonal process, letting \( R_\nu \to \infty \), and applying the compactness of the embeddings \( H^1(\Omega(R_\nu)) \subset L^2(\Omega(R_\nu)) \) together with (2.17).

**Lemma 2.3.** There exists a number \( c > 0 \) such that

\[ \| (1 + |x|)^{-1} w \|_0 \leq c \| \nabla w \|_0 \quad \text{for all} \quad w \in H^0(\nabla). \] (2.18)

**Proof:** We choose the radius \( R_0 \) such that \( \Omega^c \subset B(R_0) \) and take \( R_1 > R_0 \). Fix \( \beta > 0 \). Then there exists a function \( \phi \in C^\infty(\mathbb{R}^3) \) satisfying \( 0 \leq \phi(x) \leq 1 \) and

\[ \phi(x) = 1, \quad |x| \leq 2^{-1}(R_0 + R_1), \]

\[ = (1 + |x|)^{-\beta}, \quad |x| > R_1. \] (2.19)

Now we have \( (1 - \phi) w \in H^0_0(\nabla; E(R_0)) \) and consequently by (2.14)

\[ \| (1 - \phi)(1 + |x|)^{-1} w \|_{0, \Omega} \leq \| (1 - \phi) w \|_{0, E(R_0)} \]

\[ \leq c \| \nabla((1 - \phi) w) \|_{0, E(R_0)} \]

\[ \leq c(\| \nabla w \|_{0, \Omega} + \| (1 + |x|)^{-1-\beta} w \|_{0, \Omega}). \] (2.20)
Since, on the other hand
\[ \| \varphi (1 + |x|)^{-1} w \|_{0, \Omega} \leq c \| (1 + |x|)^{-1 - \beta} w \|_{0, \Omega}, \quad (2.21) \]
we obtain by (2.20) and (2.21)
\[ \| (1 + |x|)^{-1} w \|_{0} \leq c (\| \nabla w \|_{0} + \| (1 + |x|)^{-1 - \beta} w \|_{0}). \quad (2.22) \]

Assertion (2.18) follows from (2.22) by using a standard contradiction argument together with the compactness result of Lemma 2.2.

Our weak formulation for the exterior Neumann problem
\[ \text{div}(v^{-1} \nabla p) = \rho, \quad \text{in } \Omega, \quad n \cdot (v^{-1} \nabla p)|_{\Gamma} = \lambda \quad (2.23) \]
reads:

\( (\text{Ne}) \) Given \((\rho, \lambda) \in L^{2,1}(\Omega) \times H^{-1/2}(\Gamma)\), find the function \(p \in H^{0}(\nabla)\) such that
\[ (v^{-1} \nabla p | \nabla \varphi)_{0} = -(\rho | \varphi)_{0} + \langle \lambda | \gamma_{0} \varphi \rangle_{0}, \quad \varphi \in H^{0}(\nabla). \quad (2.24) \]

Note that the trace \(\gamma_{0} : H^{0}(\nabla) \hookrightarrow H^{1/2}(\Gamma)\) is defined in an obvious way.

As a straightforward consequence of coerciveness result (2.18), we have for the exterior Neumann problem

**Theorem 2.4.** The exterior Neumann problem \((\text{Ne})\) is uniquely solvable for all \((\rho, \lambda) \in L^{2,1}(\Omega) \times H^{-1/2}(\Gamma)\).

To solve the auxiliary problems which are necessary to find the vector potential we need some additional results. Since in the bounded domains we have continuous embeddings \(H(\text{curl } v) \cap H_{0}(\text{div}) \subset H^{1}(\Omega)^{3}\) and \(H_{0}(\text{curl}) \cap H(\text{div } \epsilon) \subset H^{1}(\Omega)^{3}\) (by the Friedrichs inequality), it is natural to ask for the corresponding results in the case of the exterior domains. We prove such extensions under certain assumptions on the coefficient matrices \(v\) and \(\epsilon\). Actually, the result will be applied here only in the case where \(v\) and \(\epsilon\) are identity matrices, but the general case is believed to have independent interest and is therefore presented.

