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# Lipschitz–Nikolskiĭ Constants for the Trotter–Feller Operator

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In this paper we determine the Lipschitz–Nikolskiĭ constants for the Trotter–Feller operator which contains various well-known operators. As corollaries to our general settings we obtain the Lipschitz–Nikolskiĭ constants for a series of concrete operators including the Bernstein, Szász, Gamma, Weierstrass, and Baskakov operators, their generalizations such as the Cheney–Sharma and Bleimann–Butzer–Hahn operators, and many others. Our results also improve Rathore's on the Meyer-König and Zeller operator and the Gamma operator of Müller as well as Rathore and Singh's on the Post–Widder operator. Throughout the paper the probabilistic method is used intensively while a result in probability theory on normal approximation plays a key role. 

## INTRODUCTION

Let $J$ be a subset of $\mathscr{R}$, and denote

$$W^{(q)}(C(J);\alpha;1) := \left\{ f \in C^{(q)}(J) : \left| f^{(q)}(x) - f^{(q)}(y) \right| \le |x-y|^{\alpha},\ \forall x, y \in J \right\},$$

$$0 < \alpha \le 1, \quad q \in \mathscr{N}_0 = \mathscr{N} \cup \{0\}, \quad \mathscr{N} = \{1, 2, \dots\}.$$

For a sequence of operators $(L_n)_{n \in \mathscr{N}}$ defined on a domain of functions containing $W^{(q)}(C(J);\alpha;1)$, consider the quantity

$$\Delta(L_n;\alpha;q;x) := \sup_{f \in W^{(q)}(C(J);\alpha;1)} \left| L_n\left( f(t) - \sum_{k=0}^{q} \frac{f^{(k)}(x)}{k!}(t-x)^k, x \right) \right|, \quad x \in J.$$



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If there exists a numerical sequence $\Psi_n \to 0$ as $n \to \infty$ such that

$$\lim_{n \to \infty} \Psi_n^{-1} \Delta(L_n; \alpha; q; x) = C(\alpha; q; x),$$

where $C(\alpha; q; x)$ is a positive number, then $C(\alpha; q; x)$is called the Lipschitz–Nikolskiĭ constant corresponding to the order $\Psi_n$ of (higher order) approximation of the function class $W^{(q)}(C(J); \alpha; 1)$ by the operator $L_n$ (see [14, 12]).

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $[A, B]$ be a (possibly infinite) interval in $\mathscr{R}$. Consider a triangular array of independent $[A, B]$-valued random variables $\{X_{nj},\ j = 1, 2, \ldots; n \in \mathscr{N}\}$ on $(\Omega, \mathscr{F}, P)$ with expectation $E(X_{nj}) = \alpha_{nj}(x)$ and finite variance $\mathrm{Var}(X_{nj}) = \sigma_{nj}^2(x) > 0$, where parameter $x \in [A, B]$. Denote $S_n = \sum_{j=1}^{n} X_{nj}$. For a sequence of positive real numbers $\{\beta_n,\ n \in \mathscr{N}\}$ with $\beta_n = o(1)$ $(n \to \infty)$, and a fixed monotone function $r(t)$ mapping $(\bar{A}, \bar{B})$ into $J$, where $(\bar{A}, \bar{B}) \supset [n\beta_n A, n\beta_n B]$ for all $n$, define the $n$th Trotter–Feller operator as

$$T_n(f, r, x) := E\big[f\big(r(\beta_n S_n)\big)\big] = \int_{-\infty}^{\infty} f(r(t))\, dF_{\beta_n S_n}(t), \tag{1}$$

where $F_{\beta_n S_n}(t) := P(\beta_n S_n \le t)$ is the distribution function of $\beta_n S_n$ and $f(r(t))$ is $F_{\beta_n S_n}$-integrable.

Hahn established in [7] the convergence theorems with rates of (1) for $r(t) = t$. For some special $r(t)$ and $X_{nj}$ with $X_{nj} \overset{\text{i.i.d.}}{\sim} X_{n1}$ for all $n$ and $j$, (1) is specified to the operator considered by Khan in [9]. If further we set $r(t) = t$, $\beta_n = 1/n$ and $X_{nj} \overset{\text{i.i.d.}}{\sim} X_{11}$, for all $n$ and $j$, then (1) becomes the Feller operator $F_n(f, x)$ [6],

$$F_n(f, x) := E\left[f\left(\frac{S_n}{n}\right)\right] = \int_{-\infty}^{\infty} f(t)\, dF_{S_n/n}(t), \tag{2}$$

which includes a number of classical operators such as the Bernstein, Szász, Baskakov, Gamma, and Weierstrass operators as special cases. However, we need the more general form (1) when considering the variations or generalizations of classical operators such as the Meyer-König and Zeller operator, Cheney–Sharma operator [4], Bleimann–Butzer–Hahn operator [2], and Gamma operator of Müller (see, e.g., [14]), as well as the special Trotter–Feller operator considered by Chen and Zeng [3].

By the aid of some results in probability theory on normal approximation we are able to determine the Lipschitz–Nikolskiĭ constants for the Trotter–Feller operator (1), and therefore simultaneously solve the Lipschitz–Nikolskiĭ constant problem for the operators mentioned as special cases of the Trotter–Feller operator. Some of those operators are for the first time considered for their Lipschitz–Nikolskiĭ constants. The results in [14–16] are improved by giving the exact constants instead of error estimates for odd $q$, while the requirement of the finiteness of all moments of $L_n$, $L_n(|x-t|^i, x)$, $i = 1, 2, \ldots,$ is removed as compared with the result of Zhou [22]. An operator induced by non-identically distributed r.v.'s due to Hahn [7] is considered at the end of the paper.

## 2. MAIN RESULTS

With $\{X_{nj}, j = 1, 2, \ldots; n \in \mathcal{N}\}$, $\{\beta_n, n \in \mathcal{N}\}$, and $r(t)$ as before, we use the following notations:

$$\alpha_n \equiv \alpha_n(x) := \frac{1}{n}\sum_{j=1}^{n} \alpha_{nj}(x), \qquad \text{where } \alpha_{nj}(x) = EX_{nj}; \tag{3}$$

$$\sigma_n^2 \equiv \sigma_n^2(x) := \frac{1}{n}\sum_{j=1}^{n} \sigma_{nj}^2(x), \qquad \text{where } \sigma_{nj}^2(x) = \operatorname{Var}(X_{nj}); \tag{4}$$

$$\rho_{sn} := \frac{1}{n}\sum_{j=1}^{n} E|X_{nj} - EX_{nj}|^s \qquad (s > 0); \tag{5}$$

$$\Psi_n \equiv \Psi_n(x) := \sqrt{n}\,\sigma_n\,\beta_n; \tag{6}$$

$$Q_n(t) := P\left(\frac{1}{\sqrt{n}}(S_n - n\alpha_n) \le t\right); \tag{7}$$

$$\Phi_{\sigma^2}(t) := \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-u^2/(2\sigma^2)}\,du \qquad (\sigma > 0); \tag{8}$$

$$\chi_E(t) := 1 \text{ for } t \in E, \ := 0 \text{ for } t \notin E, \qquad \text{where } E \text{ is a subset of } R; \tag{9}$$

$$f_q(y) := f(r(y)) - \sum_{k=0}^{q} \frac{f^{(k)}(r(x))}{k!}(r(y) - r(x))^k, \qquad q \in \mathcal{N} \cup \{0\}; \tag{10}$$

$$f_q^*(y) := \begin{cases} (r(y) - r(x))^q \displaystyle\int_0^1 \frac{(1-\tau)^{q-1}}{(q-1)!} \\ \quad \times \{f^{(q)}((r(y) - r(x))\tau + r(x)) - f^{(q)}(r(x))\}\,d\tau, & q \in \mathcal{N}, \\ f(r(y)) - f(r(x)), & q = 0; \end{cases} \tag{11}$$

$$(\alpha + 1)_q := \prod_{k=1}^{q} (\alpha + k) \text{ for } q \in \mathcal{N}, \ := 1 \text{ for } q = 0. \tag{12}$$

Our main theorem is:

THEOREM 1. *Let* $s = \max\{3, \lceil \alpha + q \rceil\}$, $\lceil\ \rceil$ *being the ceiling function, and* $T_n(f, r, x)$ *as in* (1). *If* $\{X_{nj},\ j = 1, 2, \dots;\ n \in \mathcal{N}\}$, $\{\beta_n,\ n \in \mathcal{N}\}$, *and* $r(t)$ *satisfy*:

(i) $\lim_{n\to\infty} \Psi_n = 0;$ (13)

(ii) $\lim_{n\to\infty} \dfrac{n\alpha_n\beta_n - x}{\Psi_n} = 0;$ (14)

(iii) $\lim_{n\to\infty} \rho_{sn}\sigma_n^{-s} n^{-(s-2)/2} = 0;$ (15)

(iv) $\lim_{n\to\infty} \rho_{3n}\sigma_n^{-3} n^{-1/2} \log n = 0;$ (16)

(v) $r(t) \in C^1((\bar{A}, \bar{B}))$, and $r(t)$ satisfies one of the following conditions:

(v1) $|r'(t)| \le M_r$ for some constant $M_r > 0$ independent of $n$; or (17)

(v2) $\lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \int_{|t-x| > \Psi_n^{\delta}} |r(t) - r(x)|^{\alpha+q}\, dP(\beta_n S_n \le t) = 0$, for some $\delta$ with $1 > \delta > 0$; (18)

then it holds that

$$\lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \sup_{f\in W^{(q)}(C(J);\,\alpha;\,1)} \times \left| T_n\left( f(r(t)) - \sum_{k=0}^{q} \frac{f^{(k)}(r(x))}{k!} (r(t) - r(x))^k, r, x \right) \right| \tag{19}$$
$$= \begin{cases} \dfrac{|r'(x)|^{\alpha+q}}{\sqrt{\pi}\,(\alpha+1)_q} 2^{(\alpha+q)/2} \Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \textit{when } q \textit{ is even}, \\[2ex] \dfrac{|r'(x)|^{\alpha+q}}{\sqrt{\pi}\,(\alpha+1)_q} 2^{(3\alpha+q-2)/2} \Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \textit{when } q \textit{ is odd}. \end{cases}$$

When $r(t) = t$, $\beta_n = 1/n$, and $X_{nj} \overset{\text{i.i.d.}}{\sim} X_{11}$ with $E|X_{11}|^s < \infty$, the assumptions (i)–(iv) and (v1) of Theorem 1 hold automatically. So the specification of Theorem 1 gives the following result about the Feller operator (2).

