Variational Solution
of Some Nonlinear Eigenvalue Problems. I

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1. Introduction

There has been much recent interest in variational principles for eigenvalue problems involving linear operators with the eigenvalue occurring in a nonlinear manner [1], [2]. These nonlinear eigenvalue problems occur in the theory of several important problems in engineering, hydrodynamics and astrophysics [1]–[5].

This paper examines the pattern of convergence obtained when the Ritz method is applied to a nonlinear eigenvalue problem (equation (1) of Section 2) which closely resembles that governing non-radial oscillations of stars. As is common with nonlinear eigenvalue problems [3], both problems have two cluster points of eigenvalues: zero and infinity. Equation (1) was first studied in [6].

In Section 5, a general eigenvalue problem (equation (23)), which includes both (1) and the astrophysical problem as special cases, is introduced and examined. Like many important nonlinear eigenvalue problems [1]–[4], (23) has an equivalent linear form $Tw = \lambda w$, with $T$ hermitian. This linear formulation of the astrophysical problem was introduced by Chandrasekhar [7] as a variational principle. The corresponding linear formulation of (1) and (23) is used throughout this paper.

The commonest linearisation process is that used in [2]. (See [1] and [3] for related references.) The relationship between the linear and the nonlinear forms of (23) is less simple (see Section 5). Moreover, with the problem in [2], $T$ in the linear formulation is compact. This is not so for (23). Hence Lemma A of Section 3, which justifies the commoner applications of the Ritz method, may not be applied directly to (23) in this linear formulation.

Despite the lack of theoretical studies of the reliability of the method in this case, the Ritz method has been applied on several occasions [8]–[12] to Chandrasekhar's variational formulation of the equation governing non-radial oscillations of stars. Use of the Ritz method has also been proposed [13]
for some of the various recent generalisations of Chandrasekhar's variational principle. Although the method has given useful results [8]–[10] for some artificial stellar models (especially for those closely resembling the only model for which exact solutions are known [12]), it has been less successful with more realistic models [11], [12]. Results in this case have several unusual features and depend much more crucially on the choice of coordinate functions than is usual with problems covered by Lemma A. The relationship of the present work to the astrophysical problem is examined further in Section 6.

Essentially (1) is obtained from the astrophysical problem by replacing the coefficients in the differential equation by constants and imposing new boundary conditions. This replaces a difficult problem with singularities at the boundaries by one which may be solved exactly. Despite the simplicity of (1), similarities are noted in Section 6 between results proved here for (1) and otherwise puzzling features of numerical solutions obtained for the astrophysical problem. No investigation of convergence was made in [6], and most results established there dealt with one particular choice of coordinate functions.

A discussion of the properties of (1), which is somewhat simpler than that given in [6], is included in Section 2. The main convergence results are proved in Sections 3 and 4. Two different choices of coordinate functions, which resemble those used in [11] and [12], are shown to give significantly different patterns of convergence.

A second paper will relate results proved here to some weaker results which can be proved for quite general coordinate functions.

2. THE SIMPLEST EIGENVALUE PROBLEM

Let \( a_1, \ldots, a_6 \) be real non-zero constants with \( a_1, a_5, a_6 \) and \( a_1a_6 + a_2a_5 \) all positive. Let \( H_0 \) be the Hilbert space of vector valued functions \( (u) \) such that \( u \sqrt{a_5} \) and \( v \sqrt{a_6} \) are in \( L_2[0, 1] \), with inner product

\[
\left( \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_2 \\ v_2 \end{pmatrix} , \begin{pmatrix} u'_1 \\ v'_1 \\ \vdots \\ u'_2 \\ v'_2 \end{pmatrix} \right) = \int_0^1 \left( a_1u_1(r) \, \bar{u}_2(r) + a_5v_1(r) \, \bar{v}_2(r) \right) \, dr.
\]

The subsequent analysis is valid whether \( H_0 \) is real or complex. Definitions in this section are phrased to facilitate generalisation in Section 5. The notation \( \{ u, v \} = \langle \begin{pmatrix} u \\ v \end{pmatrix} \rangle \) will frequently be employed.