Since the matrix \(\epsilon\) is symmetric positive definite, there exists a unique symmetric positive definite matrix \(\gamma(x)\) such that \(\epsilon = \gamma^{2}\). We fix the following assumptions on the behavior of \(\epsilon\) at infinity: For a positive integer \(k \in \mathbb{N}\), we assume \((\epsilon_{ij}, \gamma_{ij} \in C^{1}(\Omega))\)
\[ |\nabla \epsilon_{ij}(x)| \leq c(1 + |x|)^{-k} \]
\[ |\nabla \gamma_{ij}(x)| \leq c(1 + |x|)^{-k}, \quad x \in \Omega. \quad (\text{co } k) \]
Applying estimate (2.13), we have under \((cok)\) the inequality
\[
\|\text{curl } u\|_0 + \|\text{div } u\|_0 + \|(1 + |x|)^{-k} u\|_0 \leq c \|\nabla u\|_0, \quad u \in H^0_0(\nabla)^3.
\]
(2.25)

The following lemma shows that the norms in (2.25) are actually equivalent:

**Lemma 2.5.** Let \(\varepsilon = (\varepsilon_{ij}(x))\) be a symmetric positive definite bounded real matrix in the exterior domain \(\Omega\) such that \(\varepsilon_{ij} \in C^1(\Omega)\) and such that the growth condition \((cok)\) is valid. Then there exists a number \(c > 0\) such that
\[
\|\nabla u\|_0 \leq c(\|\text{curl } u\|_0 + \|\text{div } u\|_0 + \|(1 + |x|)^{-k} u\|_0)
\]
for all \(u \in H^0_0(\nabla)^3\). (2.26)

**Proof:** By (2.25) and (2.12), it is enough to consider the testfields \(u \in \Omega^0(\Omega)^3\). We make use of the technique employed by Leis [7] for bounded domains. Let \(\kappa\) denote the adjoint of \(\varepsilon\), \(\varepsilon \kappa = \det \varepsilon \cdot \delta\). The matrix \(\kappa\) is also symmetric positive definite, particularly
\[
\kappa_0 |\xi|^2 \leq \langle \xi | \kappa(x) \xi \rangle \leq \kappa_1 |\xi|^2, \quad (x, \xi) \in \Omega \times \mathbb{R}^3. \tag{2.27}
\]
for some positive constants \(\kappa_0, \kappa_1\).

It is illustrative to give some formulas first in the case where the matrix \(\varepsilon\) is constant. In such a case, a rough computation yields
\[
(\varepsilon \nabla) \wedge \varepsilon u = \kappa \text{curl } u. \tag{2.28}
\]
In the general case
\[
(\varepsilon \nabla) \wedge \varepsilon u = \kappa \text{curl } u - R_1(u), \tag{2.29}
\]
where \(R_1(u)\) is a sum of the terms of the type \(\varepsilon_{ij} (\varepsilon_{ik} u_{kj}) u_i\).

We therefore find
\[
\|\text{curl } u\|_0 \geq \kappa_i^{-1} (\text{curl } u \kappa \text{curl } u)_0
\]
\[
= \kappa_i^{-1} (\text{curl } u (\varepsilon \nabla) \wedge \varepsilon u)_0 + \kappa_i^{-1} (\text{curl } u |R_1(u)|)_0. \tag{2.30}
\]
where in the case of assumption \((cok)\) we are able to estimate
\[
|\kappa_i^{-1} (\text{curl } u |R_1(u)|)_0| \leq c(\eta \|\nabla u\|_0^2 + \eta^{-1} \|(1 + |x|)^{-k} u\|_0^2) \quad \text{for all } \eta > 0. \tag{2.31}
\]

Further, for a constant matrix, we have the identity
\[
\text{curl}(\varepsilon \nabla) \wedge \varepsilon u) = \varepsilon \nabla \text{div } \varepsilon u - \text{div } \varepsilon \nabla \varepsilon u. \tag{2.32}
\]
where \( \text{div} \, \nabla \varepsilon u \) is a vector defined by

\[
(\text{div} \, \nabla \varepsilon u)_i = \text{div} \, \nabla (\varepsilon u)_i = \partial_{\nu}(\varepsilon_{\nu\mu} \partial_{\mu}(\varepsilon_{\sigma\sigma} u_{\sigma})).
\] (2.33)

In the general case, instead of (2.32) we calculate

\[
\text{curl}((\varepsilon \nabla) \wedge \varepsilon u) = \varepsilon \nabla \text{div} \, \varepsilon u - \text{div} \, \varepsilon \nabla \varepsilon u + R_2(u),
\] (2.34)

where \( R_2(u) \) is a sum of the terms of type \((\partial_{\nu} \varepsilon_{\sigma\nu}) \partial_{k}(\varepsilon_{ij} u_{j})\). By (2.30), (2.31), and (2.34), we can conclude that

\[
\| \text{curl} \, u \|_0^2 \geq \kappa_1^{-1} \{ - \| \text{div} \, \varepsilon u \|_0^2 - (u \| \text{div} \, \nabla \varepsilon u \|_0) + T_1(u),
\] (2.35)

where under \((\text{co} \, k)\)

\[
|T_1(u)| \leq c(\eta^{-1} \| (1 + |x|)^{-k} u \|_0^2 + \eta \| \nabla u \|_0^2)
\] for all \( \eta > 0 \). (2.36)