THEOREM 2. *Suppose* $EX_{11} = x \in J$, $\mathrm{Var}(X_{11}) = \sigma^2(x) > 0$, *and the Feller operator* $F_n(f, x)$ *is defined as in* (2). *Further assume that* $E|X_{11}|^s < \infty$, *where s is as in Theorem* 1. *Then it holds that*

$$\lim_{n\to\infty} n^{(\alpha+q)/2} \sup_{f\in W^{(q)}(C(J);\,\alpha;1)} \left| F_n\left( f(t) - \sum_{k=0}^{q} \frac{f^{(k)}(x)}{k!}(t-x)^k, x\right)\right|$$

$$= \begin{cases} \dfrac{(\sigma(x))^{\alpha+q}}{\sqrt{\pi}(\alpha+1)_q} 2^{(\alpha+q)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is even}, \\ \dfrac{(\sigma(x))^{\alpha+q}}{\sqrt{\pi}(\alpha+1)_q} 2^{(3\alpha+q-2)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is odd}. \end{cases} \tag{20}$$

## 3. THE PROOF OF THEOREM 1

The following lemma on normal approximation plays a key role in the proof of Theorem 1.

LEMMA 1. *There are positive constants $c_i$ $(i = 1,2,3)$ depending only on s such that if $\rho_{sn} \le c_1 \sigma_n^s n^{(s-2)/2}$ holds for some integer $s \ge 3$, then for every real-valued Borel-measurable function g satisfying* $\sup_{y\in\mathscr{R}}\{(1 + |y|^r)^{-1}|g(y)|\} < \infty$ *for some integer r, $0 \le r \le s$, one has*

$$\left|\int g(t)\, d\big(Q_n(t) - \Phi_{\sigma^2}(t)\big)\right| \le c_2 \sup_{y\in\mathscr{R}} \left\{(1+|y|^r)^{-1}|g(\sigma_n y)|\right\}\left(1+\eta_n^{r+2}\right)$$
$$\times\left(\rho_{3n}\sigma_n^{-3}n^{-1/2} + \rho_{sn}\sigma_n^{-s}n^{-(s-2)/2}\right)$$
$$+ \int_{-\infty}^{\infty} 2\,\omega_g(t, \sigma_n\eta_n)\, d\Phi_{\sigma_n^2}(t), \tag{21}$$

*where* $\eta_n = c_3\,\rho_{3n}\sigma_n^{-3}([\log n] + 1)n^{-1/2}$, $[\log n]$ *being the integer part of* $\log n$, *and* $\omega_g(t,\varepsilon) := \sup\{|g(y) - g(t)|;\ |y - t| < \varepsilon\}$.

*Proof.* This follows from Corollary 15.5 on page 158 of [1] if one uses (16.3), (16.8), (16.9), the first line of (16.13), (16.16), and (16.28) (cf. also [1, p. 164, Corollary 16.5]). Q.E.D.

LEMMA 2. *Under the assumptions of Theorem* 1 *we have*

$$\lim_{n\to\infty} \sup_{f\in W^{(q)}(C(J);\,\alpha;1)} \Psi_n^{-(\alpha+q)}$$
$$\left|\int_{-\infty}^{\infty} f_q^*\left(\sqrt{n}\,\beta_n t + \alpha_n\,\beta_n\right)\chi_{[A',B']}(t)\, d\big(Q_n(t) - \Phi_{\sigma_n^2}(t)\big)\right| = 0, \tag{22}$$

*where $A'$ and $B'$ are*

$$\left.\begin{aligned} A' &= \sqrt{n}\,(A-\alpha_n) \\ B' &= \sqrt{n}\,(B-\alpha_n) \end{aligned}\right\} \qquad \text{if (v1) is satisfied;} \tag{23}$$

$$\left.\begin{aligned} A' &= \left(x - n\alpha_n\beta_n - \Psi_n^{\delta}\right)/\left(\sqrt{n}\,\beta_n\right) \\ B' &= \left(x - n\alpha_n\beta_n + \Psi_n^{\delta}\right)/\left(\sqrt{n}\,\beta_n\right) \end{aligned}\right\} \qquad \text{if (v1) fails and (v2) holds.} \tag{24}$$

*Proof.* Let $r_n(t) = (r(\sqrt{n}\,\beta_n t + n\alpha_n\beta_n) - r(x))\chi_{[A',B']}(t)$ (with the understanding that $r_n(t) = 0$ when $t \notin [A', B']$ whether $r(\cdot)$ is meaningful or not, and this convention applies to other, similar situations too) then

$$|r_n(t)| = \sqrt{n}\,\beta_n\left|t + \frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right||r'(\xi_n)|\chi_{[A',B']}(t),$$

where $\xi_n$ lies between $x$ and $\sqrt{n}\,\beta_n t + n\alpha_n\beta_n$.

If assumption (v1) is satisfied then

$$|r_n(t)| \le M_r\sqrt{n}\,\beta_n\left|t + \frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\chi_{[A',B']}(t), \tag{25}$$

where $A', B'$ are as in (23). Otherwise (v2) should be satisfied, so $A'$ and $B'$ assume the values as in (24). Since $r_n(t) = 0$ when $t \notin [A', B']$, we need only consider those $t$'s such that $A' \le t \le B'$, i.e., $(x - n\alpha_n\beta_n - \Psi_n^{\delta})/(\sqrt{n}\,\beta_n) \le t \le (x - n\alpha_n\beta_n + \Psi_n^{\delta})/(\sqrt{n}\,\beta_n)$, which leads to $|\sqrt{n}\,\beta_n t + n\alpha_n\beta_n - x| \le \Psi_n^{\delta} \to 0$, as $n \to \infty$, by assumption (i). So when $n$ is large enough $\xi_n$ falls into some fixed closed neighborhood of $x$, say, $\overline{B(x,\epsilon)} \subset [A, B]$, and

$$r'(\xi_n) \le M_\epsilon := \sup_{t \in \overline{B(x,\epsilon)}} |r'(t)| < \infty. \tag{26}$$

Thus for sufficiently large $n$ it holds that

$$|r_n(t)| \le M_\epsilon\sqrt{n}\,\beta_n\left|t + \frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\chi_{[A',B']}(t). \tag{27}$$

By (25) and (27), when $n$ is large enough, we have

$$|r_n(t)| \le M\sqrt{n}\,\beta_n\left|t + \frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\chi_{[A',B']}(t) \tag{28}$$

for some constant $M$ independent of $n$ as long as assumption (v) is satisfied.

Now in Lemma 1 assume for $f \in W^{(q)}(C(J); \alpha; 1)$ that $g(t) = f_q^*(\sqrt{n}\,\beta_n t + n\alpha_n\beta_n)\chi_{[A', B']}(t)$ and $r = s$; then by (21) we have

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} f_q^*\big(\sqrt{n}\,\beta_n t + n\alpha_n\beta_n\big)\chi_{[A', B']}(t)\,d\big(Q_n(t) - \Phi_{\sigma_n^2}(t)\big)\right| \\
&\quad \le c_2 \sup_{y\in\mathscr{R}}\Big\{\big(1 + |y|^s\big)^{-1}\big|g(\sigma_n y)\big|\Big\}\big(1 + \eta_n^{s+2}\big) \\
&\qquad \times\big(\rho_{3n}\sigma_n^{-3}n^{-1/2} + \rho_{sn}\sigma_n^{-s}n^{-(s-2)/2}\big) \\
&\qquad + \int_{-\infty}^{\infty} 2\,\omega_g(t, \sigma_n\eta_n)\,d\Phi_{\sigma_n^2}(t), \\
&\quad =: I_1 + I_2,
\end{aligned} \tag{29}$$

observing that assumption (iii) of Theorem 1 fulfills the stipulation of Lemma 1.

By (11), (12), (28), (6), the inequality $|a+b|^s \le 2^s(|a|^s + |b|^s)$, and assumption (ii) we have for large $n$ that $|g(\sigma_n t)| \le |r_n(\sigma_n t)|^{\alpha+q}\chi_{[A', B']}(\sigma_n t)/(\alpha+1)_q \le M_1\Psi_n^{\alpha+q}(|t|^s + 1)$, where $M_1 = 2^s M^{\alpha+q}/(\alpha+1)_q$, and thus

$$I_1 \le M_1\Psi_n^{\alpha+q}c_2\big(1 + \eta_n^{s+2}\big)\big(\rho_{3n}\sigma_n^{-3}n^{-1/2} + \rho_{sn}\sigma_n^{-s}n^{-(s-2)/2}\big). \tag{30}$$

Note that the right hand side of (30) is independent of $f$, and we have

$$\lim_{n\to\infty}\ \sup_{f\in W^{(q)}(C(J);\,\alpha;\,1)}\ \Psi_n^{-(\alpha+q)}I_1 = 0, \tag{31}$$

by assumptions (iii) and (iv).

