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Oscillation of second order neutral equations with distributed deviating argument

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Abstract

Oscillation criteria are established for the second order neutral delay differential equation with distributed deviating argument

$$(r(t)\psi(x(t))Z'(t))' + \int_a^b q(t,\xi)f[x(g(t,\xi))]\,\mathrm{d}\sigma(\xi) = 0, \quad t \geqslant t_0,$$

where $Z(t) = x(t) + p(t)x(t-\tau)$. These results are extensions of the integral averaging techniques due to Coles and Kamenev, and improve some known oscillation criteria in the existing literature.

MSC: 34C10; 34k11; 34K40

Keywords: Oscillation; Neutral delay differential equation; Second order; Distributed deviating argument; Integral averaging techniques

1. Introduction

In this paper we are concerned with the oscillation problem for the second order neutral delay differential equation with distributed deviating argument

$$(r(t)\psi(x(t))Z'(t))' + \int_a^b q(t,\xi)f[x(g(t,\xi))]\,\mathrm{d}\sigma(\xi) = 0, \quad t \geqslant t_0, \tag{1.1}$$

where $Z(t) = x(t) + p(t)x(t-\tau)$, $\tau \geqslant 0$, and the following conditions are assumed to hold without further mentioning:

(A1) $r, p \in C(I, \mathbb{R})$ and $0 \leqslant p(t) \leqslant 1$, $r(t) > 0$ for $t \in I$, $\int^\infty 1/r(s)\,\mathrm{d}s = \infty$, $I = [t_0, \infty)$;
(A2) $\psi \in C^1(\mathbb{R}, \mathbb{R})$, $\psi(x) > 0$ for $x \neq 0$;
(A3) $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ for $x \neq 0$;

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(A4) $q \in C(I \times [a, b], [0, \infty))$ and $q(t, \xi)$ is not eventually zero on any half-linear $[t_u, \infty) \times [a, b]$, $t_u \geqslant t_0$;

(A5) $g \in C(I \times [a, b], [0, \infty))$, $g(t, \xi) \leqslant t$ for $\xi \in [a, b]$, $g(t, \xi)$ has a continuous and positive partial derivative on $I \times [a, b]$ with respect to the first variable t and nondecreasing with respect to the second variable ξ, respectively, and $\liminf_{t\to\infty} g(t, \xi) = \infty$ for $\xi \in [a, b]$;

(A6) $\sigma \in C([a, b], \mathbb{R})$ is nondecreasing, and the integral of Eq. (1.1) is in the sense of Riemann–Stieltjes.

We restrict our attention to those solutions $x(t)$ of Eq. (1.1) which exist on some half linear $[t_x, \infty)$ with $\sup\{x(t) : t \geqslant T\} \neq 0$ for any $T \geqslant t_x$, and satisfy Eq. (1.1). As usual, a solution $x(t)$ of Eq. (1.1) is called oscillatory if the set of its zeros is unbounded from above, otherwise, it is called nonoscillatory. Eq. (1.1) is called oscillatory if all solutions are oscillatory.

We note that second order neutral delay differential equations have various applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems. For further applications and questions concerning existence and uniqueness of solutions of neutral delay differential equations, see [8].

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions of second order linear and nonlinear neutral delay differential equations with distributed deviating arguments (see, for example, [4,10,15] and the references therein). Very recently, in [12,13], the results of Philos [11] for second order linear ordinary differential equation have been extended to the neutral delay differential equations

$$(r(t)\psi(x(t))Z'(t))' + q(t)f(x(\tau(t))) = 0 \tag{1.2}$$

and

$$(r(t)Z'(t))' + \int_a^b q(t, \xi)x[g(t, \xi))]\,d\sigma(\xi) = 0, \tag{1.3}$$

which are not applicable to Eq. (1.1). Therefore it will be of great interest to improve the results of Sahiner [12] and Wang [13].