In the case of constant \( \varepsilon \), we obtain by setting \( \varepsilon = \gamma^2 \) that (summation convention)

\[
-(u \| \text{div} \, \nabla \varepsilon u \|_0) = \sum_{\mu = 1}^{3} (\nabla [\gamma u]_{\mu} \| \varepsilon \nabla [\gamma u]_{\mu})_0 \\
\geq \varepsilon_0 \sum_{\mu = 1}^{3} \| \nabla [\gamma u]_{\mu} \|_0^2 = \varepsilon_0 (\varepsilon_{\sigma\nu} \partial_{k} u_{\sigma} | \partial_{k} u_{\nu})_0 \\
\geq \varepsilon_0^2 \| \nabla u \|_0^2.
\] (2.37)

In general we can obtain, instead of (2.37), the inequality

\[
-(u \| \text{div} \, \nabla \varepsilon u \|_0) \geq \varepsilon_0^2 \| \nabla u \|_0^2 + T_3(u),
\] (2.38)

where \( T_3(u) \) has the same bound (2.36) as \( T_1(u) \). From (2.35), (2.36), and (2.38) it now follows that

\[
\varepsilon_0^2 \| \nabla u \|_0^2 \leq \kappa_1 \| \text{curl} \, u \|_0^2 + \| \text{div} \, \varepsilon u \|_0^2 + c \eta^{-1} \| (1 + |x|)^{-k} u \|_0^2 + c \eta \| \nabla u \|_0^2,
\] (2.39)

which gives the required estimate when the parameter \( \eta \) is chosen to be small enough.

For the magnetic type problem we obtain a similar result.

**Lemma 2.6.** Let \( v(x) = (v_{ij}(x)) \) be a symmetric positive definite bounded real matrix in the exterior domain \( \Omega \) such that \( v_{ij} \in C^1(\Omega) \) and such that \( v
satisfies the growth condition \((\alpha k)\) \((v\text{ instead of }\varepsilon)\). Then there exists a constant \(c > 0\) such that

\[
\|\nabla u\|_0 \leq c(\|\text{curl} \, v u\|_0 + \|\text{div} \, u\|_0 + \| (1 + |x|)^{-k} u\|_0) \quad \text{for all } u \in H^0_0(\nabla)^3.
\]  

(2.40)

\textbf{Proof.} The matrix \(v^{-1}\) also satisfies condition \((\alpha k)\). With \(w = v u\) we therefore obtain, by the previous lemma,

\[
\|\nabla w\|_0 \leq c(\|\text{curl} \, w\|_0 + \|\text{div} \, v^{-1} \, w\|_0 + \| (1 + |x|)^{-k} w\|_0)
\]

\[
\leq c(\|\text{curl} \, v u\|_0 + \|\text{div} \, u\|_0 + \| (1 + |x|)^{-k} u\|_0).
\]  

(2.41)

Furthermore, the following holds:

\[
\sum_{j=1}^{s} \| v_{j\alpha} \nabla u_{\alpha}\|_0^2 \leq 2 \sum_{j=1}^{s} \| \nabla (v_{j\alpha} u_{\alpha})\|_0^2 + \| u_{\alpha} \nabla v_{j\alpha}\|_0^2
\]

\[
\leq c(\|\nabla w\|_0^2 + \| (1 + |x|)^{-k} u\|_0^2).
\]  

(2.42)

where in the case of a constant matrix \(v\),

\[
\sum_{j=1}^{3} \| v_{j\alpha} \nabla u_{\alpha}\|_0^2 = \sum_{j=1}^{3} \| \nabla (v_{j\alpha} u_{\alpha})\|_0^2 = \sum_{j=1}^{3} \sum_{k=1}^{3} \| v_{j\alpha} \partial_k u_{\alpha}\|_0^2
\]

\[
= (v_{j\alpha} \partial_k u_{\alpha}) v_{j\alpha} \partial_k u_{\alpha}\|_0
\]

\[
= v_0 \sum_{k, \alpha} \| \partial_k u_{\alpha}\|_0^2 = v_0 \| \nabla u\|_0^2.
\]

In the general case we have, instead of the above, the inequality

\[
\sum_{j=1}^{3} \| v_{j\alpha} \nabla u_{\alpha}\|_0^2 \geq v_0 \| \nabla u\|_0^2 + T(u),
\]  

(2.43)

where

\[
|T(u)| \leq c(\eta^{-1} \| (1 + |x|)^{-k} u\|_0^2 + \eta \| \nabla u\|_0^2)
\]  

for all \(\eta > 0\).  

(2.44)

Combining inequalities (2.41)-(2.44), we get the required estimate (2.40).  