In order to estimate $I_2$, we write

$$f_q^{**}(t) = \begin{cases} \displaystyle\int_0^1 \frac{(1-\tau)^{q-1}}{(q-1)!}\big\{f^{(q)}(r_n(t)\tau + r(x)) & \\ -f^{(q)}(r(x))\big\}\,d\tau\,\chi_{[A', B']}(t), & q \in \mathscr{N}, \\ \big(f(r(t)) - f(r(x))\big)\chi_{[A', B']}(t), & q = 0. \end{cases} \tag{32}$$

Then $g(t) = (r_n(t))^q f_q^{**}(t)$ and for $t \in [A', B']$ it holds that

$$
\begin{aligned}
|g(y) - g(t)| &\le |g(y) - g(t)|\chi_{[A',B']}(y) + |g(t)|\big(\chi_{[A',A'+|y-t|]}(t) \\
&\qquad + \chi_{[B'-|y-t|,B']}(t)\big) \\
&\le \left|(r_n(y))^q - (r_n(t))^q\right| \left|f_q^{**}(y)\right| + |r_n(t)|^q \big| f_q^{**}(y) \\
&\qquad - f_q^{**}(t)\big|\chi_{[A',B']}(y) \\
&\quad + |g(t)|\big(\chi_{[A',A'+|y-t|]}(t) + \chi_{[B'-|y-t|,B']}(t)\big) \\
&=: I_{21} + I_{22} + I_{23}.
\end{aligned} \tag{33}
$$

Here $I_{21} = 0$ if $q = 0$. When $q \ge 1$ by the Mean Value Theorem we get for large $n$ that

$$
\begin{aligned}
I_{21} &= \left|q(r_n(\zeta_n))^{q-1}\sqrt{n}\,\beta_n r'(\zeta_n)(y-t)\right| \left|(r_n(y))^\alpha \int_0^1 \frac{(1-\tau)^{q-1}}{(q-1)!}\tau^\alpha\,d\tau\right| \\
&\qquad (\zeta_n \text{ lies between } t \text{ and } y.) \\
&\le qM'M^{\alpha+q-1}\big(\sqrt{n}\,\beta_n\big)^{\alpha+q}\left(|t| + |y-t| + \left|\frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\right)^{\alpha+q-1} \frac{|y-t|}{(\alpha+1)_q} \\
&\qquad \big(\text{By (28), (17) (or (26) if (v1) fails) and noting that } |\zeta_n| \\
&\qquad \text{and } |y| \text{ are less than } |t| + |y-t|. \text{ Here } M' = M_r, \text{ if (v1) holds;} \\
&\qquad M' = M_\epsilon, \text{ otherwise.}\big) \\
&= K_1\big(\sqrt{n}\,\beta_n\big)^{\alpha+q}\left(|t| + |y-t| + \left|\frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\right)^{\alpha+q-1}|y-t|,
\end{aligned} \tag{34}
$$

where the constant $K_1$ is independent of $f$, $n$, and $t$.

Likewise, we can show for large $n$ that

$$
\begin{aligned}
I_{22} &\le |r_n(t)|^q |r_n(y) - r_n(t)|^\alpha/(\alpha+1)_q \\
&= |r_n(t)|^q \left|\sqrt{n}\,\beta_n r'(\zeta_n')(y-t)\right|^\alpha/(\alpha+1)_q \qquad (\zeta_n' \text{ lies between } t \text{ and } y) \\
&\le K_2\big(\sqrt{n}\,\beta_n\big)^{\alpha+q}\left(|t| + \left|\frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\right)^q |y-t|^\alpha,
\end{aligned} \tag{35}
$$

where the constant $K_2$ is independent of $f$, $n$, and $t$.

Furthermore, it holds for large $n$ that

$$
\begin{aligned}
I_{23} &\le \left|r_n(t)\right|^{\alpha+q} \frac{1}{(\alpha+1)_q}\left(\chi_{[A', A'+|y-t|]}(t) + \chi_{[B'-|y-t|, B']}(t)\right) \\
&\le K_3\left(\sqrt{n}\,\beta_n\right)^{\alpha+q}\left(|t| + \left|\frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\right)^{\alpha+q}\left(\chi_{[A', A'+|y-t|]}(t)\right. \\
&\qquad\left. + \chi_{[B'-|y-t|, B']}(t)\right), \qquad (36)
\end{aligned}
$$

where the constant $K_3$ is independent of $f$, $n$, and $t$.

Combining (33), (34), (35), and (36) we have for large $n$ that

$$
\begin{aligned}
\omega_g(t, \sigma_n\eta_n) &= \sup_{|y-t|<\sigma_n\eta_n} \left|g(y) - g(t)\right| \\
&\le \bar{\delta}_{q0} K_1\left(\sqrt{n}\,\beta_n\right)^{\alpha+q}\left(|t| + \sigma_n\eta_n + \left|\frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\right)^{\alpha+q-1}\sigma_n\eta_n \\
&\quad + K_2\left(\sqrt{n}\,\beta_n\right)^{\alpha+q}\left(|t| + \left|\frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\right)^{q}(\sigma_n\eta_n)^{\alpha} \\
&\quad + K_3\left(\sqrt{n}\,\beta_n\right)^{\alpha+q}\left(|t| + \left|\frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\right)^{\alpha+q} \\
&\quad \times\left(\chi_{[A', A'+\sigma_n\eta_n]}(t) + \chi_{[B'-\sigma_n\eta_n, B']}(t)\right),
\end{aligned}
$$

where $\bar{\delta}_{q0} = 0$ for $q = 0$, $= 1$ otherwise. Thus

$$
\begin{aligned}
\Psi_n^{-(\alpha+q)} I_2 \le 2\sigma_n^{-(\alpha+q)}\int_{-\infty}^{\infty} &\left\{\bar{\delta}_{q0} K_1\left(|t| + \sigma_n\eta_n + \left|\frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\right)^{\alpha+q-1}\sigma_n\eta_n\right. \\
&+ K_2\left(|t| + \left|\frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\right)^{q}(\sigma_n\eta_n)^{\alpha} \\
&+ K_3\left(|t| + \left|\frac{n\alpha_n\beta_n - x}{\sqrt{n}\,\beta_n}\right|\right)^{\alpha+q} \\
&\left.\times\left(\chi_{[A', A'+\sigma_n\eta_n]}(t) + \chi_{[B'-\sigma_n\eta_n, B']}(t)\right)\right\}\frac{1}{\sqrt{2\pi}\,\sigma_n}e^{-t^2/(2\sigma_n^2)}\,dt
\end{aligned}
$$

$$\leq \int_{-\infty}^{\infty} 2\left\{\bar{\delta}_{q0} K_1\left(|t| + \eta_n + \left|\frac{n\alpha_n \beta_n - x}{\Psi_n}\right|\right)^{\alpha+q-1} \eta_n\right.$$

$$+K_2\left(|t| + \left|\frac{n\alpha_n \beta_n - x}{\Psi_n}\right|\right)^{q} \eta_n^{\alpha} + K_3\left(|t| + \left|\frac{n\alpha_n \beta_n - x}{\Psi_n}\right|\right)^{\alpha+q}$$

$$\left.\times\left( \chi_{[A', A'+\sigma_n\eta_n]}(\sigma_n t) + \chi_{[B'-\sigma_n\eta_n, B']}(\sigma_n t)\right)\right\}$$

$$\times \frac{1}{\sqrt{2\pi}} e^{t^2/2}\, dt. \tag{37}$$

By assumption (iv) we have $\eta_n \to 0$ as $n \to \infty$ and consequently $\chi_{[A', A'+\sigma_n\eta_n]}(\sigma_n t) \to 0$ and $\chi_{[B'-\sigma_n\eta_n, B']}(\sigma_n t) \to 0$ as $n \to \infty$. Those facts together with assumption (ii) imply that the above integrand goes to 0 as $n \to \infty$ for each $t$. Furthermore we can take, e.g., $2\{\bar{\delta}_{g0} K_1(|t| + 2)^{\alpha+q-1} + K_2(|t| + 1)^q + 2K_3(|t| + 1)^{\alpha+q}\}e^{-t^2/2}/\sqrt{2\pi}$ as the dominating function. Therefore the last integral goes to 0 as $n \to \infty$ by Lebesgue's Convergence Theorem. Since the right-most part of the last inequality (37) is independent of $f$, it holds that

$$\lim_{n\to\infty} \sup_{f\in W^{(q)}(C(J);\, \alpha;\, 1)} \Psi_n^{-(\alpha+q)} I_2 = 0. \tag{38}$$

Finally, (22) follows from (29), (31), and (38), and the proof is complete. Q.E.D.

LEMMA 3. *Under the assumptions of Theorem* 1 *we have*

$$\lim_{n\to\infty} \sup_{f\in W^{(q)}(C(J);\, \alpha;\, 1)} \Psi_n^{-(\alpha+q)}\left|\int_{-\infty}^{\infty} f_q^*\left(\sqrt{n}\,\beta_n t + \alpha_n \beta_n\right)\chi_{[A', B']}(t)\, d\Phi_{\sigma_n^2}(t)\right|$$

$$= \begin{cases} \dfrac{|r'(x)|^{\alpha+q}}{\sqrt{\pi}\,(\alpha+1)_q} 2^{(\alpha+q)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is even}, \\ \dfrac{|r'(x)|^{\alpha+q}}{\sqrt{\pi}\,(\alpha+1)_q} 2^{(3\alpha+q-2)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is odd}, \end{cases} \tag{39}$$

*where $A'$ and $B'$ are as in Lemma* 2.

*Proof.* For $f \in W^{(q)}(C(J); \alpha; 1)$ write

$$P_n(f) = \Psi_n^{-(\alpha+q)} \left| \int_{-\infty}^{\infty} f_q^*\left(\sqrt{n}\,\beta_n t + n\alpha_n \beta_n\right) \chi_{[A', B']}(t) \frac{1}{\sqrt{2\pi}\,\sigma_n} e^{-t^2/(2\sigma_n^2)}\, dt \right|. \tag{40}$$

We proceed with our proof according to the parity of $q$.

• *Case* 1: $q$ even. Consider $h_{\alpha,q,x}(t) := |t - r(x)|^{\alpha+q}/(\alpha+1)_q$, $0 < \alpha \le 1$; we have $h^{(q)}_{\alpha,q,x}(t) = |t - r(x)|^{\alpha}$ and $h_{\alpha,q,x}(t) \in W^{(q)}(C(J); \alpha; 1)$ by Lemma 1 of [14]. It is easy to see that (cf. [12])

$$\sup_{f \in W^{(q)}(C(J);\, \alpha;\, 1)} P_n(f) = P_n\left(h_{\alpha,q,x}\right). \tag{41}$$

Moreover

$$\begin{aligned} P_n\left(h_{\alpha,q,x}\right) &= \int_{-\infty}^{\infty} \Psi_n^{-(\alpha+q)} \left| r\left(\sqrt{n}\,\sigma_n \beta_n t + n\alpha_n \beta_n\right) - r(x) \right|^{\alpha+q} \chi_{[A', B']}(\sigma_n t) \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \frac{1}{(\alpha+1)_q} \\ &= \int_{-\infty}^{\infty} \Psi_n^{-(\alpha+q)} \left| r'(\xi_n)\left(\sqrt{n}\,\sigma_n \beta_n t + n\alpha_n \beta_n - x\right) \right|^{\alpha+q} \chi_{[A', B']}(\sigma_n t) \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \frac{1}{(\alpha+1)_q}. \end{aligned}$$

Here $\xi_n$ lies between $x$ and $\sqrt{n}\,\sigma_n \beta_n t + n\alpha_n \beta_n$, and $\xi_n \to x$ as $n \to \infty$ for each fixed $t$ by assumption (ii). Also by (ii), $\lim_{n\to\infty} \chi_{[A', B']}(\sigma_n t) = 1$, $\forall t \in \mathscr{R}$ for both choices of the pair $A'$ and $B'$ in (23) and (24). Meanwhile by (17) (or (26) if (v1) fails) and (ii) the above integrand is dominated by $M(|t|+1)^{\alpha+q} e^{-t^2/2}/\sqrt{2\pi}$ for large $n$, where $M = M_r$ if (v1) holds, $= M_\epsilon$ (as in (26)) if (v1) fails. Now Lebesgue's Convergence Theorem yields

$$\begin{aligned} \lim_{n\to\infty} P_n\left(h_{\alpha,q,x}\right) &= \frac{|r'(x)|^{\alpha+q}}{(q-1)!} \int_{-\infty}^{\infty} |t|^{\alpha+q} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \frac{1}{(\alpha+1)_q} \\ &= \frac{|r'(x)|^{\alpha+q}}{\sqrt{\pi}\,(\alpha+1)_q} 2^{(\alpha+q)/2} \Gamma\left(\frac{\alpha+q+1}{2}\right); \end{aligned} \tag{42}$$

therefore (39) follows for even $q$ in view of (41).