In this paper, by using a generalized Riccati technique and the integral averaging technique and following the results of Coles [5] and Wong [14], we establish some oscillation criteria for Eq. (1.1), which complement and extend the results in [13,15]. The relevance of our results becomes clear due to some carefully selected examples. The obtained oscillation criteria are essentially new even for Eq. (1.3). Our methodology is somewhat different from that of previous authors. We believe that our approach is simpler and also providers a more unified account for study of Coles and Kamenev type oscillation theorems.

For other oscillation results of various neutral functional differential equation we refer the reader to the monographs [1–3,6,7].

2. Notations and lemmas

For the simplicity of the proofs of the main results we present the following notations and lemmas.

Let $\Phi(t_0, t)$ denote the class of positive and locally integrable functions, but not integrable, which contains all the bounded functions for $t \geqslant t_0$. For arbitrary functions $\phi \in \Phi(t_0, t)$, $h \in C(I, \mathbb{R}^+ = (0, \infty))$ and $\theta \in C(I, \mathbb{R})$. For $t \geqslant T \geqslant t_0$, we define

$$\alpha(T, t) = \int_T^t \phi(s)\,ds, \quad \beta(h; T, t) = \frac{1}{\phi(t)} \int_T^t h(s)\phi^2(s)\,ds$$

and

$$A(\theta; T, t) = \frac{1}{\alpha(T, t)} \int_T^t \phi(s) \int_T^s \theta(u)\,du\,ds.$$

Following Philos [11], we introduce a class of functions $\mathfrak{I}$ as follows. Let

$$D_0 = \{(t, s) : t > s \geqslant t_0\} \quad \text{and} \quad D = \{(t, s) : t \geqslant s \geqslant t_0\}.$$

A kernel function $k \in C(D, \mathbb{R})$ is said to belong to the function class $\mathfrak{I}$ (written by $k \in \mathfrak{I}$) if

(H1) $k(t,t) = 0$ for $t \geqslant t_0$, $k(t,s) > 0$ on D_0;
(H2) k has a continuous and nonpositive partial derivative on D_0 with respect to the second variable such that the condition

$$-\frac{\partial k}{\partial s}(t,s) = \lambda(t,s)k(t,s) \quad \text{for all } (t,s) \in D_0$$

is satisfied for some $\lambda \in C(D, \mathbb{R})$.

Let $\rho \in C(I, \mathbb{R})$, we define an integral operator B, which is defined in [14] in terms of $k(t,s)$ and $\rho(s)$ as

$$B(\theta; T, t) = \int_T^t k(t,s)\theta(s)\rho(s)\,\mathrm{d}s \quad \text{for } t \geqslant t \geqslant t_0,$$

where $\theta \in C(I, \mathbb{R})$.

Let us state three sets of conditions commonly used as in [12] which we rely on:

(S1) $f'(x)$ exists, $f'(x) \geqslant k_1$ and $\psi(x) \leqslant L^{-1}$ for $x \neq 0$;
(S2) $f'(x)$ exists, $f'(x)/\psi(x) \geqslant k_2$ for $x \neq 0$;
(S3) $f(x)/x \geqslant k_3$ and $\psi(x) \leqslant L^{-1}$ for $x \neq 0$,

where k_1, k_2, k_3 and L are positive real numbers.

It is clear that assumption (S1) implies (S2), but not converse. For example, the functions $f(x) = x^3$ and $\psi(x) = x^2$ do not satisfy (S1), but (S2) holds. In (S1) and (S2), we need f to be differentiable. Clearly this condition is not required in (S3). The above facts force us to study Eq. (1.1) under conditions (S1), (S2) and (S3) separately.

In addition, we will make use of the following conditions:

(N1) There exists a positive real number M such that $\pm f(\pm uv) \geqslant Mf(u)f(v)$ for $uv > 0$;
(N2) $u\psi'(u) > 0$ for $u \neq 0$.

The following three lemmas will be useful for establishing oscillation criteria for Eq. (1.1).