It should be noted that when \(v\) (or \(\varepsilon\)) is the identity matrix, the simple relation

\[
\| \nabla u\|_0^2 = \| \text{curl} \, u\|_0^2 + \|\text{div} \, u\|_0^2
\]  

(2.45)

holds for all \(u \in H^0_0(\nabla)^3\).
Now we are able to prove an embedding result of the Friedrichs type for exterior domains. We abbreviate for \( \alpha \geq 0 \)
\[
\begin{align*}
H^0_0(\text{div}) &= \{ u \in H^0(\text{div}) \mid n \cdot u = 0 \}, \\
H^0_0(\text{curl}) &= \{ u \in H^0(\text{curl}) \mid n \wedge u = 0 \}.
\end{align*}
\]

**Theorem 2.1.** Let \( \epsilon = (\epsilon_{ij}) \) and \( \nu = (\nu_{ij}) \) be symmetric positive definite bounded real matrices such that \( \epsilon_{ij}, \nu_{ij} \in C^2(\overline{\Omega}) \) and such that \( \epsilon \) and \( \nu \) satisfy condition \((co k)\) for some \( k \in \mathbb{N} \). Then the embeddings
\[
\begin{align*}
(i) \quad &H^0_0(\text{curl}) \cap H^0(\text{div } \epsilon) \subset H^0(\nabla)^3, \\
(ii) \quad &H^0(\text{curl } \nu) \cap H^0_0(\text{div}) \subset H^0(\nabla)^3
\end{align*}
\]
are true and
\[
\| q \|_{1,0} \leq C(\| \text{curl } q \|_0 + \| \text{div } \epsilon q \|_0 + \| (1 + |x|)^{-k} q \|_0) \quad (2.46)
\]
for all \( q \in H^0_0(\text{curl}) \cap H^0(\text{div } \epsilon) \) and
\[
\| q \|_{1,0} \leq C(\| \text{curl } \nu q \|_0 + \| \text{div } q \|_0 + \| (1 + |x|)^{-k} q \|_0) \quad (2.47)
\]
for all \( q \in H^0(\text{curl } \nu) \cap H^0_0(\text{div}) \).

**Proof.** We consider the electric type embedding (i) with (2.46). For (i) it is enough to show that \( \nabla q \) is square integrable. We choose the balls \( B(R_i) \), \( l = 0, 1, 2, 3 \) such that \( R_l < R_{l+1} \) and such that the complement \( \Omega^c \) is contained in \( B(R_0) \).

Let \( \xi \in \mathcal{D}(\mathbb{R}^3) \) be a cutoff function satisfying \( 0 \leq \xi(x) \leq 1 \) and
\[
\begin{align*}
\xi(x) &= 1, \quad 0 \leq |x| \leq R_1, \\
&= 0, \quad R_2 \leq |x|.
\end{align*}
\]
Then we have \( \xi q \in H_0(\text{curl}; \Omega(R_3)) \cap H(\text{div } \epsilon; \Omega(R_3)) \) and consequently by the assumption \( \epsilon_{ij} \in C^2(\overline{\Omega}) \), we have \( \xi q \in H^1(\Omega(R_3))^3 \)\(^7\) with
\[
\begin{align*}
\| \nabla q \|_{0, \Omega(R_3)} &\leq \| \nabla (\xi q) \|_{0, \Omega(R_3)} \\
&\leq c(\| \text{curl } (\xi q) \|_{0, \Omega(R_3)} + \| \text{div } (\epsilon\xi q) \|_{0, \Omega(R_3)} \\
&\quad + \| \xi q \|_{0, \Omega(R_3)}) \\
&\leq c(\| \text{curl } q \|_{0, \Omega} + \| \text{div } \epsilon q \|_{0, \Omega} \\
&\quad + \| (1 + |x|)^{-k} q \|_{0, \Omega}). \quad (2.48)
\end{align*}
\]
It remains to show that \( \nabla q \in (L^2(E(R_1))^3)^* \). We use a new cutoff function...
\( \zeta \) such that \( \zeta \in \mathcal{D}(\mathbb{R}), \ 0 \leq \zeta(t) \leq 1, \ \zeta(t) = 1 \) for \( R_1 \leq t \leq R_2 \) and \( \zeta \in \mathcal{D}((R_0, R_3)) \). For every \( R > R_2 \), we consider the mapping \( \tau(R) : \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[
\tau(R)(t) = R_1 + \left( (R_2 - R_1)/ (R - R_1) \right) \cdot (t - R_1).
\]

We then have \( \tau(R)(R_1) = R_1, \ \tau(R)(R) = R_2 \). Moreover, at

\[
R_\ast = R_1 + \left( (R_3 - R_1)/ (R_2 - R_1) \right) \cdot (R - R_1)
\]

we have \( \tau(R)(R_\ast) = R_1 \). The composite function \( \varphi(R) = \zeta \circ \tau(R) \) satisfies \( \varphi(R) \in C^\infty(\mathbb{R}), \ \varphi(R) \in \mathcal{D}((R_0, R_\ast)), \) and \( 0 \leq \varphi(R) \leq 1, \ \varphi(R)(t) = 1 \).