• *Case* 2: $q$ odd. Consider $k_{\alpha,q,x}(t) := 2^{\alpha-1}|t-r(x)|^{\alpha+q}/(\alpha+1)_q$, $0<\alpha\le 1$; we have $k_{\alpha,q,x}^{(q)}(t) = 2^{\alpha-1}\operatorname{sgn}(t-r(x))|t-r(x)|^{\alpha}$ and $k_{\alpha,q,x}(t) \in W^{(q)}(C(J);\alpha;1)$ by Lemma 1 of [14]. Hence

$$\begin{aligned}\lim_{n\to\infty} &\sup_{f\in W^{(q)}(C(J);\,\alpha;1)} P_n(f)\\ &\ge \lim_{n\to\infty} P_n\big(k_{\alpha,q,x}\big)\\ &\ge \lim_{n\to\infty} 2^{\alpha-1}\int_{-\infty}^{\infty}\Big|r\big(\sqrt{n}\,\beta_n + n\alpha_n\beta_n\big) - r(x)\Big|^{\alpha+q}\\ &\quad \times \chi_{[A',B']}(t)\,dQ_n(t)\,\frac{1}{(\alpha+1)_q}\\ &= \frac{|r'(x)|^{\alpha+q}}{\sqrt{\pi}\,(\alpha+1)_q}2^{(3\alpha+q-2)/2}\Gamma\left(\frac{\alpha+q+1}{2}\right)\quad (\text{cf. }(42)).\end{aligned}\tag{43}$$

On the other hand, for any $f\in W^{(q)}(C(J);\alpha;1)$, it holds that

$$\begin{aligned}P_n(f) &\le \left|\int_{-\infty}^{\infty}\Psi_n^{-(\alpha+q)}f_q^*\big(\sqrt{n}\,\beta_n t + n\alpha_n\beta_n\big)\chi_{[-C',C']}(t)\,d\Phi_{\sigma_n^2}(t)\right|\\ &\quad+\left|\int_{-\infty}^{\infty}\Psi_n^{-(\alpha+q)}f_q^*\big(\sqrt{n}\,\beta_n t + n\alpha_n\beta_n\big)\chi_{[A',-C']}(t)\,d\Phi_{\sigma_n^2}(t)\right|\\ &\quad+\left|\int_{-\infty}^{\infty}\Psi_n^{-(\alpha+q)}f_q^*\big(\sqrt{n}\,\beta_n t + n\alpha_n\beta_n\big)\chi_{[C',B']}(t)\,d\Phi_{\sigma_n^2}(t)\right|\\ &:= J_1+J_2+J_3,\end{aligned}\tag{44}$$

where $C' = \min\{-A', B'\}$. Moreover

$$\begin{aligned}J_1 &= \bigg|\int_0^{\infty}\Psi_n^{-(\alpha+q)}\Big\{f_q^*\big(\sqrt{n}\,\sigma_n\beta_n t + n\alpha_n\beta_n\big) + f_q^*\big(-\sqrt{n}\,\sigma_n\beta_n t + n\alpha_n\beta_n\big)\Big\}\\ &\qquad\qquad \times\chi_{[0,C']}(\sigma_n t)\,\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt\bigg|\\ &\le \bigg|\int_0^{\infty}\Psi_n^{-(\alpha+q)}\Big(r\big(\sqrt{n}\,\sigma_n\beta_n t + n\alpha_n\beta_n\big) - r(x)\Big)^q\int_0^1\frac{(1-\tau)^{q-1}}{(q-1)!}\\ &\qquad \times\Big\{f^{(q)}\Big(\Big(r\big(\sqrt{n}\,\sigma_n\beta_n t + n\alpha_n\beta_n\big) - r(x)\Big)\tau + r(x)\Big)\end{aligned}$$

$$-f^{(q)}\Big(\Big(r\big(-\sqrt{n}\,\sigma_n\,\beta_n t + n\alpha_n\,\beta_n\big) - r(x)\Big)\tau + r(x)\Big)\Big\}\,d\tau$$

$$\times\chi_{[0,C']}(\sigma_n t)\,\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt\Bigg|$$

$$+\Bigg|\int_0^\infty \Psi_n^{-(\alpha+q)}\Big\{\Big(r\big(\sqrt{n}\,\sigma_n\,\beta_n t + n\alpha_n\,\beta_n\big) - r(x)\Big)^q$$

$$+\Big(r\big(-\sqrt{n}\,\sigma_n\,\beta_n t + n\alpha_n\,\beta_n\big) - r(x)\Big)^q\Big\}$$

$$\times\int_0^1 \frac{(1-\tau)^{q-1}}{(q-1)!}\Big\{f^{(q)}\Big(\Big(r\big(\sqrt{n}\,\sigma_n\,\beta_n t + n\alpha_n\,\beta_n\big) - r(x)\Big)\tau + r(x)\Big)$$

$$-f^{(q)}(r(x))\Big\}$$

$$\times d\tau\,\chi_{[0,C']}(\sigma_n t)\,\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt\Bigg| =: J_{11} + J_{12}. \qquad (45)$$

By a similar argument leading to (42) we have

$$\lim_{n\to\infty} J_{11} \le \lim_{n\to\infty}\int_0^\infty \Psi_n^{-(\alpha+q)}\Big|r\big(\sqrt{n}\,\sigma_n\,\beta_n t + n\alpha_n\,\beta_n\big) - r(x)\Big|^q\int_0^1 \frac{(1-\tau)^{q-1}}{(q-1)!}$$

$$\times\Big|r\big(\sqrt{n}\,\sigma_n\,\beta_n t + n\alpha_n\,\beta_n\big) - r\big(-\sqrt{n}\,\sigma_n\,\beta_n t + n\alpha_n\,\beta_n\big)\Big|^\alpha \tau^\alpha\,d\tau$$

$$\times\chi_{[0,C']}(\sigma_n t)\,\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt$$

$$= \lim_{n\to\infty}\int_0^\infty |r'(\xi_n)|^q\left|t + \frac{n\alpha_n\,\beta_n - x}{\Psi_n}\right|^q |2r'(\xi_n')t|^\alpha$$

$$\times\chi_{[0,C']}(\sigma_n t)\,\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt\,\frac{1}{(\alpha+1)_q}$$

$\Big($Here $\xi_n$ lies between $x$ and $\sqrt{n}\,\sigma_n\,\beta_n t + n\alpha_n\,\beta_n$, and $\xi_n'$ lies between $\sqrt{n}\,\sigma_n\,\beta_n t + n\alpha_n\,\beta_n$ and $-\sqrt{n}\,\sigma_n\,\beta_n t + n\alpha_n\,\beta_n$ for $t\in[0,C']$.$\Big)$

$$= 2^\alpha|r'(x)|^{\alpha+q}\int_0^\infty t^{\alpha+q}\,\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt\,\frac{1}{(\alpha+1)_q}$$

$$= \frac{|r'(x)|^{\alpha+q}}{\sqrt{\pi}(\alpha+1)_q}2^{(3\alpha+q-2)/2}\Gamma\left(\frac{\alpha+q+1}{2}\right) \qquad (46)$$

and

$$\lim_{n\to\infty} J_{12} \le \lim_{n\to\infty}\int_0^\infty \Psi_n^{-(\alpha+q)}\Big|\Big(r\big(\sqrt{n}\,\sigma_n\beta_n t + n\alpha_n\beta_n\big) - r(x)\Big)^q$$

$$+\Big(r\big(-\sqrt{n}\,\sigma_n\beta_n t + n\alpha_n\beta_n\big) - r(x)\Big)^q\Big|$$

$$\times\int_0^1 \frac{(1-\tau)^{q-1}}{(q-1)!}\Big|r\big(\sqrt{n}\,\sigma_n\beta_n t + n\alpha_n\beta_n\big)$$

$$-r(x)\Big|^\alpha \chi_{[0,C']}(\sigma_n t)\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt$$

$$= \lim_{n\to\infty}\int_0^\infty \left|\left(t + \frac{n\alpha_n\beta_n - x}{\Psi_n}\right)^q (r'(\xi_n))^q\right.$$

$$\left.-\left(t - \frac{n\alpha_n\beta_n - x}{\Psi_n}\right)^q (r'(\xi_n'))^q\right|$$

$$\times\left|t + \frac{n\alpha_n\beta_n - x}{\Psi_n}\right|^\alpha |r'(\xi_n)|^\alpha \chi_{[0,C']}(\sigma_n t)\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\,dt|$$

(Here $\xi_n$ lies between $x$ and $\sqrt{n}\,\sigma_n\beta_n t + n\alpha_n\beta_n$, $\xi_n'$ lies between $x$ and $-\sqrt{n}\,\sigma_n\beta_n t + n\alpha_n\beta_n$ for $t \in [0, C']$.)

$$= 0. \tag{47}$$

From (45), (46), and (47) we derive that

$$\lim_{n\to\infty} J_1 \le \frac{|r'(x)|^{\alpha+q}}{\sqrt{\pi}(\alpha+1)_q} 2^{(3\alpha+q-2)/2}\Gamma\left(\frac{\alpha+q+1}{2}\right). \tag{48}$$

Since $\lim_{n\to\infty}\chi_{[A', -C']}(\sigma_n t) = 0$, and $\lim_{n\to\infty}\chi_{[C', B']}(\sigma_n t) = 0$, $\forall t \in \mathscr{R}$, by Lebesgue's Convergence Theorem again, we have $\lim_{n\to\infty} J_2 = 0$ and $\lim_{n\to\infty} J_3 = 0$. These facts together with (44) and (48) imply that

$$\lim_{n\to\infty}\sup_{f\in W^{(q)}(C(J);\,\alpha;\,1)} P_n(f) \le \frac{|r'(x)|^{\alpha+q}}{\sqrt{\pi}(\alpha+1)_q} 2^{(3\alpha+q-2)/2}\Gamma\left(\frac{\alpha+q+1}{2}\right). \tag{49}$$

Now (39) follows from (43) and (49) for odd $q$, and the proof is complete. Q.E.D.