We consider the eigenvalue problem

\[
A_0 \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}
\] (1)
where \( A_0 \) is defined by \( A_0(u) = (u) \) and
\[
\begin{align*}
  a_5 u &= a_1 u + a_2 v - a_1 a_2 v', \\
  a_6 v &= [a_4(a_1 u + a_2 v - a_5 a_4 v')]' + a_4 u + a_5 v - a_2 a_4 v'.
\end{align*}
\]
The domain \( \text{Dom}(A_0) \) of \( A_0 \) is defined as all \( \{u, v\} \) in \( H_0 \) for which \( \{u, v\} \) is continuous in \([0, 1]\) and which satisfy
\[
v(0) = v(1) = 0. \tag{2}
\]
Clearly \( A_0 \) is hermitian and (1) has variational formulation
\[
\int_0^1 (a_1 | u(r) + (a_2 | a_4) v(r) - a_4 v'(r)|^2 + (a_3 - (a_2 | a_4)) | v(r)|^2) \, dr \\
= \lambda \int_0^1 (a_5 | u(r)|^2 + a_6 | v(r)|^2) \, dr. \tag{3}
\]
It is easily shown that (1) is satisfied (for suitable \( u \)) if and only if (2) and
\[
\lambda v'' = -f(\lambda)v \tag{4}
\]
are satisfied where
\[
a_1 a_4^2 a_5 f(\lambda) = a_1 a_3 - a_2^2 - (a_1 a_6 + a_5 a_6) \lambda + a_5 a_6 \lambda^2.
\]
This nonlinear formulation shows, rather more simply than in [6], that (1) has two sequences of eigenvalues, \( \{\lambda_{k+}\} \) and \( \{\lambda_{k-}\} \), with
\[
f(\lambda_{k\pm}) = k^2 \pi^2 \lambda_{k\pm} \tag{5}
\]
so that
\[
\lambda_{k+}, \lambda_{k-} = (a_1 a_3 - a_2^2)/a_5 a_6, \tag{6}
\]
and
\[
2a_5 a_6 \lambda_{k\pm} = a_1 a_6 + a_5 a_5 + a_4 a_4 a_5 k^2 \pi^2 \pm [(a_3 a_5 - a_1 a_6 + a_4 a_4 a_5 k^2 \pi^2)^2 \\
+ 4a_5 a_6 (a_2^2 + a_4 a_4 k^2 \pi^2)]^{1/2}.
\]
The eigenfunctions \( \{u_{k\pm}, v_{k\pm}\} \) corresponding to \( \lambda_{k\pm} \) satisfy
\[
v_{k\pm}(r) = c \sin k\pi r, \tag{7}
\]
\[
(a_1 - \lambda_{k\pm} a_3) u_{k\pm}(r) = a_4 a_4 v_{k\pm}'(r) - a_2 v_{k\pm}(r). \tag{8}
\]
Clearly these are the only eigenfunctions of (1) in which \( v \) is not identically zero.

An important difference between (1) and the system (2), (4) is that (1) has the additional eigenvalue \( \lambda_0 = a_1/a_5 \) with corresponding eigenfunction
\{u_0, v_0\} \text{ where } u_0(r) = c \exp(-a r^2/a_t, a_t), v_0 = 0. \text{ This is the only eigenfunction of (1) with } v = 0. \text{ As noted in [6]}
\begin{align*}
\lambda_{k-} < \lambda_0 < \lambda_{k+} < \lambda_{(k+1)+}.
\end{align*}

In this paper the Ritz method is applied to (1) by extremising \( \lambda \) given by (3) when \( \left( \begin{array}{c}
u_r \\ \nu_t
\end{array} \right) \) is restricted to a finite-dimensional subspace of \( H_0 \). This is essentially the procedure used for the astrophysical problem in [8]-[12]. A completely different method, which would be no help in explaining the results obtained for the astrophysical problem, would be to extremise \( f(\lambda)/\lambda \) given by
\begin{align*}
\lambda \int_0^1 |v'(r)|^2 \, dr = f(\lambda) \int_0^1 |v(r)|^2 \, dr,
\end{align*}
where \( v \) satisfies (2). This latter method would merely be equivalent to solving the problem
\begin{align*}
Bv = \lambda v
\end{align*}
where \( Bv = -v'' \), and the domain of \( B \) is those functions \( v \) in \( L_2[0, 1] \) satisfying (2) for which \( v'' \) is also in \( C[0, 1] \). In general the two methods give significantly different results (see for example Theorems 2 and 3(ii)), but in certain cases (see Theorem 1) they give the same results for the non-zero \( v_{k\pm} \).