For the field \( \varphi(R) q \), where \( (\varphi(R) q)(x) := \varphi(R)(|x|) q(x) \), we have \( \varphi(R) q \in H^0_0(\nabla: E(R_0))^3 \). An application of Lemma 2.5 yields \( (B(R_1, R) = \{ |x| | R_1 < |x| < R \}) \)

\[
\| \nabla q \|_{0,B(R_1,R)} \leq \| \nabla(\varphi(R) q) \|_{0,E(R_0)}
\]

\[
\leq c\left(\| \text{curl}(\varphi(R) q) \|_{0,E(R_0)} + \| \text{div}(\varphi(R) q) \|_{0,E(R_0)}\right)
\]

\[
+ \| (1 + |x|)^{-k} \varphi(R) q \|_{0,E(R_0)}
\]

\[
\leq c\left(\| \text{curl} q \|_{0,\Omega} + \| \text{div} q \|_{0,\Omega} + \| (1 + |x|)^{-k} q \|_{0,\Omega}\right)
\]

\[
+ \| (\nabla \varphi(R)) \wedge q \|_{0,E(R_0)} + \| (\nabla \varphi(R)) \cdot q \|_{0,E(R_0)}\right). \quad (2.50)
\]

In the above, as well as in the sequel, \( c \) denotes a generic constant independent of \( q \) and \( R \).

By observing that

\[
|\nabla \varphi(R)(x)| \leq c, \quad 0 \leq |x| \leq R_1,
\]

\[
\leq 0, \quad R_1 \leq |x| \leq R \quad \text{or} \quad |x| \geq R_\ast, \quad (2.51)
\]

we conclude that

\[
\| (\nabla \varphi(R)) \wedge q \|_{0,E(R_0)} + \| (\nabla \varphi(R)) \cdot q \|_{0,E(R_0)}
\]

\[
\leq c\left(\| q \|_{0,B(R_1)} + \| (1 + |x|)^{-1} q \|_{0,B(R_1)}\right). \quad (2.52)
\]

Since \( (1 + |x|)^{-1} q \in L^2(\Omega)^3 \), estimates (2.50), (2.52) imply

\[
\| \nabla q \|_{0,E(R_1)} = \lim_{R \rightarrow \infty} \| \nabla q \|_{0,B(R_1,R)}
\]

\[
\leq c\left(\| \text{curl} q \|_{0} + \| \text{div} q \|_{0} + \| (1 + |x|)^{-k} q \|_{0}\right). \quad (2.53)
\]
By (2.48), (2.53), the embedding (i) is valid. Combining (2.48), (2.53) with (2.17), we obtain inequality (2.46).

The proof for the magnetic type boundary condition is similar and is therefore omitted.

Let us introduce the weighted spaces of the \( v \)-harmonic fields of magnetic type and \( \varepsilon \)-harmonic fields of electric type.

\[
\mathcal{H}^\alpha(m|v) = \{ u \in H^\alpha(\text{curl } v) \cap H_0^\alpha(\text{div}) | \text{ curl } vu = 0, \quad \text{div } u = 0 \}, \quad (2.54)
\]
\[
\mathcal{H}^\alpha(\varepsilon|\varepsilon) = \{ u \in H_0^\alpha(\text{curl}) \cap H^\alpha(\text{div } \varepsilon) | \text{ curl } u = 0, \quad \text{div } \varepsilon u = 0 \}. \quad (2.55)
\]

Accordingly, \( \mathcal{H}^\alpha(m|v) \subset \mathcal{H}^\beta(m|v) \) and \( \mathcal{H}^\alpha(\varepsilon|\varepsilon) \subset \mathcal{H}^\beta(\varepsilon|\varepsilon) \) hold for \( \beta < \alpha \).

As a consequence of the previous theorem we obtain

**Theorem 2.8.** Let \( v \), respectively \( \varepsilon \), be a symmetric positive definite bounded real matrix satisfying \( v_{ij} \in C^2(\Omega) \), respectively \( \varepsilon_{ij} \in C^2(\Omega) \), with growth condition (co 2). Then the space \( \mathcal{H}^0(m|v) \), respectively \( \mathcal{H}^0(\varepsilon|\varepsilon) \), is finite dimensional.