*Proof of Theorem* 1. Denote the left hand side and the right hand side of (19) by $L$ and $R$, respectively. With the notations presented at the beginning of Section 2 we have

$$L = \lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \sup_{f\in W^{(q)}(C(J);\,\alpha;1)} \left| \int_{-\infty}^{\infty} f_q(t)\, dP(\beta_n S_n \le t) \right|$$

$$= \lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \sup_{f\in W^{(q)}(C(J);\,\alpha;1)} \left| \int_{-\infty}^{\infty} f_q^*(t)\, dP(\beta_n S_n \le t) \right|$$

$$\text{(by Taylor's formula)}.$$

We want to show $L \le R$ first.

If assumption (v1) is satisfied then, by substitution, it holds that

$$L = \lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \sup_{f\in W^{(q)}(C(J);\,\alpha;1)} \times \left| \int_{-\infty}^{\infty} f_q^*\left(\sqrt{n}\,\beta_n t + n\alpha_n \beta_n\right) \chi_{[A',\,B']}(t)\, dQ_n(t) \right|$$

$$\left(\text{Here } A' = \sqrt{n}\,(A - \alpha_n) \text{ and } B' = \sqrt{n}\,(B - \alpha_n).\right)$$

$$\le \lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \sup_{f\in W^{(q)}(C(J);\,\alpha;1)} \times \left| \int_{-\infty}^{\infty} f_q^*\left(\sqrt{n}\,\beta_n t + n\alpha_n \beta_n\right) \chi_{[A',\,B']}(t)\, d\Phi_{\sigma_n^2}(t) \right|$$

$$+ \lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \sup_{f\in W^{(q)}(C(J);\,\alpha;1)} \left| \int_{-\infty}^{\infty} f_q^*\left(\sqrt{n}\,\beta_n t + n\alpha_n \beta_n\right) \chi_{[A',\,B']}(t) \times d\left(Q_n - \Phi_{\sigma_n^2}\right)(t) \right| = R,$$

observing that in the penultimate part the first term is $R$ and the second term is 0 by Lemma 3 and Lemma 2, respectively.

If assumption (v1) fails then (v2) should be satisfied. Now we have

$$L \le \lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \sup_{f\in W^{(q)}(C(J);\,\alpha;1)} \left| \int_{|t-x|>\Psi_n^{\delta}} f_q^*(t)\, dP(\beta_n S_n \le t) \right|$$

$$+ \lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \sup_{f\in W^{(q)}(C(J);\,\alpha;1)} \left| \int_{|t-x|\le\Psi_n^{\delta}} f_q^*(t)\, dP(\beta_n S_n \le t) \right|$$

$$=: L_1 + L_2, \tag{50}$$

$$L_1 \le \lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \int_{|t-x|>\Psi_n^{\delta}} |r(t) - r(x)|^{\alpha+q} \frac{1}{(\alpha+1)_q}\, dP(\beta_n S_n \le t)$$

$$= 0, \tag{51}$$

by assumption (v2), and

$$
\begin{aligned}
L \le\ & \lim_{n\to\infty} \sup_{f\in W^{(q)}(C(J);\,\alpha;\,1)} \Psi_n^{-(\alpha+q)} \\
&\times \left| \int_{-\infty}^{\infty} f_q^*\left(\sqrt{n}\,\beta_n t + n\alpha_n\,\beta_n\right) \chi_{[A',\,B']}(t)\, d\Phi_{\sigma_n^2}(t) \right| \\
&+ \lim_{n\to\infty} \sup_{f\in W^{(q)}(C(J);\,\alpha;\,1)} \Psi_n^{-(\alpha+q)} \\
&\times \left| \int_{-\infty}^{\infty} f_q^*\left(\sqrt{n}\,\beta_n t + n\alpha_n\,\beta_n\right) \chi_{[A',\,B']}(t)\, d\left(Q_n(t) - \Phi_{\sigma_n^2}(t)\right) \right| \\
&\qquad (\text{Here } A' \text{ and } B' \text{ take the values as in } (24).) \\
=\ & R,
\end{aligned}
\tag{52}
$$

By Lemmas 3 and 2 again. It follows from (50), (51), and (52) that $L \le R$ is also true in the case in which assumption (v1) fails while (v2) holds.

It remains to show $L \ge R$.

Define function $l_{(\alpha,q,x)}(t) := h_{(\alpha,q,x)}(t)$ when $q$ is even, $:= k_{(\alpha,q,x)}(t)$ when $q$ is odd. As shown in the proof of Lemma 3 we have $l_{(\alpha,q,x)}(t) \in W^{(q)}(C(J);\alpha;1)$, thus

$$
\begin{aligned}
L \ge\ & \lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \left| \int_{-\infty}^{\infty} l_{\alpha,q,x}\left(\sqrt{n}\,\beta_n t + n\alpha_n\,\beta_n\right) \chi_{[A',\,B']}(t)\, dQ_n(t) \right| \\
&\qquad (\text{Here } A' \text{ and } B' \text{ are as in Lemma 2.}) \\
\ge\ & \lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \left| \int_{-\infty}^{\infty} l_{\alpha,q,x}\left(\sqrt{n}\,\beta_n t + n\alpha_n\,\beta_n\right) \chi_{[A',\,B']}(t)\, d\Phi_{\sigma_n^2}(t) \right| \\
&- \lim_{n\to\infty} \Psi_n^{-(\alpha+q)} \\
&\times \left| \int_{-\infty}^{\infty} l_{\alpha,q,x}\left(\sqrt{n}\,\beta_n t + n\alpha_n\,\beta_n\right) \chi_{[A',\,B']}(t)\, d\left(Q_n(t) - \Phi_{\sigma_n^2}(t)\right) \right| \\
=:\ & H_1 - H_2.
\end{aligned}
\tag{53}
$$

Now $H_1 \ge R$ by (42) when $q$ is even and by (43) when $q$ is odd, while $H_2 = 0$ by (22). Therefore $L \ge R$ and the proof of Theorem 1 is complete. Q.E.D.

## 4. SPECIAL CASES

When we specialize the quantities involved in the Trotter−Feller operator (1) and the Feller operator (2), we will get various concrete operators and their Lipschitz−Nikolskiĭ constants can be derived from Theorem 1 or Theorem 2 accordingly.

### 4.1. *Classical Operators*

We shall apply Theorem 2 first to a number of well-known classical operators. A sufficient condition for $E|X_{11}|^s < \infty$ in Theorem 2 is the existence of the moment generating function $E(e^{\delta X_{11}})$ for some $\delta > 0$. It happens to be the case for the operators listed below (cf. [10]).

| Operator | Distribution of $X_{11}$ | $\sigma^2(x)$ | $E(e^{\delta X_{11}})$ |
|---|---|---|---|
| Bernstein | Bernoulli | $x(1-x)\ (0 \le x \le 1)$ | $1 - x + xe^{\delta}$ |
| Szász | Poisson | $x\ (x \ge 0)$ | $e^{x(e^{\delta}-1)}$ |
| Gamma | Gamma | $x^2\ (x > 0)$ | $(1 - x\delta)^{-1}$ |
| Weierstrass | Normal | $1\ (-\infty < x < \infty)$ | $e^{x\delta + \delta^2/2}$ |
| Baskakov | Geometric $p = (1+x)^{-1}$ | $x(1+x)\ (x \ge 0)$ | $(1 + x - e^{\delta x})^{-1}$ |

By applying Theorem 2 we get the corresponding Lipschitz-Nikolskiĭ constants as follows.

| Operator | Order $\Psi_n$ | Lipschitz−Nikolskiĭ Constants $C(\alpha; q; x)$ |
|---|---|---|
| Bernstein | $n^{-(\alpha+q)/2}$ | $\begin{cases} (x(1-x))^{(\alpha+q)/2} 2^{(\alpha+q)/2} \Gamma\left(\frac{\alpha+q+1}{2}\right) \Big/ \left(\sqrt{\pi}(\alpha+1)_q\right) & q \text{ even} \\ (x(1-x))^{(\alpha+q)/2} 2^{(3\alpha+q-2)/2} \Gamma\left(\frac{\alpha+q+1}{2}\right) \Big/ \left(\sqrt{\pi}(\alpha+1)_q\right) & q \text{ odd} \end{cases}$ |
| Szász | $n^{-(\alpha+q)/2}$ | $\begin{cases} x^{(\alpha+q)/2} 2^{(\alpha+q)/2} \Gamma\left(\frac{\alpha+q+1}{2}\right) \Big/ \left(\sqrt{\pi}(\alpha+1)_q\right) & q \text{ even} \\ x^{(\alpha+q)/2} 2^{(3\alpha+q-2)/2} \Gamma\left(\frac{\alpha+q+1}{2}\right) \Big/ \left(\sqrt{\pi}(\alpha+1)_q\right) & q \text{ odd} \end{cases}$ |
| Gamma | $n^{-(\alpha+q)/2}$ | $\begin{cases} x^{\alpha+q} 2^{(\alpha+q)/2} \Gamma\left(\frac{\alpha+q+1}{2}\right) \Big/ \left(\sqrt{\pi}(\alpha+1)_q\right) & q \text{ even} \\ x^{\alpha+q} 2^{(3\alpha+q-2)/2} \Gamma\left(\frac{\alpha+q+1}{2}\right) \Big/ \left(\sqrt{\pi}(\alpha+1)_q\right) & q \text{ odd} \end{cases}$ |

$$\text{Weierstrass } n^{-(\alpha+q)/2}\begin{cases} 2^{(\alpha+q)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right)\Big/\left(\sqrt{\pi}(\alpha+1)_q\right) & q \text{ even} \\ 2^{(3\alpha+q-2)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right)\Big/\left(\sqrt{\pi}(\alpha+1)_q\right) & q \text{ odd} \end{cases}$$

$$\text{Baskakov } n^{-(\alpha+q)/2}\begin{cases} (x(1+x))^{(\alpha+q)/2}2^{(\alpha+q)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right)\Big/\left(\sqrt{\pi}(\alpha+1)_q\right) & q \text{ even} \\ (x(1+x))^{(\alpha+q)/2}2^{(3\alpha+q-2)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right)\Big/\left(\sqrt{\pi}(\alpha+1)_q\right) & q \text{ odd} \end{cases}$$

## 4.2. *Multiplier Enlargement*

The so-called "multiplier enlargement" technique has been employed to approximate unbounded functions by modifying existing operators with a sequence of positive multipliers $\{a_n\}_{n=1}^{\infty}$ (see [8, 5, 20]). We will demonstrate that our general settings are also applicable to the multiplier enlargement modifications of certain operators and their Lipschitz–Nikolskiĭ constants are obtained consequently.