Denote solutions (eigenvalues and eigenfunctions) obtained by the Ritz method, Ritz approximate solutions (eigenvalues and eigenfunctions). In [6] it was shown that, for all \( \{\xi, \eta\} \) in \( \text{Dom} \,(A_0) \) and all \( n > m \), the trial functions
\begin{align*}
u = \sum_{i=1}^n c_i \xi_i, \quad v = \sum_{j=1}^m c_{n+j} \eta_j
\end{align*}
yield at least \( n - m \) linearly independent Ritz approximate eigenfunctions with \( v = v_0 \), each corresponding to the Ritz approximate eigenvalue \( \lambda_0 \). In the special case
\begin{align*}
\xi_i(r) = r^{i-1}, \quad \eta_j(r) = r^j(1 - r)
\end{align*}
it was shown that when \( n \geq m + 2 \), the Ritz approximate eigenvalues other than \( \lambda_0 \) are independent of \( n \), and occur in pairs \( A_{k\pm}(m) \) satisfying
\begin{align*}
A_{k+(m)} A_{k-(m)} = (a_1 a_3 - a_2^2)/a_5 a_6
\end{align*}
where \( A_{(k+1)+}(m) \geq A_{k+(m)} \geq A_{k-(m)} \), and that
\begin{align*}
V_{km+} = V_{km-}
\end{align*}
and
\begin{align*}
(a_1 - A_{k\pm}(m)) U_{km\pm} = a_4 a_2 V_{km\pm} - a_2 V_{km\pm},
\end{align*}
where \( \{U_{km\pm}, V_{km\pm}\} \) is the (suitably normalised) Ritz approximate eigenfunction corresponding to \( A_{k\pm}(m) \).

3. POLYNOMIAL COORDINATE FUNCTIONS

**Theorem 1.**

(i) \( \lambda_{k+} \leq A_{k+}(m + 1) \leq A_{k+}(m) \) and
\[
|A_{k-}(m)| \leq |A_{k-}(m + 1)| \leq |A_{k-}|.
\]

(ii) \( \lim_{m \to \infty} A_{k\pm}(m) = \lambda_{k\pm} \).

(iii) With suitable normalisation of \( \{U_{km\pm}, V_{km\pm}\} \),
\[
\lim_{m \to \infty} \|\{U_{km\pm}, V_{km\pm}\} - \{u_{k\pm}, v_{k\pm}\}\| = 0.
\]

The proof uses three known Lemmas.

**Lemma A.** Let \( T \) be a strictly positive, compact, hermitian, linear operator, whose domain is (the whole of) a Hilbert space \( S \). Let \( \{\phi_j\} \) be a complete sequence of linearly independent elements in \( S \). Let \( \mu_1 \geq \mu_2 \geq \ldots \) be the eigenvalues of \( T \) and let \( \mu_1(n) \geq \cdots \geq \mu_n(n) \) be the Ritz approximate eigenvalues of \( T \) obtained using as coordinate functions the first \( n \) elements of \( \{\phi_j\} \). Then for all \( i \) and \( n \)

(i) \( \mu_i(n) \leq \mu_i(n + 1) \).

(ii) \( \lim_{n \to \infty} \mu_i(n) = \mu_i \).

(iii) If \( \mu_i \) is a simple eigenvalue of \( T \) then for suitably normalised eigen-elements \( \psi_i \) and \( \psi_{in} \) corresponding to \( \mu_i \) and \( \mu_i(n) \), \( \lim_{n \to \infty} \|\psi_{in} - \psi_i\| = 0 \).

A proof of (i) and (ii) of Lemma A is given, for example, in [14], and (iii) follows from a result proved in [15].

**Lemma B.** Let the \( m \times m \) matrices \( P = (p_{ij}) \) and \( Q = (q_{ij}) \) satisfy
\[
p_{ij} = \frac{2ij}{(i+j-1)(i+j)(i+j+1)},
q_{ij} = \frac{2}{(i+j+1)(i+j+2)(i+j+3)}.
\]
Let \( x_{km} = \{x_{km1}, \ldots, x_{kmm}\} \) where \( V_{km}(r) = \sum_{i=1}^{m} x_{km}(1 - r) \). Then
\[
A_{k\pm}(m)P_{x_{km}} = f(A_{k\pm}(m))Q_{x_{km}}.
\]

(16)

Although Lemma B is not stated explicitly in [6], it is a direct consequence of equations (24), (27) and (28) of [6] and the reasoning of Theorem 4 of that paper.
LEMMA C. If \( k_2 > k_1 > 0 \) then \( k_1 \lambda^*(k_1) < k_1 \lambda^*(k_2) < k_2 \lambda^*(k_1) \) where \( \lambda^*(k) \) is the greater root of \( f(\lambda) = k \lambda \).

This result is proved in [6], in a section throughout which the extra hypothesis \( a_1 a_3 > a_2^2 \) is made. This extra hypothesis is not used in the proof.