**Proof:** This is shown by standard techniques using estimate (2.46), resp. (2.47), for \( k = 2 \) and the compactness of the embedding \( H^0(\text{curl } v) \) \( \subset L^{2,-2}(\Omega)^3 \) guaranteed by Lemma 2.2.

In the following, we denote by \( \mathcal{H}^0(\varepsilon|\varepsilon)^\perp \) the orthogonal complement of the space \( \mathcal{H}^0(\varepsilon|\varepsilon) \) in \( L^{2,-1}(\Omega)^3 \) with respect to the inner product \( \langle u, v \rangle = \langle (1 + |x|)^{-1} u, (1 + |x|)^{-1} v \rangle \). Using Theorem 2.7 and the compactness of the inclusion \( H^0(\text{curl } v) \) \( \subset L^{2,-2}(\Omega)^3 \), we obtain

**Theorem 2.9.** With the previous assumptions we have

\[
\| q \|_{1,0} \leq c(\| \text{curl } q \|_0 + \| \text{div } q \|_0), \quad q \in \mathcal{H}^0(m|v)^\perp \cap H_0^0(\text{curl } v) \cap H^0(\text{div}), \quad (2.56)
\]
\[
\| q \|_{1,0} \leq c(\| \text{curl } q \|_0 + \| \text{div } \varepsilon q \|_0), \quad q \in \mathcal{H}^0(\varepsilon|\varepsilon)^\perp \cap H_0^0(\text{curl}) \cap H_0^0(\text{div } \varepsilon). \quad (2.57)
\]

We have also to consider the spaces \( \mathcal{H}^1(m|v) \) and \( \mathcal{H}^1(\varepsilon|\varepsilon) \), which include all harmonic fields lying in \( L^{2}(\Omega)^3 \). For these spaces, a stronger result than in Theorem 2.8 is true.

**Theorem 2.10.** Let \( v \), respectively \( \varepsilon \), be a symmetric positive definite measurable bounded real matrix in \( \Omega \). Then the space \( \mathcal{H}^1(m|v) \), respectively \( \mathcal{H}^1(\varepsilon|\varepsilon) \), is finite dimensional and the dimension is independent of \( v \), respectively of \( \varepsilon \).
Proof. Theorem 2.8 implies that the spaces $\mathcal{H}^1(m|v)$ and $\mathcal{H}^1(e|1)$ are finite dimensional. The assertion follows by the same argument as used in the proof of Theorem 1.1, since we have the $L^2(\Omega)^3$-decompositions

$$
H^1_0(\text{div}) \cap H^1(\text{div}|0) = \mathcal{H}^1(m|v) \oplus \text{curl } H^0_0(\text{curl}),
$$

$$
H^1_0(\text{curl}) \cap H^1(\text{curl}|0) = \mathcal{H}^1(e|1) \oplus \nabla H^0_0(\nabla)
$$

(where in fact $\nabla H^0_0(\nabla) = \nabla H^0_0(\nabla)$).

We are now able to discuss the determination of the vector potential $q$ in the case of the magnetic type problem (Me).

Because of inequality (2.57), the form $(\nu \text{ curl } q | \text{ curl } \phi)_0$ is a coercive form in the space $H^1_0(\text{curl}) \cap H^1(\text{div}|0) \cap (\mathcal{H}^0(e|1)^{-1})$. Therefore the vector potential $q$ is uniquely defined by the requirement: $q \in H^1_0(\text{curl}) \cap H^1(\text{div}|0) \cap (\mathcal{H}^0(e|1)^{-1})$ such that

$$
(\nu \text{ curl } q | \text{ curl } \phi)_0 = (j | \phi)_0, \quad \phi \in H^1_0(\text{curl}) \cap H^1(\text{div}|0) \cap (\mathcal{H}^0(e|1)^{-1}).
$$

(2.58)

Note that since $j \in L^{2,1}(\Omega)$ and $\phi \in L^{2,-1}(\Omega)$, the right side is well defined and continuous in $(H^1_0(\text{curl}) \cap H^1(\text{div}|0)) \cap (\mathcal{H}^0(e|1)^{-1})$ with respect to the norm $(\| \text{ curl } \phi \|^2 + \|(1 + |x|)^{-1} \phi \|^2)^{1/2}$.

Our final result concerning the magnetic type problem (Me) for exterior domains is

**THEOREM 2.11.** Suppose that the real matrix $\nu = (\nu_{ij})$ is symmetric positive definite, measurable, and bounded. For the mapping $M(\nu): H^1(\text{curl } v) \cap H^1(\text{div}) \mapsto L^{2,1}(\Omega)^3 \times L^{2,1}(\Omega) \times H^{-1/2}(F) =: Y^1$ defined by (2.3), the following assertions are true:

(a) The range $R(M(\nu))$ of $M(\nu)$ is given by

$$
R(M(\nu)) = \{(j, \rho, \lambda) \in Y^1 | \text{div } j = 0, (j | h)_0 = 0, \phi \in \mathcal{H}^0(e|1)\}.
$$

(2.59)

(b) The dimension of the null space $N(M(\nu)) = \mathcal{H}^1(m|v)$ is finite and independent of $\nu$.