EXAMPLE 4.2.1. In Theorem 1 let $r(t) = t$, $\beta_n = a_n/n$, and $X_{nj} \overset{\text{i.i.d.}}{\sim} X_{11}$, $\forall n, j$, where $X_{11}$ follows Bernoulli distribution: $P(X_{11} = 1) = x/a_n$, $P(X_{11} = 0) = 1 - x/a_n$, where $x$ is a parameter satisfying $0 < x/a_n < 1$ for all $n$. Then

$$\alpha_n = EX_{11} = \frac{x}{A_n}, \qquad \sigma_n^2(x) = \mathrm{Var}(X_{11}) = \frac{x}{a_n}\left(1 - \frac{x}{a_n}\right),$$

$$\rho_{sn} = E|X_{11} - EX_{11}|^s = \frac{x}{a_n}\left(1 - \frac{x}{a_n}\right)\left(\left(1 - \frac{x}{a_n}\right)^{s-1} + \left(\frac{x}{a_n}\right)^{s-1}\right)$$

and the corresponding Trotter–Feller operator (1) is reduced to the multiplier enlargement of Bernstein operator (cf. [8])

$$\overline{B}_n(f, x) = \sum_{k=0}^{n} f\left(\frac{a_n k}{n}\right)\binom{n}{k}\left(\frac{x}{a_n}\right)^k\left(1 - \frac{x}{a_n}\right)^k. \tag{54}$$

If further it is assumed that $\liminf_{n\to\infty} a_n > 0$ and $\lim_{n\to\infty}(a_n/n)^{1/2}\log n = 0$ then it is easy to verify that the assumptions (i)–(iv) and (v1) of Theorem 1 are satisfied with $\Psi_n = (x(a_n - x)/n)^{1/2}$. Therefore by Theorem 1 we have

COROLLARY 4.2.1. *If* $\liminf_{n\to\infty} a_n > 0$ *and* $\lim_{n\to\infty}(a_n/n)^{1/2}\log n = 0$ *then*

$$\lim_{n\to\infty}\left(\frac{x(a_n-x)}{n}\right)^{-(\alpha+q)/2} \times \sup_{f\in W^{(q)}(C(J);\,\alpha;1)}\left|\bar{B}_n\left(f(t)-\sum_{k=0}^{q}\frac{f^{(k)}(x)}{k!}(t-x)^k, x\right)\right| = \begin{cases} \dfrac{1}{\sqrt{\pi}(\alpha+1)_q}2^{(\alpha+q)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is even},\\ \dfrac{1}{\sqrt{\pi}(\alpha+1)_q}2^{(3\alpha+q-2)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is odd}.\end{cases}$$

EXAMPLE 4.2.2. Suppose $a_n \to \infty$ as $n \to \infty$. In Theorem 1 let $r(t) = t$, $\beta_n = a_n/n$, and $X_{nj} \overset{\text{i.i.d.}}{\sim} X_{11}$, $\forall n, j$, where $X_{11}$ follows Poisson distribution $P(X_{11} = k) = e^{-x/a_n}(x/a_n)^k/k!$, $k = 0, 1, 2, \ldots,$ where $x > 0$ is a parameter, then $\alpha_n = EX_{11} = x/a_n$, $\sigma_n^2(x) = \mathrm{Var}(X_{11}) = x/a_n$, and $\rho_{sn} = E|X_{11} - EX_{11}|^s \le C_s x/a_n$, where $C_s$ is a constant independent of $n$. Now the corresponding Trotter–Feller operator (1) is reduced to the multiplier enlargement of Szász operator (cf. [20, p. 91])

$$\bar{S}_n(f, x) = e^{-nx/a_n}\sum_{k=0}^{\infty} f\left(\frac{a_n k}{n}\right)\frac{(nx/a_n)^k}{k!}. \tag{55}$$

If further it is assumed that $\lim_{n\to\infty}(a_n/n)^{1/2}\log n/\sqrt{n} = 0$ then it is easy to verify that the assumptions (i)–(iv) and (v1) of Theorem 1 are satisfied with $\Psi_n = (xa_n/n)^{1/2}$. Therefore by Theorem 1 we have

COROLLARY 4.2.2. *If* $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty}(a_n/n)^{1/2}\log n/\sqrt{n} = 0$ *then*

$$\lim_{n\to\infty}\left(\frac{xa_n}{n}\right)^{-(\alpha+q)/2} \times \sup_{f\in W^{(q)}(C(J);\,\alpha;1)}\left|\bar{S}_n\left(f(t)-\sum_{k=0}^{q}\frac{f^{(k)}(x)}{k!}(t-x)^k, x\right)\right| = \begin{cases} \dfrac{1}{\sqrt{\pi}(\alpha+1)_q}2^{(\alpha+q)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is even},\\ \dfrac{1}{\sqrt{\pi}(\alpha+1)_q}2^{(3\alpha+q-2)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is odd}.\end{cases}$$

### 4.3. *Node Shifting*

Frequently the nodes of the operator in question are shifted to meet the requirement of functions approximated at ending points (cf. [17]). This can be achieved by assigning different $\beta_n$ to special cases of the Trotter–Feller operator (1). The first of the following examples, the Post–Widder operator, is studied by Rathore and Singh in [16] for their Lipschitz–Nikolskiĭ constant problem among others. We improve their result by giving the exact value for odd $q$. A similar result is obtained by Zhou in [22].

EXAMPLE 4.3.1. For a fixed integer $p$ the Post–Widder operator (see [16]) $P_n^*(f; x)$ is defined by

$$P_n^*(f;x) = \frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1}\int_0^\infty f(t)t^{n+p}e^{-nt/x}\,dt. \tag{56}$$

In Theorem 1 let $r(t) = t$, $\beta_n = 1/(n-p-1)$, and $X_{nj} \overset{\text{i.i.d.}}{\sim} X_{11}$, $\forall n, j$, where $X_{11}$ follows Gamma distribution $P(X_{11} \le t) = (1/x)\int_0^t e^{-u/x}\,du$, $t \ge 0$. Then

$$P_n^*(f,x) = E\left\{f\left(\frac{1}{n}S_{n+p+1}\right)\right\} = E\left\{f\left(\beta_{n+p+1}S_{n+p+1}\right)\right\} = T_{n+p+1}(f,r,x). \tag{57}$$

Using the quantities found in the first part of the present section concerning the Gamma operator it can be shown that the requirements of Theorem 1 are fulfilled with $\Psi_n = \sqrt{n}\,x/(n-p-1)$. So Theorem 1 gives

COROLLARY 4.3.1.

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{\sqrt{n+p+1}}{n}x\right)^{-(\alpha+q)}\\ &\quad\times \sup_{f\in W^{(q)}(C(J);\,\alpha;\,1)}\left|P_n^*\left(f(t) - \sum_{k=0}^{q}\frac{f^{(k)}(x)}{k!}(t-x)^k, x\right)\right|\\ &= \begin{cases}\dfrac{1}{\sqrt{\pi}(\alpha+1)_q}2^{(\alpha+q)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is even},\\[2ex] \dfrac{1}{\sqrt{\pi}(\alpha+1)_q}2^{(3\alpha+q-2)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is odd.}\end{cases}\end{aligned} \tag{58}$$

EXAMPLE 4.3.2. A class of interesting special Trotter–Feller operators $A_n(f,x)$ was investigated by Chen and Zeng in a series of papers (cf. [3, 21] and the references cited there). For two fixed integers $l_1$ and $l_2$ and a sequence of random variables $X_n \overset{\text{i.i.d.}}{\sim} X$ with $EX = x$, $\sigma^2(X) = \sigma^2 > 0$, $A_n(f,x)$ is defined by

$$A_n(f,x) = E\left\{ f\left( \frac{1}{n+l_1} \left( \sum_{k=1}^{n} X_k + l_2 \right) \right) \right\}. \tag{59}$$

With suitable conditions on $X$ we are able to determine the corresponding Lipschitz–Nikolskiĭ constants. Note that $A_n(f,x)$ is a special case of (1). When assuming $r(t) = t$, $\beta_n = 1/(n + l_1)$ and $X_{nj} \overset{\text{i.i.d.}}{\sim} X_{n1} \equiv X + l_2/n$, $j = 1, 2, \ldots, \forall n$, we have $\alpha_n = x + l_2/n$, $\sigma_n^2(x) = \sigma^2$, $\Psi_n = \sqrt{n}\,\sigma/(n + l_2)$ and the conditions of Theorem 1 will be satisfied if $E|X|^s < \infty$. Now Theorem 1 implies

COROLLARY 4.3.2. *If $E|X|^s < \infty$ then*

$$\lim_{n\to\infty} \left( \frac{\sqrt{n}\,\sigma}{n+l_2} \right)^{-(\alpha+q)} \sup_{f\in W^{(q)}(C(J);\,\alpha;\,1)} \left| A_n\left( f(t) - \sum_{k=0}^{q} \frac{f^{(k)}(x)}{k!} (t-x)^k, x \right) \right|$$

$$= \begin{cases} \dfrac{1}{\sqrt{\pi}\,(\alpha+1)_q} 2^{(\alpha+q)/2} \Gamma\left( \dfrac{\alpha+q+1}{2} \right), & \text{when } q \text{ is even,} \\[2ex] \dfrac{1}{\sqrt{\pi}\,(\alpha+1)_q} 2^{(3\alpha+q-2)/2} \Gamma\left( \dfrac{\alpha+q+1}{2} \right), & \text{when } q \text{ is odd.} \end{cases} \tag{60}$$

In fact the Post–Widder operator (56) is a subcase of $A_n(f,x)$ when $X$ follows Gamma distribution, $l_1 = -p - 1$ and $l_2 = 0$. Moreover, most of the examples considered by Shaw and Yeh in [18] are also the subcases of $A_n(f,x)$ and can be handled similarly.