Proof of Theorem 1. Since

\[
p_{ij} = \int_0^1 \frac{d}{dr} \left[ r^i(1 - r) \right] \frac{d}{dr} \left[ r^j(1 - r) \right] dr
\]

and

\[
q_{ij} = \int_0^1 r^i(1 - r) r^j(1 - r) dr,
\]

the matrix equation (16) is precisely that obtained in application of the Ritz method to (10) with \( v(r) = \sum_{i=1}^{\infty} c_i r^i(1 - r) (\in \text{Dom}(B)) \). The eigenvalues of (11) and the corresponding eigenfunctions are given by \( \lambda = k^2 x^2 \), \( v(r) = c \sin kmr, k = 1, 2, ... \) The inverse of \( B \) in (11), an integral operator with a Green's function as kernel, is a strictly positive, compact, hermitian, linear operator in the energy space \( H_B \) of \( B \) [19; p. 150]. Hence Lemma A may be applied to \( \text{Inv}(Bx) = h^{-1}(Bx) \). The sequence \( \{r^i(1 - r)\} \) is mapped by \( B \) onto a sequence complete in \( L_2[0,1] \). Hence, (i) and (ii) follow from (5), (6), Lemma A (i) and (ii) and Lemma C. Since the eigenvalues of (11) are simple, (iii) follows from Lemma A (iii) and equations (7), (8) and (15).

Remark 1. The above proof shows that with this special choice of coordinate functions, the two variational formulations (3) and (10) yield the same Ritz approximate values of the non-zero \( v_{k\pm} \) and the corresponding \( \lambda_{k\pm} \). However, the proof [6] of the vital Lemma B depends on the fact that (for all \( j \)) \( q_j \) and \( q_j' \) in (13) are both linear combinations of \( \xi_j, \xi_{j+1} \) and \( \xi_{j+2} \). Since most other choices of \( q_i, q_i' \) in (12) do not have this property, (3) and (10) will normally yield quite distinct results. Even with (13), the Ritz approximate eigenvalues, other than \( a_1/a_5 \), obtained from (3) with \( n = 2, m = 1 \) in (12) satisfy

\[
a_5 a_6 \lambda^2 - \lambda(10 a_1 a_2 a_5 + a_2 a_5 + a_1 a_6) + a_1 a_3 - (5 a_2^2/6) = 0,
\]

and hence do not satisfy (6).

4. PIECEWISE LINEAR COORDINATE FUNCTIONS

For an integer \( N \geq 2 \), define \( \phi_{0N}, ..., \phi_{NN} \) as those functions, continuous in \([0,1]\), which satisfy

\[
\phi_{iN}(x) = 0 \quad \text{for} \quad x \neq j/N \quad \text{and} \quad \phi_{iN}(j/N) = \delta_{ij},
\]
for all integers $i, j = 0, \ldots, N$, where $\delta_{ij}$ is the Kronecker delta. We consider application of the Ritz method to (3) using (12) with

$$n = N + 1, \quad m = N - 1, \quad \xi_i = \phi_{i-1,N}, \quad \eta_j = \phi_{jN}.$$  \hspace{1cm} (17)

A slight and easily removed [16] difference between this choice and the usual formulation of the Ritz method is that $\xi_i, \eta_j$ depend on $N$, but this is essentially the method used for the astrophysical problem in [11] (see Section 6).

**Theorem 2.** Application of the Ritz method to (3) using (12) and (17) yields exactly two linearly independent Ritz approximate eigenfunctions with $v = 0$ (and hence $\lambda = \lambda_0$). The remaining Ritz approximate eigenvalues occur in pairs, $\lambda_{KN+} > \lambda_{KN-}$, satisfying

$$f(\lambda_{KN\pm}) = 3N^2\beta_{KN}(2\lambda_{KN\pm} - \beta_{KN}a_1/a_0)$$  \hspace{1cm} (18)

where $\beta_{KN} = (1 - \cos(k\pi/N))/(2 + \cos(k\pi/N))$. The corresponding Ritz approximate eigenfunctions $\{u_{KN\pm}, v_{KN\pm}\}$ satisfy

$$v_{KN\pm}(i/N) = c \sin(ki\pi/N),$$  \hspace{1cm} (19a)

$$(a_1 - \lambda_{KN\pm}a_0)u_{KN\pm}(i/N) = c(a_{KN}Na_4\cos(ki\pi/N) - a_2\sin(ki\pi/N)),$$  \hspace{1cm} (19b)

$i = 0, \ldots, N$, where

$$a_{KN} = 3 \sin(k\pi/N)/(2 + \cos(k\pi/N)).$$

**Proof.** Direct calculation shows that the Ritz approximate solutions satisfy the matrix equation

$$P(\lambda)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (a_1 - \lambda a_0)P_1 & P_2 \\ P_3^T & (a_1 - \lambda a_0)P_3 + P_4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$  \hspace{1cm} (20)

with $P_r = (p_{rij}), r = 1, \ldots, 4$ where $P_1$ is $(N + 1) \times (N + 1)$, $P_3$ and $P_4$ are $(N - 1) \times (N - 1)$, $P_2$ is $(N + 1) \times (N - 1)$,