(c) The field $u$ is a solution of the problem (Me) if and only if $u$ has the representation

$$
u u = u_0 + v^{-1} \nabla p + \text{curl } q,
$$

where $u_0 \in \mathcal{H}^1(m|v)$, the scalar potential $p \in H^0(\nabla)$ is defined by (2.23), and the vector potential $q \in H^1_0(\text{curl}) \cap H^1(\text{div}|0) \cap (\mathcal{H}^0(e|1)^{-1})$ satisfies (2.58).
Let \( u \in H'(\text{curl } v) \cap H'(\text{div}) \) with \( M(v) u = (j, \rho, \lambda) \) be given. Then we have \( \text{div} j = \text{div}(\text{curl } v) u = 0 \). Further if \( h \in \mathcal{D}'(\Omega) \) and if we choose a cutoff function \( \zeta \in \mathcal{D}(\mathbb{R}^3) \) such that \( \zeta(x) = 1, \ |x| \leq 1 \), then it follows, with \( \zeta_n(x) := \zeta(n^{-1}x) \), that
\[
(j|h)_0 = \lim_{n \to \infty} (\text{curl } vu|\zeta_n h)_0 = \lim_{n \to \infty} (vu|\text{curl}(\zeta_n h))_0
\]
\[
= \lim_{n \to \infty} (vu| (\nabla \zeta_n \wedge h)_0)
\]
where
\[
| (vu| (\nabla \zeta_n \wedge h)_0 | \leq c \| u \|_{0, \infty(n)} \| (1 + |x|)^{-1} h \|_{0, \infty(n)} \to 0
\]
for \( n \to \infty \) by \( u \in L^2(\Omega)^3, h \in L^{1,-1}(\Omega)^3 \). Accordingly,
\[
R(M(v)) \subset \{(j, \rho, \lambda) \in Y^1|\text{div} j = 0, \ (j|h)_0 = 0, \ h \in \mathcal{D}'(\Omega) \}.
\]

On the other hand, let \( (j, \rho, \lambda) \in Y^1 \) with \( \text{div} j = 0, \ (j|h)_0 = 0, \) and \( h \in \mathcal{D}'(\Omega) \) be given. By Theorem 2.4, we can define uniquely the function \( p \in H^0(\nabla) \) such that
\[
(v^{-1} \nabla | (\nabla \varphi \wedge h)_0 | \rho_0 + \langle \lambda | \gamma \varphi \rangle \varphi \in H^0(\nabla).
\]

Similarly, there exists a unique field \( q \in H^0_0(\text{curl}) \cap H^0(\text{div} | 0) \cap (\mathcal{D}'(\Omega)^3) \) such that
\[
(v \text{curl } q | (\nabla \varphi)_0) = (j|\varphi)_0
\]
for every \( \varphi \in H^0_0(\text{curl}) \cap H^0(\text{div} | 0) \cap (\mathcal{D}'(\Omega)^3) \).

We will verify that the field \( w := v^{-1} \nabla p + \text{curl } q \) satisfies \( M(v) w = (j, \rho, \lambda) \). We have \( w \in L^2(\Omega)^3 \). Further, if \( \eta \in \mathcal{D}(\Omega) \), then we obtain by (2.63)
\[
(w| \nabla \eta)_0 = (v^{-1} \nabla p + \text{curl } q | \nabla \eta)_0 = (v^{-1} \nabla p | \nabla \eta)_0 = -(\rho | \eta)_0,
\]
which yields \( \text{div} w = \rho \).

In order to show the relation \( \text{curl } vw = j \), we first note that equation (2.64) together with \( (j|h)_0 = 0 \) and \( h \in \mathcal{D}'(\Omega) \) yields
\[
(v \text{curl } q | (\nabla \varphi)_0) = (j|\varphi)_0
\]
for all \( \varphi \in H^0_0(\text{curl}) \cap H^0(\text{div} | 0) \) (apply the decomposition \( H = H \cap (\mathcal{D}'(\Omega)^3) + \mathcal{D}'(\Omega) \) with \( H = H^0_0(\text{curl}) \cap H^0(\text{div} | 0) \)). Now let \( \varphi \in \mathcal{D}(\Omega)^3 \) be given. We decompose it in the form \( \varphi = \varphi_0 + \varphi_1 \), where \( \varphi_1 = \nabla w, \ \varphi_0 \in H^0_0(\nabla) \) solves the exterior Dirichlet problem \( \Delta w = \varphi \). Then \( \varphi_0 \in H \) and \( (j|\varphi_0)_0 = 0 \) by \( \text{div} j = 0 \). Accordingly, by (2.65)
\[
(v \text{curl } q | (\nabla \varphi)_0) = (v \text{curl } q | (\nabla \varphi_0)_0 = (j|\varphi_0)_0 = (j|\varphi)_0,
\]
which implies