Lemma 2.1. *Suppose that* (S1) *and* (N1) *are satisfied. Let $x(t)$ be an eventually positive solution of Eq.* (1.1); *then there exists a $T_0 \geqslant t_0$ such that*

$$Z(t) > 0, \quad Z'(t) > 0 \quad \textit{and} \quad (r(t)\psi(x(t))Z'(t))' \leqslant 0, \quad t \geqslant T_0. \tag{2.1}$$

Moreover,

$$(r(t)\psi(x(t))Z'(t))' + Mf[Z(g(t,a))] \int_a^b q(t,\xi) f[1 - p(g(t,\xi))]\,\mathrm{d}\sigma(\xi) \leqslant 0, \quad t \geqslant T_0. \tag{2.2}$$

Proof. Let $x(t)$ be an eventually positive solution of Eq. (1.1). Note that in view of (A5), there exists a $T_0 \geqslant t_0$ such that

$$x(t) > 0, \quad x(t-\tau) > 0 \quad \text{and} \quad x[g(t,\xi)] > 0, \quad t \geqslant T_0, \quad \xi \in [a,b]. \tag{2.3}$$

From (1.1), we also have $Z(t) > 0$ and $(r(t)\psi(x(t))Z'(t))' \leqslant 0$ for $t \geqslant T_0$.

Next, we show that $Z'(t) > 0$ for $t \geqslant T_0$. In fact, if there exists a $t_1 \geqslant T_0$ with $Z'(t_1) < 0$, then, noting that $r(t)\psi(x(t))$ $Z'(t)$ is decreasing, we have, for $t \geqslant t_1$,

$$r(t)\psi(x(t))Z'(t) \leqslant r(t_1)\psi(x(t_1))Z'(t_1) =: \delta < 0 \quad \text{for } t \geqslant t_1.$$

Dividing both sides by $r(t)\psi(x(t))>0$, we obtain

$$Z'(t)\leqslant \frac{\delta}{\psi(x(t))}\frac{1}{r(t)}. \tag{2.4}$$

Integrating (2.4) from t_1 to t leads to

$$Z(t)\leqslant Z(t_1)+L\delta\int_{t_1}^{t}\frac{\mathrm{d}s}{r(s)}. \tag{2.5}$$

In view of (A1), it follows from (2.5) that $Z(t)$ takes on negative values for sufficiently large values of t. Since this contradicts the fact that $Z(t)$ is eventually positive, we must have $Z'(t)>0$ for $t\geqslant T_0$. Using this fact together with $x(t)\leqslant Z(t)$, we see that

$$x(t)\geqslant[1-p(t)]Z(t),\quad t\geqslant T_0. \tag{2.6}$$

In view of (S1), (N1) and (2.6), we also see that

$$f[x(g(t,\xi))]\geqslant Mf[1-p(g(t,\xi))]f[Z(g(t,\xi))],\quad t\geqslant T_0,$$

thus, from (1.1), we get

$$\begin{aligned}0&=(r(t)\psi(x(t))Z'(t))'+\int_a^b q(t,\xi)f[x(g(t,\xi))]\,\mathrm{d}\sigma(\xi)\\&\geqslant(r(t)\psi(x(t))Z'(t))'+M\int_a^b q(t,\xi)f[1-p(g(t,\xi))]f[Z(g(t,\xi))]\,\mathrm{d}\sigma(\xi).\end{aligned} \tag{2.7}$$

Further, observing that $g(t,\xi)$ is nondecreasing with respect to ξ and $Z'(t)>0$ for $t\geqslant T_0$, we have

$$Z[g(t,\xi)]\geqslant Z[g(t,a)],\quad t\geqslant T_0,\quad \xi\in[a,b]. \tag{2.8}$$

So, $f[Z(g(t,\xi))]\geqslant f[Z(g(t,a))]$ for $t\geqslant T_0$ and $\xi\in[a,b]$. Thus (2.7) implies that (2.2) holds. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Suppose that* (S2) *and* (N1) *are satisfied. Let* $x(t)$ *be an eventually positive solution of Eq.* (1.1); *then there exists a* $T_0\geqslant t_0$ *such that* (2.1) *and* (2.2) *hold.*