$$x_1 = \{x_{11}, \ldots, x_{1,N+1}\}, \quad x_2 = \{x_{21}, \ldots, x_{2,N-1}\},$$

$$x_{1i} = u\left(\frac{i-1}{N}\right), \quad x_{2i} = v\left(\frac{i}{N}\right), \quad \text{and}$$

$$p_{1ii} = 2/3N, \quad i = 2, \ldots, N - 1 \quad \text{and} \quad p_{111} = p_{1,N+1,N+1} = 1/3N,$$

$$p_{1,i+1} = p_{1,i+1,i} = 1/6N, \quad p_{1ij} = 0 \quad \text{if} \quad |i - j| > 1,$$

$$p_{2ii} = a_2/6N - a_4a_2/2, \quad p_{2,i+1,i} = 2a_2/3N,$$

$$p_{2,i+2,i} = a_2/6N + a_4a_2/2 \quad \text{and} \quad p_{2i,j} = 0 \quad \text{if} \quad |i - j - 1| > 1,$$

$$p_{3ii} = 2/3N, \quad p_{3,i+1} = p_{3,i+1,i} = 1/6N,$$

$$p_{3,i+2} = 2Na_4a_2^2, \quad p_{3,i,i+1} = p_{3,i+1,i} = -Na_4a_2^2,$$

and $p_{3ij} = p_{4ij} = 0 \quad \text{if} \quad |i - j| > 1.$
With the above $p_{ij}$, substitution of

$$
x_{1i} = \alpha_{kN} N a_4 a_5 \cos(k\pi(i - 1)/N) - a_2 \sin(k\pi(i - 1)/N),
$$

$$
x_{2i} = (a_1 - \lambda_{kN+} a_5) \sin(k\pi i/N)
$$

in the $j$th line of (20) in each of the cases $j = 1, j = N + 1, 2 \leq j \leq N$ and $N + 2 \leq j \leq 2N$ shows, after some simplification, that this is a solution for $k = 1, \ldots, N - 1$ when $\lambda_{kN\pm}$ satisfies (18). By (18),

$$
\lambda_{kN+} \lambda_{kN-} - (\lambda_{kN+} + \lambda_{kN-}) a_1/a_5 + a_1^2/a_5^2
$$

$$
= -(a_2^2 + (2 - \beta_{kN}) 3N^2\beta_{kN} a_1^2 a_4^2)/a_5 a_6 < 0,
$$

since $0 < \beta_{kN} < 2$.

It follows that $\lambda_{kN-} < a_1/a_5 < \lambda_{kN+}$. Clearly the space spanned by the \{u_{kN+}, v_{kN+}\} and that spanned by the \{u_{kN-}, v_{kN-}\} each have dimension $N - 1$. Hence, as $P(\lambda)$ is $2N \times 2N$, there are at most two linearly independent Ritz approximate eigenfunctions which are not given by (19). But [6] there are at least $n - m = 2$ linearly independent Ritz approximate solutions with $v = 0$ (and hence, by (3), $\lambda = a_1/a_5$). The result follows.

**Corollary 1.** $\lambda_{k+1,N\pm} - \lambda_{kN\pm}$ has the same sign as

$$
g(k + 1) + g(k) + 2(a_3 a_5 - a_2 a_5) + a_4(3N^2 a_1^2 a_5 - a_6)(\beta_{k+1N} + \beta_{kN})
$$

where

$$
g(k) = ((6a_4 a_5^2 a_1 N^2\beta_{kN} + a_6 a_5 - a_1 a_5)^2 + 4a_1^2 a_5 a_6
$$

$$
+ (2 - \beta_{kN}) 12N^2\beta_{kN} a_4^2 a_5 a_6)^{1/2} > 0.
$$

The proof is a straightforward calculation using (18).

The Ritz approximate eigenfunctions (19) and the convergence results of Theorem 3 below make it natural to regard $\lambda_{kN\pm}$ as the Ritz approximate eigenvalue corresponding to $\lambda_{k\pm}$. Yct Corollary 1 shows that the sign of $\lambda_{k+1,N\pm} - \lambda_{kN\pm}$ varies with $N$, $k$ and the $a_i$, so that, for given $N$ and $a_i$, the $\lambda_{kN\pm}$ do not always occur in the same order as the corresponding $\lambda_{k\pm}$. This contrasts with the result in the classical case (Lemma A) and with the results of Theorem 1 which show that $A_{(k+1)\pm}(m) - A_{k\pm}(m)$ always has the same sign as $\lambda_{(k+1)\pm} - \lambda_{k\pm}$. Clearly $\lambda_{k+1,N+} > \lambda_{kN+}$ whenever $a_6 \leq 3N^2 a_4^2 a_5$. (Also the existence of both $\lambda_{k+1,N+}$ and $\lambda_{kN+}$ implies $N \geq 3$.)