\[
(vw|\text{curl } \varphi)_0 = (\nabla p + v \text{ curl } q|\text{curl } \varphi)_0 = (v \text{ curl } q|\text{curl } \varphi)_0 = (j|\varphi)_0 \tag{2.66}
\]

for all \( \varphi \in \mathcal{D}(\Omega)^3 \). Hence the relation \( vw = j \) is valid.

We have \( n \cdot \text{curl } q = 0 \) for \( q \in H^0_0(\text{curl}) \) and \( n \cdot (v^{-1} \nabla p) = \lambda \) for the solution \( p \in H^0_0(\nabla) \) of Eq. (2.63). Hence

\[
n \cdot w = n \cdot (v^{-1} \nabla p + \text{curl } q) = \lambda
\]

is satisfied and we have \( (j, \rho, \lambda) \in R(M(v)) \). Thus assertion (a) has been proved.

Assertion (b) follows from Theorem 2.10 and (c) was established in the proof of (a).

Without going into details, we briefly discuss the exterior problem (Ee) of electric type. The vector potential \( q \in H^0_0(\text{curl}) \cap H^0_0(\text{div}), \text{div } q = 0, q \in \mathcal{H}^0(m|1)^- \) is defined as the unique solution of the problem

\[
(\varepsilon^{-1} \text{curl } q|\text{curl } \varphi)_n = (j|\varphi)_n - \langle n \wedge \sigma | n \wedge \varphi \rangle_0 \tag{2.67}
\]

for all \( \varphi \in H^0_0(\text{curl}) \cap H^0_0(\text{div}), \text{div } \varphi = 0, \varphi \in \mathcal{H}^0(m|1)^- \). Note that the last term on the right side is well defined since \( H^0_0(\text{curl}) \cap H^0_0(\text{div}) \subset H^0_0(\nabla)^3 \) (Theorem 2.7), which implies \( n \wedge \varphi \in H^{1/2}(\Gamma)^3 \). Furthermore \( \varphi \mapsto \langle n \wedge \sigma | n \wedge \varphi \rangle_0 \) is continuous in \( H^0_0(\text{curl}) \cap H^0_0(\text{div}) \cap H^0_0(\text{div}|0) \cap (\mathcal{H}^0(m|1)^-) \) with respect to the norm \( \|\varphi\| = (\|\text{curl } \varphi\|_0^2 + \|(1 + |x|)^{-1} \varphi\|_0^3)^{1/2} \) by (2.57).

Comparing the above proof and the argument for the corresponding theorem in the case of bounded domains, we can finally establish

**Theorem 2.12.** Assume that the real matrix \( \varepsilon = (\varepsilon_{ij}) \) is symmetric positive definite, measurable, and bounded in the exterior domain \( \Omega \). Then for the mapping \( E(\varepsilon): H^1(\text{curl}) \cap H^1(\text{div}) \rightarrow L^{2,1}(\Omega)^3 \times L^{2,1}(\Omega) \times (H^{-1/2}(\Gamma)^3) =: Z^1 \) defined by (2.4), the following assertions are true:

(a) The range \( R(E(\varepsilon)) \) of \( E(\varepsilon) \) is given by

\[
R(E(\varepsilon)) = \{(j, \rho, \sigma) \in Z^1 \mid \text{div } j = 0, \rho = n \wedge v \text{ for some } v \in H^1(\text{curl}), (j|h)_0 - \langle \sigma | h \rangle_0 = 0 \text{ for all } h \in \mathcal{H}^0(m|1)\} \tag{2.68}
\]

(b) The dimension of the null space \( N(E(\varepsilon)) = \mathcal{H}^1 e \) is finite and independent of \( \varepsilon \).
The field $u$ is a solution of the problem (Ee) if and only if it has the representation

$$u = u_0 + \nabla p + \varepsilon^{-1} \text{curl } q,$$

where $u_0 \in \mathscr{H}^1(e | \varepsilon)$, the scalar potential $p \in H^0_0(\nabla)$ is defined by (2.16), and the vector potential $q \in H^0(\text{curl}) \cap H^0_0(\text{div})$ is the unique solenoidal field satisfying $q \in \mathscr{H}^0(m | 1) \perp$ with equation (2.67).

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REFERENCES