Proof. Let $x(t)$ be an eventually positive solution of Eq. (1.1). As in the proof of Lemma 2.1, there exists a $T_0\geqslant t_0$ such that (2.3) holds. Further, we also see that $Z(t)>0$, $(r(t)\psi(x(t))Z'(t))'\leqslant 0$ and (2.4) hold for $t\geqslant T_0$. Now we may claim $Z'(t)>0$ for $t\geqslant T_0$. In fact, in view of $x(t)\leqslant Z(t)$ for $t\geqslant T_0$, multiplication of (2.4) by $f'(Z(t))>0$ gives

$$f'(Z(t))Z'(t)\leqslant\frac{k_2\,\delta}{r(t)}.$$

Clearly,

$$f(Z(t))\leqslant f(Z(T_0))+k_2\delta\int_{T_0}^{t}\frac{\mathrm{d}s}{r(s)}\quad \text{for } t\geqslant T_0.$$

Letting $t\to\infty$ in the above inequality and using (A1), $f(Z(t))\to-\infty$. Since this contradicts (A3) we must have $Z'(t)>0$ for $t\geqslant T_0$. Next, by following the same steps in the proof of Lemma 2.1, we get that (2.2) holds. This completes the proof. □

Lemma 2.3. *Suppose that* (S3) *is satisfied. Let* $x(t)$ *be an eventually positive solution of Eq.* (1.1); *then there exists a* $T_0\geqslant t_0$ *such that* (2.1) *holds. Moreover*,

$$(r(t)\psi(x(t))Z'(t))'+k_3Z[g(t,a)]\int_a^b q(t,\xi)[1-p(g(t,\xi))]\,\mathrm{d}\sigma(\xi)\leqslant 0,\quad t\geqslant T_0. \tag{2.9}$$

Proof. Let $x(t)$ be an eventually positive solution of Eq. (1.1). As in the proof of Lemma 2.1, there exists a $T_0 \geqslant t_0$ such that (2.1) and (2.3) hold. Thus, from (1.1) and (S3), we have

$$\begin{aligned} 0 &= (r(t)\psi(x(t))Z'(t))' + \int_a^b q(t,\xi) f[x(g(t,\xi))]\,\mathrm{d}\sigma(\xi) \\ &\geqslant (r(t)\psi(x(t))Z'(t))' + k_3 \int_a^b q(t,\xi) x[(g(t,\xi))]\,\mathrm{d}\sigma(\xi) \\ &\geqslant (r(t)\psi(x(t))Z'(t))' + k_3 \int_a^b q(t,\xi)\{Z[g(t,\xi)] - p[g(t,\xi)]x[g(t,\xi)-\tau]\}\,\mathrm{d}\sigma(\xi). \end{aligned} \tag{2.10}$$

Note that

$$Z[g(t,\xi)] \geqslant Z[g(t,\xi)-\tau] \geqslant x[g(t,\xi)-\tau].$$

Thus, (2.10) implies that

$$(r(t)\psi(x(t))Z'(t))' + k_3 \int_a^b q(t,\xi)[1 - p(g(t,\xi))]Z[g(t,\xi)]\,\mathrm{d}\sigma(\xi) \leqslant 0, \quad t \geqslant T_0. \tag{2.11}$$

Observing (2.8), it follows from (2.11) that (2.9) holds. This completes the proof. □

3. Main results

In this section, we will give and show the main results of this paper. First of all, we establish Coles-type oscillation criteria for Eq. (1.1).