### 4.4. *Rational Transformations*

The probabilistic method has been shown very effective in recent studies of the Cheney–Sharma operator [4] and the Bleimann–Butzer–Hahn operator [2] (cf. [9, 11]). These operators can be derived from the Trotter–Feller operator (1) by choosing appropriate $r(t)$ (rational transformations). Another example studied here is the Gamma operator of Müller, which is derived from (1) by reciprocal transformation $r(t) = 1/t$. We obtain their Lipschitz–Nikolskiĭ constants as corollaries to Theorem 1.

Our results improve Rathore's for the Meyer-König and Zeller operator since the latter is a special case of the Cheney–Sharma operator. Also, a result by Rathore regarding the Lipschitz–Nikolskiĭ constants of the Gamma operator of Müller is sharpened for odd $q$.

EXAMPLE 4.4.1. The Cheney–Sharma operator [4] is defined by

$$P_n(f,v) = (1-v)^{n+1}\exp\left(\frac{\omega v}{1-v}\right)\sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(\alpha)}(\omega) v^k, \quad (61)$$

where $0 \le v \le a < 1$, $\omega \le 0$, and $L_k^{(\alpha)}$ $(\alpha > -1)$ denotes the Laguerre polynomial [19] of degree $k$ with parameter $\alpha$. The Cheney–Sharma operator is a special case of Trotter–Feller operator (1).

For each row of the triangular array $\{X_{nj},\ j = 1,2,\dots,\ n \in \mathcal{N}\}$ considered in Theorem 1, let $X_{nj} \overset{\text{i.i.d.}}{\sim} X_{n1}$, where $X_{n1}$ follows the Laguerre distribution (cf. [9])

$$P(X_{n1} = k) = \frac{1}{1+x}\exp(\omega^* x) L_k^{(0)}(\omega^*)\left(\frac{x}{1+x}\right)^k,$$

for $k = 1,2,\dots,\ x \ge 0,\ \omega^* = \omega/n,\ x = v/(1-v)$. Then (see [9, p. 100, p. 92])

$$\alpha_n = EX_{n1} = x - x(1+x)\omega^*,$$

$$\sigma_n^2 = \operatorname{Var}(X_{n1}) = x(1+x)(1-\omega^* - 2\omega^* x)$$

and (see [19, p. 101])

$$E(e^{\delta X_{n1}}) = \frac{1}{1+x-xe^{\delta}}\exp\left(\frac{\omega^* x(1+x)(1-e^{\delta})}{1+x-xe^{\delta}}\right). \quad (62)$$

Further let $r(t) = t/(1+t)$, $\beta_n = 1/(n-1)$ in Theorem 1; then

$$P_n(f,v) = E\left\{f\left(\frac{S_{n+1}}{n+S_{n+1}}\right)\right\} = E\{f(r(\beta_{n+1}S_{n+1}))\} = T_{n+1}(f,r,x).$$

We verify in the following that the assumptions of Theorem 1 are satisfied. Assumptions (i) and (ii) hold because $\Psi_n = \sqrt{n}\,\sigma_n\,\beta_n = (\sqrt{n}/(n-1))(x(1+x)(1-\omega/n-2\omega x/n))^{1/2} \to 0$ and $n\alpha_n\beta_n - x = (n/(n-1))(x - x(1+x)\omega^*) - x = o(\Psi_n)$ as $n \to \infty$. Next since (cf. [13, p. 274])

$$E(|X_{n1}|^{\alpha}) \le \left(\frac{\alpha}{\delta e}\right)^{\alpha} E(e^{\delta X_{n1}})$$

$$\overset{(\text{by }(62))}{\le} \left(\frac{\alpha}{\delta e}\right)^{\alpha}\frac{1}{1+x-xe^{\delta}}\exp\left(\frac{\omega x(1+x)(1-e^{\delta})}{1+x-xe^{\delta}}\right) < \infty$$

when $\delta < \log((1+x)/x)$, there exists a constant $C$ such that $\rho_{mn} \le C < \infty$ for all $m, n$ and thus assumptions (iii) and (iv) are satisfied. Last, assumption (v1) is true due to the fact that $r'(t) = 1/(1+t)^2 \le 1$ for $t \ge 0$.

Now by Theorem 1 we get the Lipschitz−Nikolskiĭ constants for the Cheney−Sharma operator as follows.

COROLLARY 4.4.1.

$$\lim_{n\to\infty}\left(\frac{(n+1)v}{n^2(1-v)^2}\left(1-\frac{\omega}{n+1}-\frac{2\omega v}{(n+1)(1-v)}\right)\right)^{-(\alpha+q)/2}$$
$$\times \sup_{f\in W^{(q)}(C(J);\,\alpha;1)}\left|P_n\left(f(t)-\sum_{k=0}^{q}\frac{f^{(k)}(v)}{k!}(t-v)^k, v\right)\right|$$
$$= \begin{cases} \dfrac{(1-v)^{2(\alpha+q)}}{\sqrt{\pi}(\alpha+1)_q}2^{(\alpha+q)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is even},\\ \dfrac{(1-v)^{2(\alpha+q)}}{\sqrt{\pi}(\alpha+1)_q}2^{(3\alpha+q-2)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is odd}. \end{cases} \tag{63}$$

*When* $\omega = 0$, (61) *reduces to the Meyer-König and Zeller operator*

$$M_n(f,v) = (1-v)^{n+1}\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}v^k, \qquad 0 \le v < 1,$$

*and the corresponding Lipschitz−Nikolskiĭ constants are*

COROLLARY 4.4.2.

$$\lim_{n\to\infty}\left(\frac{(n+1)v}{n^2(1-v)^2}\right)^{-(\alpha+q)/2}$$
$$\times \sup_{f\in W^{(q)}(C(J);\,\alpha;1)}\left|M_n\left(f(t)-\sum_{k=0}^{q}\frac{f^{(k)}(v)}{k!}(t-v)^k, v\right)\right|$$
$$= \begin{cases} \dfrac{(1-v)^{2(\alpha+1)}}{\sqrt{\pi}(\alpha+1)_q}2^{(\alpha+q)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \textit{when } q \textit{ is even},\\ \dfrac{(1-v)^{2(\alpha+q)}}{\sqrt{\pi}(\alpha+1)_q}2^{(3\alpha+q-2)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \textit{when } q \textit{ is odd}. \end{cases} \tag{64}$$

EXAMPLE 4.4.2. The Bleimann–Butzer–Hahn operator [2] is defined by

$$B_n^*(f,v) = \left(\frac{1}{1+v}\right)^n \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k} v^k, \qquad v \geq 0. \tag{65}$$

It is also a special case of Trotter–Feller operator (1). In Theorem 1 let $r(t) = t/(1-t)$, $\beta_n = 1/(n+1)$, and $X_{nj} \overset{\text{i.i.d.}}{\sim} X_{11}$, $\forall n, j$, where $X_{11}$ follows Bernoulli distribution $P(X_{11} = 1) = x$, $P(X_{11} = 0) = 1 - x$, $x = v/(1+v)$. Then

$$\alpha_n(x) = EX_{11} = x, \qquad \sigma_n^2(x) = \operatorname{Var}(X_{11}) = x(1-x), \tag{66}$$

$$\rho_{sn} = E|X_{11} - EX_{11}|^s = x(1-x)\left((1-x)^{s-1} + x^{s-1}\right) \tag{67}$$

and

$$B_n^*(f,v) = E\left\{f\left(\frac{S_n}{n - S_n + 1}\right)\right\} = E\{f(r(\beta_n S_n))\} = T_n(f,r,x). \tag{68}$$

Since $\Psi_n = \sqrt{n}\,\sigma_n\,\beta_n = (nx(1-x))^{1/2}/(n+1) \to 0$ and $n\alpha_n\beta_n - x = -x/(n+1) = o(\Psi_n)$ as $n \to \infty$, assumptions (i) and (ii) are satisfied. By (66) and (67) conditions (iii) and (iv) hold too. It remains to show that (v2) is true. Denote

$$\Delta := \Psi_n^{-(\alpha+q)} \int_{|t-x| > \Psi_n^{\delta}} |r(t) - r(x)|^{\alpha+q}\, dP(\beta_n S_n \leq t)$$

$$= \Psi_n^{-(\alpha+q)} \int_{|t-x| > \Psi_n^{\delta}} \frac{|t-x|^{\alpha+q}}{(1-t)^{\alpha+q}(1-x)^{\alpha+q}}\, dP(\beta_n S_n \leq t).$$

Then for some integer $m > \alpha + q$ it holds that

$$\Delta \leq \Psi_n^{-(\alpha+q)} \int_{|t-x| > \Psi_n^{\delta}} \frac{|t-x|^m}{\Psi_n^{\delta(m-\alpha-q)}} \frac{1}{|1-t|^m |1-x|^m}\, dP(\beta_n S_n \leq t)$$

$$\leq \Psi_n^{(\delta-1)(\alpha+q)-\delta m} \frac{1}{(1-x)^m} \left(\int_0^1 (t-x)^{2m}\, dP(\beta_n S_n \leq t)\right)^{1/2}$$

$$\times \left(\int_0^1 \frac{1}{(1-t)^{2m}}\, dP(\beta_n S_n \leq t)\right)^{1/2},$$

by Hölder's inequality. (69)

Further we have

$$\begin{aligned}\int_0^1 (t-x)^{2m}\, dP(\beta_n S_n \le t) &= E\left(\left|\frac{1}{n+1} S_n - x\right|^{2m}\right) \\ &\le 2^{2m} E\left|\frac{1}{n+1} S_{n+1} - x\right|^{2m} + 2^{2m}\left(\frac{1}{n+1}\right)^{2m} E|X_{11}|^{2m}. \quad (70)\end{aligned}$$

and by Theorem 3.1 of [13] (set $\delta = 1$ there) when $n \ge 30m/E(e^{X_{11}})$ it holds that

$$E\left(\left|\frac{1}{n+1} S_{n+1} - x\right|^{2m}\right) \le 2n^{-m}\left(2mE(e^{X_{11}})/e\right)^m. \qquad (71)$$