The condition $\lambda_{k+1,N-} < \lambda_{kN-}$ is stronger than the condition $\lambda_{k+1,N-} > \lambda_{kN-}$. Consequently, in the case $a_3 a_5 > a_2^2$ (when $\lambda_{(k+1)\pm} < \lambda_{k\pm}$ for all $k$) the $\lambda_{kN\pm}$ are less likely to occur in the same order as the corresponding exact eigenvalues than are the $\lambda_{kN\pm}$. (For example, when $a_3 = 2$ and $a_1 = a_2 = a_4 = a_5 = a_6 = 1$, $\lambda_{k+1,N+} > \lambda_{kN+}$ for all $k$ and $N$, but $\lambda_{1N-} < 0.153 < 0.495 < \lambda_{2N-}$ although Theorem 3(i) below shows that $\lambda_{1N-} > \lambda_{2N-}$ for sufficiently large $N$). In a
more complicated problem where the true solution is not known in advance, such unusual ordering of the Ritz approximate eigenvalues could lead to difficulty in deciding which Ritz approximate solutions correspond to which exact solutions. Consequently, totally incorrect conclusions about the true solutions could be reached.

Remark 2. The Ritz approximate eigenvalues obtained for (11) with \( v - \sum_{i=1}^{n} c_i \phi_{iN} \) are \( 6N^2 \beta_{kN} \), \( k = 1, \ldots, N - 1 \), and the corresponding Ritz approximate eigenfunctions have \( v(i/N) = \sin(k\pi i/N) \), \( i = 0, \ldots, N \). The extra term \( -3N^2 \beta_{kN} a_1/a_6 \) in (18) shows that, with this choice of coordinate functions, application of the Ritz method to (10) and to (3) yields different results.

Theorem 3.

(i) \( \lim_{N \to \infty} \lambda_{kN} = \lambda_k \pm \) with
\[ \lambda_{kN} - \lambda_k = O(k^4/N^2) \quad \text{as} \quad N \to \infty. \]

(ii) For \( N \) sufficiently large, \( \lambda_{kN} \geq \lambda_k \).

(iii) When \( \{u_{kN} \pm, v_{kN} \pm\} \) and \( \{u_k \pm, v_k \pm\} \) are suitably normalised
\[ \lim_{N \to \infty} \sup_{r \in [0,1]} \left( |u_{kN} \pm(r) - u_k \pm(r)| + |v_{kN} \pm(r) - v_k \pm(r)| \right) = 0. \]

Proof. By (5) and (18),
\[ \lambda_{kN} - \lambda_k = \frac{(d_1 - d_2 \pm (d_3 - d_4))/2a_5a_6}{d_1 = a_1a_6 + a_3a_5 + 6N^2 \beta_{kN} a_1a_4^2a_5,} \]
\[ d_2 = a_1a_6 + a_3a_5 + k^2 \pi^2 a_1a_4^2a_5, \]
\[ d_3 = (d_1^3 - 4a_3a_6(a_1a_3 - a_2^2 + 3N^2 \beta_{kN} a_1a_4^2a_5))^{1/2}, \]
\[ d_4 = (d_2^2 - 4a_3a_6(a_1a_3 - a_2^2))^{1/2}. \]

By Taylor's theorem
\[ d_1 - d_2 = \frac{k^4 \pi^4 a_1a_4^2a_5}{4N^2(2 + \cos(k\pi/N))} \left( 2 - \cos \frac{k\pi \theta_k}{N} - \frac{k^2 \pi^2}{6N^2} \cos \frac{k\pi \theta_k}{N} \right), \]
\[ d_3 - d_4 = (d_3 + d_4)^{-1} \left\{ \left[ (a_1a_6 + a_3a_5) + (6N^2 \beta_{kN} + k^2 \pi^2) a_1a_4^2a_5 \right] (d_1 - d_2) \right\} \]
\[ \frac{3a_1^2a_4^2a_5a_6k^4 \pi^4 \cos^2(k\pi \theta_k/N)}{N^2(2 + \cos(k\pi/N))^2}, \quad 0 < \theta_k < 1. \]
After some simplification it follows that, as \( N \to \infty \),

\[
\lambda_{kN^+} - \lambda_{k^\pm} \approx \frac{a_1a_2^2k^2\pi^4}{24a_6N^2} \left( 1 \pm \frac{d_6}{(d_6^2 + 4a_5(a_2^2 + a_1^2a_4^2k^2\pi^2))^{1/2}} \right) \tag{21}
\]

where \( d_6 = a_2a_6 - a_2a_8 + a_2a_4^2a_5k^2\pi^2 \). Since \( a_5a_6(a_2^2 + a_1^2a_4^2k^2\pi^2) > 0 \) and is independent of \( N \), (i) and (ii) follow. Hence, since \( u_{kN^\pm} \) and \( v_{kN^\pm} \) are linear in each of the intervals \([(i - 1)/N, i/N] (i = 1, \ldots, N) \) and continuous in \([0, 1]\), (iii) follows from (19), (7) and (8).