Theorem 3.1. *Let assumptions* (S1) *and* (N1) *hold. If there exist functions* $\phi \in \Phi(t_0, t)$, $\varphi \in C^1(I, \mathbb{R}^+)$ *and* $R \in C^1(I, \mathbb{R})$ *such that*

$$l_1(t) \geqslant 0 \quad \text{for } t \geqslant t_0, \tag{3.1}$$

$$\int_{t_0}^{\infty} \frac{\alpha^v(T,s)}{\beta(h_1; T, s)}\,\mathrm{d}s = \infty, \quad 0 \leqslant v < 1, \quad T \geqslant t_0, \tag{3.2}$$

and

$$\lim_{t\to\infty} A(\Theta_1 - \tfrac{1}{4}h_1 l_1^2; T, t) = \infty, \quad T \geqslant t_0, \tag{3.3}$$

where

$$l_1(t) = \frac{\varphi'(t)}{\varphi(t)} + \frac{2k_1 L g'(t,a) R(t)}{r[g(t,a)]}, \quad h_1(t) = \frac{r[g(t,a)]\varphi(t)}{k_1 L g'(t,a)},$$

and

$$\Theta_1(t) = \varphi(t)\left\{ M \int_a^b q(t,\xi) f[1 - p(g(t,\xi))]\,\mathrm{d}\sigma(\xi) + \frac{k_1 L g'(t,a) R^2(t)}{r[g(t,a)]} - R'(t) \right\},$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1) on I. Without loss of generality we assume that $x(t) \neq 0$ for $t \geqslant t_0$. Furthermore, we suppose that $x(t) > 0$ for $t \geqslant t_0$, since the substitution $u = -x$ transforms Eq. (1.1) into an equation of the same form subject to the assumptions of the theorem. Then, by Lemma 2.1, there exists a $T_0 \geqslant t_0$ such that (2.1) and (2.2) hold for $t \geqslant T_0$. Define

$$v(t) = \varphi(t)\left[\frac{r(t)\psi(x(t))Z'(t)}{f[Z(g(t,a))]} + R(t) \right] \quad \text{for all } t \geqslant T_0. \tag{3.4}$$

Then, differentiating (3.4) and using (2.2), it follows that

$$v'(t) \leqslant \frac{\varphi'(t)}{\varphi(t)} v(t) - \varphi(t) \left[M \int_a^b q(t,\xi) f[1-p(g(t,\xi))] \, d\sigma(\xi) + \frac{r(t)\psi(x(t))Z'(t)}{f^2[Z(g(t,a))]} f'[Z(g(t,a))]Z'[g(t,a)]g'(t,a) - R'(t) \right].$$

Since $g(t,a) \leqslant t$ and $(r(t)\psi(x(t))Z'(t))' \leqslant 0$ for $t \geqslant T_0$, we have

$$r(t)\psi(x(t))Z'(t) \leqslant r[g(t,a)]\psi[x(g(t,a))]Z'[g(t,a)].$$

Therefore, we have

$$\begin{aligned} v'(t) &\leqslant \frac{\varphi'(t)}{\varphi(t)} v(t) - \varphi(t) \left[M \int_a^b q(t,\xi) f[1-p(g(t,\xi))] \, d\sigma(\xi) - R'(t) \right] \\ &\quad - \frac{k_1 \varphi(t) g'(t,a)}{r[g(t,a)]\psi[x(g(t,a))]} \left(\frac{r(t)\psi(x(t))Z'(t)}{f[Z(g(t,a))]} \right)^2 \\ &\leqslant \frac{\varphi'(t)}{\varphi(t)} v(t) - \varphi(t) \left[M \int_a^b q(t,\xi) f[1-p(g(t,\xi))] \, d\sigma(\xi) - R'(t) \right] \\ &\quad - \frac{k_1 L \varphi(t) g'(t,a)}{r[g(t,a)]} \left(\frac{v(t)}{\varphi(t)} - R(t) \right)^2 \\ &= -\Theta_1(t) + l_1(t) v(t) - \frac{1}{h_1(t)} v^2(t), \end{aligned} \tag{3.5}$$

that is,

$$v'(t) \leqslant -\Theta_1(t) + \frac{1}{4} h_1(t) l_1^2(t) - \frac{1}{h_1(t)} \left[v(t) - \frac{1}{2} h_1(t) l_1(t) \right]^2. \tag{3.6}$$