Note that the second term of (70) is of order $n^{-2m}$. When $n \to \infty$ the above gives

$$\int_0^1 (t-x)^{2m}\, dP(\beta_n S_n \le t) = O(n^{-m}), \qquad \text{as } n \to \infty. \qquad (72)$$

Moreover

$$\begin{aligned}\int_0^1 \frac{1}{(1-t)^{2m}}\, dP(\beta_n S_n \le t) &= \sum_{k=0}^{n} \left(\frac{n+1}{n+1-k}\right)^{2m} \binom{n}{k} x^k (1-x)^{n-k} \\ &\le \sum_{k=0}^{n} \frac{(n+1)^{2m}}{(n+1-k)^{2m}} \frac{(n+2m-k)\cdots(n+1-k)}{(n+2m)\cdots(n+1)} \\ &\quad \times \frac{(n+2m)!}{(n+2m-k)!k!} x^k (1-x)^{n+2m-k} (1-x)^{-2m} \\ &\le (1-x)^{-2m} \frac{(n+1)^{2m}}{(n+2m)\cdots(n+1)} (2m)! \sum_{k=0}^{n} \frac{(n+2m)!}{(n+2m-k)!k!} \\ &\quad \times x^k (1-x)^{n+2m-k}\end{aligned}$$

$$\left(\text{Note that } \frac{(n+2m-k)\cdots(n+1-k)}{(n+1-k)^{2m}} \le (2m)! \text{ for all } 0 \le k \le n.\right)$$

$$\le (1-x)^{-2m} \frac{(n+1)^{2m}}{(n+2m)\cdots(n+1)}(2m)! \le M_m < \infty, \tag{73}$$

where the constant $M_m$ is independent of $n$.

From (69), (72), (73), and the fact that $\Psi_n = O(1/\sqrt{n})$ as $n \to \infty$ we conclude that

$$\Delta = O\left(\left(\frac{1}{n}\right)^{(1-\delta)(m-\alpha-q)/2}\right)$$

and $\Delta \to 0$ as $n \to \infty$. So assumption (v2) is satisfied. Now Theorem 1 leads to

COROLLARY 4.4.3.

$$\lim_{n\to\infty}\left(\frac{\sqrt{nv}}{(n+1)(1+v)}\right)^{-(\alpha+q)} \times \sup_{f\in W^{(q)}(C(J);\,\alpha;\,1)}\left|B_n^*\left(f(t) - \sum_{k=0}^{q}\frac{f^{(k)}(v)}{k!}(t-v)^k, v\right)\right|$$

$$= \begin{cases} \dfrac{(1+v)^{2(\alpha+q)}}{\sqrt{\pi}(\alpha+1)_q} 2^{(\alpha+q)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \textit{when } q \textit{ is even}, \\[2ex] \dfrac{(1+v)^{2(\alpha+q)}}{\sqrt{\pi}(\alpha+1)_q} 2^{(3\alpha+q-2)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \textit{when } q \textit{ is odd}. \end{cases} \tag{74}$$

EXAMPLE 4.4.3. The Gamma operator of Müller (cf. [15]) is defined by

$$G_n^*(f,v) = \frac{v^{n+1}}{n!}\int_0^\infty t^n e^{-vt} f\left(\frac{n}{t}\right)dt, \qquad v > 0. \tag{75}$$

In Theorem 1 let $r(t) = 1/t$, $\beta_n = 1/(n-1)$, and $X_{nj} \overset{\text{i.i.d.}}{\sim} X_{11}$, $\forall n, j$, where $X_{11}$ follows Gamma distribution $P(X_{11} \le t) = (1/x)\int_0^t e^{-u/x}\,du$, $t \ge 0$, $x = 1/v$. Then

$$G_n^*(f,x) = E\left\{f\left(\frac{n}{S_{n+1}}\right)\right\} = E\{f(r(\beta_{n+1}S_{n+1}))\} = T_{n+1}(f,r,x). \tag{76}$$

Using the quantities found in the first part of the present section concerning the Gamma operator it is not difficult to show that the assumptions (i)–(iv) of Theorem 1 are satisfied with $\Psi_n = \sqrt{n}x/(n-1)$. We still need to show that (v2) is true.

Note that for some integer $m > \alpha + q$ it holds that

$$\begin{aligned}
\Delta &:= \Psi_n^{-(\alpha+q)} \int_{|t-x|>\Psi_n^\delta} \left| r(t) - r(x) \right|^{\alpha+q} dP(\beta_n S_n \le t) \\
&= \Psi_n^{-(\alpha+q)} \int_{|t-x|>\Psi_n^\delta} \frac{|t-x|^{\alpha+q}}{|tx|^{\alpha+q}} dP(\beta_n S_n \le t) \\
&\le \Psi_n^{-(\alpha+q)} \int_{|t-x|>\Psi_n^\delta} \frac{|t-x|^m}{\Psi_n^{\delta(m-\alpha-q)}} \frac{1}{|tx|^{\alpha+q}} dP(\beta_n S_n \le t) \\
&\le \left(\frac{1}{x}\right)^{\alpha+q} \Psi_n^{(\delta-1)(\alpha+q)-\delta m} \left( E\left( \left| \frac{1}{n-1} S_n - x \right|^{2m} \right) \right)^{1/2} \\
&\quad \times \left( \int_0^\infty \frac{1}{t^2(\alpha+q)} dP(\beta_n S_n \le t) \right)^{1/2}.
\end{aligned}$$

The last integral is uniformly bounded for $n$ due to the fact (see [14]) that

$$\int_0^\infty \frac{1}{t^{2(\alpha+q)}} dP(\beta_n S_n \le t) = G_n^*(t^{2(\alpha+q)}, x) \to x^{2(\alpha+q)}, \qquad \text{as } n \to \infty.$$

Now follow an argument similar to that in the previous example (cf. (71)), and we can show that $\Delta \to 0$ as $n \to \infty$. So by applying Theorem 1 we get

COROLLARY 4.4.4.

$$\begin{aligned}
&\lim_{n\to\infty} \left( \frac{\sqrt{n}}{(n-1)v} \right)^{-(\alpha+q)} \\
&\quad \times \sup_{f \in W^{(q)}(C(J);\, \alpha\,;1)} \left| G_n^*\left( f(t) - \sum_{k=0}^{q} \frac{f^{(k)}(v)}{k!} (t-v)^k, v \right) \right| \\
&= \begin{cases} \dfrac{v^{2(\alpha+q)}}{\sqrt{\pi}(\alpha+1)_q} 2^{(\alpha+q)/2} \Gamma\left( \dfrac{\alpha+q+1}{2} \right), & \textit{when } q \textit{ is even}, \\[2ex] \dfrac{v^{2(\alpha+q)}}{\sqrt{\pi}(\alpha+1)_q} 2^{(3\alpha+q-2)/2} \Gamma\left( \dfrac{\alpha+q+1}{2} \right), & \textit{when } q \textit{ is odd}. \end{cases}
\end{aligned} \tag{77}$$

### 4.5. *Non-identically Distributed r.v.'s*

EXAMPLE 4.5.1. In [7], Hahn introduced an operator constructed from non-identically distributed random variables. We will give its Lipschitz–Nikolskiĭ constants here.

In Theorem 1, let $r(t) = t$ and $X_{nj}$ follow the distribution

$$P_{X_{nj}} = x\epsilon_j + (1-x)\epsilon_0 \qquad (0 \le x \le 1),$$

where $\epsilon_a$ denotes the point measure for $a \in \mathscr{R}$. Then (see [7, p. 318])

$$P_{S_n} = \sum_{k=0}^{n(n+1)/2} \sum_{m=0}^{n} Q(k,m,n) x^m (1-x)^{n-m} \epsilon_k,$$

where $Q(k,m,n)$, $k,m,n \in \mathscr{N}$, is the number of partitions of $k$ into exactly $m$ unequal parts which do not exceed $n$, and for $k = 0$, $Q(0,m,n) = 0$ for $m \neq 0$, $= 1$ for $m = 0$, $n \in \mathscr{N}$. If we denote the corresponding operator by $H_n$, then

$$H_n(f,x) = \sum_{k=0}^{n(n+1)/2} \sum_{m=0}^{n} f\left(\frac{2k}{n(n+1)}\right) Q(k,m,n) x^m (1-x)^{n-m}. \quad (78)$$

Since $\alpha_n = (1/n)\sum_{j=1}^{n} E(X_{nj}) = \sum_{j=1}^{n} jx = xn(n+1)/2$, $\sigma_n^2 = (1/n)\sum_{j=1}^{n} \mathrm{Var}(X_{nj}) = x(1-x)(n+1)(2n+1)/6$, and $\rho_{mn} = (1/n)\sum_{j=1}^{n} E(|X_{nj} - EX_{nj}|^m) \le 2^m (1/n)\sum_{j=1}^{n} E|X_{nj}|^m = 2^m(1/n)\sum_{j=1}^{n} j^m x \le x2^m n^m$ by choosing $\beta_n = 2/(n(n+1)$ it is straightforward to verify that the assumptions in Theorem 1 are satisfied with $\Psi_n = (2x(1-x)(2n+1)/3n(n+1))^{1/2}$. Now an application of Theorem 1 yields

COROLLARY 4.5.1.

$$\begin{aligned}
&\lim_{n\to\infty} \left(\frac{2x(1-x)(2n+1)}{3n(n+1)}\right)^{-(\alpha+q)/2} \\
&\quad \times \sup_{f\in W^{(q)}(C(J);\,\alpha;1)} \left| H_n\left(f(t) - \sum_{k=0}^{q} \frac{f^{(k)}(x)}{k!}(t-x)^k, x\right)\right| \\
&= \begin{cases} \dfrac{1}{\sqrt{\pi}(\alpha+1)_q} 2^{(\alpha+q)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is even}, \\[2ex] \dfrac{1}{\sqrt{\pi}(\alpha+1)_q} 2^{(3\alpha+q-2)/2}\Gamma\left(\dfrac{\alpha+q+1}{2}\right), & \text{when } q \text{ is odd}. \end{cases}
\end{aligned} \quad (79)$$

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