Remark 3. Numerical results indicate that, at least for some \( a_i \), Theorem 3(ii) holds even without the restriction “for sufficiently large \( N \)”. Moreover, for all \( N \), the weaker results \( \lambda_{kN^+}, \lambda_{kN^-} > \lambda_{k^+}, \lambda_{k^-} \) and

\[
\lambda_{kN^+} + \lambda_{kN^-} > \lambda_{k^+} + \lambda_{k^-}
\]

follow immediately from (18), (5) and the inequality \( 6N^2\beta_{kN} > k^2\pi^2, \ 0 < k < N \). Theorem 3(ii) contrasts with the result obtained with (13) where, by (14) and Theorem 1(i), \( \lambda_{k^+}(m) < \lambda_{k^-} \) for all \( k \) and \( m \) whenever \( a_1a_3 > a_5^2 \).

Differences in the pattern of results for (1) obtained by polynomial coordinate functions and by piecewise linear coordinate functions have analogues in results obtained for the astrophysical problem (see Section 6).

5. A MORE GENERAL EIGENVALUE PROBLEM

Denote by \( H \) and \( A \) the Hilbert space and the operator obtained when, in the definitions of \( H_0 \) and of \( A_0 \) in (1), the definitions of the \( a_i \) and the boundary condition (2) are replaced by the following. The \( a_i \) are functions mapping \([0, 1]\) into the real line, with \( a_1', a_2', a_4', a_3, a_5, a_6 \) continuous and \( a_5 > 0, a_6 > 0 \) in \((0, 1)\). Any boundary conditions may be specified which ensure that for \( \{u_1, v_1\}, \{u_2, v_2\} \) in \( \text{Dom} (A) \)

\[
[a_4(a_2u_1 + a_2v_1 - a_1a_4v_1^*)v_2]_0 = 0. \tag{22}
\]

Clearly \( A \) is hermitian and

\[
A \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} \tag{23}
\]

has variational formulation (3) where the \( a_i \) are no longer necessarily constant. Equation (1) is a special case.
Straightforward calculations (used in special cases in [6]) convert (23) into the nonlinear form
\[
\begin{pmatrix}
\lambda a_1 a_4 a_5 \\
\lambda a_1 a_5 \\
\lambda a_2 a_3 \\
\lambda a_2 a_5
\end{pmatrix}
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix}
= \begin{pmatrix}
-\lambda (a_2 a_5 + a_1 a_6') \\
\lambda a_1 a_6 + a_2^2 - a_1 a_6' \\
\lambda a_2 a_5
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix},
\] (24)

In this form, quadratic in \(\lambda\), (24) is a second order system of differential equations. In the linear form (23), it is of third order. Both formulations are in the same Hilbert space, \(H\). The linear formulation of the problem in [2] is not in the same Hilbert space as the nonlinear formulation but in a product space. Thus an \(n\)th order differential equation, quadratic in \(\lambda\), becomes a \(2n\)th order equation, linear in \(\lambda\). The linearization procedure used in [2] does not convert (24) into a form to which Lemma A may be applied.

Convergence of the Ritz method for many eigenvalue problems involving differential operators may be proved by applying Lemma A to the inverse operator (as in the proof of Theorem 1). This cannot be done with (23). The inverse of the third order differential operator \(A\), which exists whenever \(a_1(a_2 - a_5^2) > 0\) on \((0, 1)\), is another third order differential operator, similar in form to \(A\) (compare [4; Section II]). Results of Eisenfeld [4; Section 17], [3; Remark 4] suggest that there may be another linear formulation of (23) involving a linear operator which is both hermitian and compact. However, as already noted (Remarks 1 and 2), application of the Ritz method to different variational formulations of the same problem usually leads to different results. The only variational formulation of the astrophysical problem which has yet been used in numerical work is that analogous to (3). The present study concerns application of the Ritz method to that particular variational formulation, not the discovery of other formulations.