Hence, for $t \geqslant T \geqslant T_0$,

$$v(t) + \int_T^t \frac{1}{h_1(s)} \left[v(s) - \frac{1}{2} h_1(s) l_1(s) \right]^2 ds \leqslant v(T) - \int_{T_0}^t \left[\Theta_1(s) - \frac{1}{4} h_1(s) l_1^2(s) \right] ds. \tag{3.7}$$

Multiplying relation (3.7) by $\phi(t)$ and integrating from T to t, we get

$$\begin{aligned} &\int_T^t \phi(s) v(s) \, ds + \int_T^t \phi(s) \int_T^s \frac{1}{h_1(u)} \left[v(u) - \frac{1}{2} h_1(u) l_1(u) \right]^2 du \, ds \\ &\quad \leqslant \alpha(T,t) \left[v(T) - A\left(\Theta_1 - \frac{1}{4} h_1 l_1^2; T, t \right) \right]. \end{aligned}$$

From condition (3.3), there exists a $T_1 \geqslant T$ such that

$$v(T) - A(\Theta_1 - \tfrac{1}{4} h_1 l_1^2; T, t) < 0 \quad \text{for all } t \geqslant T_1.$$

Then, for every $t \geqslant T_1$,

$$\begin{aligned} F(t) &= \int_T^t \phi(s) \int_T^s \frac{1}{h_1(u)} \left[v(u) - \frac{1}{2} h_1(u) l_1(u) \right]^2 du \, ds \\ &\leqslant -\int_T^t \phi(s) v(s) \, ds, \end{aligned}$$

and by condition (3.1), we obtain

$$F(t) \leqslant F(t) + \int_T^t \frac{1}{2}\,\phi(s)h_1(s)l_1(s)\,\mathrm{d}s < -\int_T^t \phi(s)\left[v(s) - \frac{1}{2}\,h_1(s)l_1(s)\right]\mathrm{d}s.$$

Since F is nonnegative, we have

$$F^2(t) \leqslant \left\{\int_T^t \phi(s)\left[v(s) - \frac{1}{2}\,h_1(s)l_1(s)\right]\mathrm{d}s\right\}^2, \quad t \geqslant T_1.$$

By the Schwarz inequality, we obtain, for $t \geqslant T_1$,

$$\begin{aligned} F^2(t) &\leqslant \left\{\int_T^t \sqrt{h_1(s)}\phi(s)\left(\frac{1}{\sqrt{h_1(s)}}\left[v(s) - \frac{1}{2}h_1(s)l_1(s)\right]\right)\mathrm{d}s\right\}^2 \\ &\leqslant \left(\int_T^t h_1(s)\phi^2(s)\,\mathrm{d}s\right)\int_T^t \frac{1}{h_1(s)}\left[v(s) - \frac{1}{2}\,h_1(s)l_1(s)\right]^2 \mathrm{d}s \\ &= \beta(h_1; T, t)F'(t). \end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned} F(t) &= \int_T^t \phi(s)\int_T^s \frac{1}{h_1(u)}\left[v(u) - \frac{1}{2}\,h_1(u)l_1(u)\right]^2 \mathrm{d}u\,\mathrm{d}s \\ &\geqslant \int_T^t \phi(s)\int_T^{T_1} \frac{1}{h_1(u)}\left[v(u) - \frac{1}{2}\,h_1(u)l_1(u)\right]^2 \mathrm{d}u\,\mathrm{d}s \\ &= C\alpha(T, t), \end{aligned} \tag{3.9}$$

where $C = \int_T^{T_1}(1/h_1(u))[v(u) - \frac{1}{2}h_1(u)l_1(u)]^2\,\mathrm{d}u$. From (3.8) and (3.9), for all $t \geqslant T_1$ and some v, $0 \leqslant v < 1$, we get

$$C^v\,\frac{\alpha^v(T, t)}{\beta(h_1; T, t)} \leqslant F^{v-2}(t)F'(t). \tag{3.10}$$

Integrating (3.10) from T_1 to t, we obtain

$$C^v \int_{T_1}^t \frac{\alpha^v(T, s)}{\beta(h_1; T, s)}\,\mathrm{d}s \leqslant \frac{1}{1-v}\,\frac{1}{F^{1-v}(T_1)} < \infty,$$

and this contradicts (3.2). Hence, we complete the proof of Theorem 3.1. □

Theorem 3.2. *Let assumptions* (S2), (N1) *and* (N2) *hold. If there exist functions* $\phi \in \Phi(t_0, t)$, $\varphi \in C^1(I, \mathbb{R}^+)$ *and* $R \in C^1(I, \mathbb{R})$ *such that*

$$l_2(t) \geqslant 0 \quad \textit{for } t \geqslant t_0, \tag{3.11}$$

$$\int_{t_0}^\infty \frac{\alpha^v(T, s)}{\beta(h_2; T, s)}\,\mathrm{d}s = \infty, \quad 0 \leqslant v < 1, \quad T \geqslant t_0, \tag{3.12}$$

and

$$\lim_{t\to\infty} A(\Theta_2 - \tfrac{1}{4}\,h_2 l_2^2; T, t) = \infty, \quad T \geqslant t_0, \tag{3.13}$$

where

$$l_2(t) = \frac{\varphi'(t)}{\varphi(t)} + \frac{2k_2\,g'(t, a)R(t)}{r[g(t, a)]}, \quad h_2(t) = \frac{r[g(t, a)]\varphi(t)}{k_2\,g'(t, a)},$$

and

$$\Theta_2(t) = \varphi(t)\left\{M\int_a^b q(t,\xi)f[1-p(g(t,\xi))]\,\mathrm{d}\sigma(\xi) + \frac{k_2\,g'(t,a)R^2(t)}{r[g(t,a)]} - R'(t)\right\}.$$

Then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1) on I, say $x(t)>0$ for $t\geqslant t_0$. Then, by Lemma 2.2, there exists a $T_0\geqslant t_0$ such that (2.1) and (2.2) hold. We consider the function $v(t)$ defined by (3.4), and obtain

$$\begin{aligned} v'(t) \leqslant{}& \frac{\varphi'(t)}{\varphi(t)}\,v(t) - \varphi(t)\left[M\int_a^b q(t,\xi)f[1-p(g(t,\xi))]\,\mathrm{d}\sigma(\xi) - R'(t)\right] \\ &- \frac{\varphi(t)g'(t,a)}{r[g(t,a)]}\,\frac{f'[Z(g(t,a))]}{\psi[x(g(t,a))]}\left(\frac{r(t)\psi(x(t))Z'(t)}{f[Z(g(t,a))]}\right)^2. \end{aligned}$$

Now, we use $x[g(t,a)]\leqslant Z[g(t,a)]$ and (N2) to obtain that

$$\frac{f'[Z(g(t,a))]}{\psi[x(g(t,a))]} \geqslant \frac{f'[Z(g(t,a))]}{\psi[Z(g(t,a))]} \geqslant k_2.$$

Therefore, we have

$$\begin{aligned} v'(t) \leqslant{}& \frac{\varphi'(t)}{\varphi(t)}\,v(t) - \varphi(t)\left[M\int_a^b q(t,\xi)f[1-p(g(t,\xi))]\,\mathrm{d}\sigma(\xi) - R'(t)\right] \\ &- \frac{k_2\varphi(t)g'(t,a)}{r[g(t,a)]}\left(\frac{r(t)\psi(x(t))Z'(t)}{f[g(t,a)]}\right)^2 \\ ={}& -\Theta_2(t) + l_2(t)v(t) - \frac{1}{h_2(t)}\,v^2(t). \end{aligned} \tag{3.14}$$