6. NON-RADIAL OSCILLATIONS OF STARS

The eigenvalue problem governing small adiabatic non-radial oscillations of stars [5; p. 520] and the simpler eigenvalue problem obtained when (as is frequently done) the Eulerian perturbation of gravitational potential is neglected, will be denoted here, as in [6], by \(IA\) and \(IB\) respectively. When the radius of the star is taken as the unit of length, \(IB\) is the special case of (23) in which, with the notation of [3],

\[
\begin{align*}
  a_1(r) &= \gamma \rho r^2, \\
  a_2(r) &= -p' r^2, \\
  a_3(r) &= p' r^2 \rho, \\
  a_4(r) &= 1, \\
  a_5(r) &= \rho \mu l(l + 1), \\
  a_6(r) &= \rho r^2, \\
  u &= \chi, \\
  v &= \psi, \\
  \lambda &= \sigma^2.
\end{align*}
\]

The boundary conditions required by physical considerations are that \(v(r)/r^2, u(r)/r\) and \(r \delta p(r)\) be everywhere bounded and that \(\delta p(1) = 0\), where
\[ \delta p = a_1 u + a_2 v - a_4 q u' \]. These are more than enough to ensure that (22) is satisfied. (Implications of minimum boundary requirements are studied in [3].) The \( a_i \) depend on the stellar model, but for all realistic models, and all those used in [8]–[12], the \( a_i \) are all strictly positive in \((0, 1)\). The fact that both \( IA \) and \( IB \) are special cases of the slight generalisation (23) was not noted in [6].

Neither \( IA \) nor \( IB \) has been solved exactly except for an unrealistic model of constant density [17] (for which the solutions closely resemble those of (1)). However asymptotic analysis of \( IB \) [18] and plentiful numerical results for \( IA \) and \( IB \) (see [6], [12], [17] for some references) indicate the following pattern, associated with each spherical harmonic \( l \). There is an infinite sequence of normal modes (\( p \)-modes) with eigenvalues tending to \(+\infty\). For each interval with \( a_1 a_3 - a_2^2 > 0 \) [respectively \(<0, =0\)] there is an infinite sequence of normal modes (\( g \)-modes) with positive [respectively negative, zero] eigenvalues with limit zero. (Physically, negative eigenvalues correspond to dynamical instability.) There is a single extra normal mode (the \( f \)-mode) analogous to the eigenfunction \( \{u_0, v_0\} \) of (1) with (positive) eigenvalue less than those of the \( p \)-modes and greater than those of the \( g \)-modes. Eigenfunctions of \( IA \) and \( IB \) are both similar to those of (1).

For a model in which \( a_1 a_3 > a_2^2 \) throughout, the Ritz method was applied to \( IA \) and to \( IB \) in [12], with \( (\psi) \) restricted to the subspace given by

\[ v(r) = \sum_{i=0}^{n} c_i r^{l+2l+1}, \quad u(r) = (l + 1) c_0 r^l + \sum_{i=1}^{m} c_{m+i} r^{l+2i}. \]  

(The same coordinate functions, with smaller \( m \) and \( n \), were used exclusively in [8]–[10].) For both \( IA \) and \( IB \), eigenfunctions obtained for \( g \)-modes were much less accurate than those obtained for \( f \)- or \( p \)-modes. In some cases addition of extra coordinate functions decreased the accuracy of results, and with \( m < n \) some apparently spurious modes were obtained. For \( IB \), the Ritz method, with \( u(r)/r^l \) and \( v(r)/r^{l+1} \) assumed continuous and piecewise linear, gave good estimates of \( f \)- and \( p \)-modes but no trace of \( g \)-modes, yielding instead some spurious modes [11]. Yet these piecewise linear coordinate functions were very successful with the problem of radial stellar oscillations [16], a singular Sturm-Liouville problem with a relationship to \( IB \) similar to that of (11) to (1). Numerical tests in [12] indicate that even if the calculations in [11] and [12] had been carried out without rounding errors, results would not have been significantly better, although rounding errors can be very important with the Ritz method [19; p. 229–240].

Interesting similarities may be noted between results proved here for (1) and numerical results for \( IA \) and \( IB \) given in [11] and [12]. In both cases polynomial coordinate functions gave results more akin to those obtained...
for the corresponding linear problem ((11) and the problem of radial stellar oscillations respectively) than did piecewise linear coordinate functions. A second resemblance is perhaps more significant. Ritz approximate solutions obtained for the sequence of modes with eigenvalues tending to zero were more often unsatisfactory, and were much more sensitive to changes in coordinate functions, than were those of the sequence with eigenvalues tending to infinity. It would be interesting to compare results given by the variational principle obtained from the inverse operator.

With both IA and IB, Robe and Brandt [9], using (25), obtained over-estimates for the eigenvalues of the p-modes and under-estimates for the modulus of those of the g-modes. They therefore conjectured that the p-modes might correspond to minima and the g-modes to maxima. Remark 3 suggests that the pattern in the Ritz approximate solutions which led to this conjecture depends critically on the coordinate functions used.

REFERENCES