The rest of the proof runs as in Theorem 3.1. □

Theorem 3.3. *Let assumption* (S3) *hold. If there exist functions* $\phi\in\Phi(t_0,t)$, $\varphi\in C^1(I,\mathbb{R}^+)$ *and* $R\in C^1(I,\mathbb{R})$ *such that*

$$l_3(t)\geqslant 0 \quad \text{for } t\geqslant t_0, \tag{3.15}$$

$$\int_{t_0}^{\infty} \frac{\alpha^{\nu}(T,s)}{\beta(h_3;T,s)}\,\mathrm{d}s = \infty, \quad 0\leqslant \nu<1,\ \ T\geqslant t_0, \tag{3.16}$$

and

$$\lim_{t\to\infty} A(\Theta_3 - \tfrac{1}{4}h_3 l_3^2;T,t) = \infty, \quad T\geqslant t_0, \tag{3.17}$$

where

$$l_3(t) = \frac{\varphi'(t)}{\varphi(t)} + \frac{2\,L\,g'(t,a)R(t)}{r[g(t,a)]}, \quad h_3(t) = \frac{r[g(t,a)]\varphi(t)}{L\,g'(t,a)},$$

and

$$\Theta_3(t) = \varphi(t)\left\{k_3\int_a^b q(t,\xi)\{1-p[g(t,\xi)]\}\,\mathrm{d}\sigma(\xi) + \frac{L\,g'(t,a)R^2(t)}{r[g(t,a)]} - R'(t)\right\},$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1) on I, say $x(t)>0$ for $t \geqslant t_0$. Then, by Lemma 2.3, there exists a $T_0 \geqslant t_0$ such that (2.1) and (2.9) hold. We define the function $v(t)$ by

$$v(t)=\varphi(t)\left[\frac{r(t)\psi(x(t))Z'(t)}{Z[g(t,a)]}+R(t)\right] \quad \text{for all } t \geqslant T_0. \tag{3.18}$$

Differentiating (3.18) and using (2.9), we obtain

$$\begin{aligned} v'(t) &\leqslant \frac{\varphi'(t)}{\varphi(t)}v(t)-\varphi(t)\left[k_3\int_a^b q(t,\xi)\{1-p(g(t,\xi))\}\,\mathrm{d}\sigma(\xi)-R'(t)\right] \\ &\quad -\frac{\varphi(t)g'(t,a)}{r[g(t,a)]\psi[x(g(t,a))]}\left(\frac{r(t)\psi(x(t))Z'(t)}{Z(g(t,a))}\right)^2 \\ &\leqslant \frac{\varphi'(t)}{\varphi(t)}v(t)-\varphi(t)\left[k_3\int_a^b q(t,\xi)\{1-p(g(t,\xi))\}\,\mathrm{d}\sigma(\xi)-R'(t)\right] \\ &\quad -\frac{L\varphi(t)g'(t,a)}{r[g(t,a)]}\left(\frac{v(t)}{\varphi(t)}-R(t)\right)^2 \\ &= -\Theta_3(t)+l_3(t)v(t)-\frac{1}{h_3(t)}v^2(t). \end{aligned} \tag{3.19}$$

Inequality (3.19) is of the same type as inequality (3.5). Hence, we can use a similar procedure to complete the proof of Theorem 3.3. □

Next, we present some new oscillation results for Eq. (1.1), by using integral average conditions of Philos-type.

Theorem 3.4. *Let assumptions* (S1) *and* (N1) *hold. If there exist functions* $\varphi \in C^1(I,\mathbb{R}^+)$, $R \in C^1(I,\mathbb{R})$, $\rho \in C^1(I,\mathbb{R}^+)$ *and* $k \in \mathfrak{I}$ *such that*

$$\limsup_{t\to\infty}\frac{1}{k(t,t_0)}B\left(\Theta_1-\frac{1}{4}h_1(\lambda-l_1-\rho^{-1}\rho')^2;t_0,t\right)=\infty, \tag{3.20}$$

then Eq. (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.1, we see that (3.5) holds for all $t \geqslant T \geqslant T_0$. Applying the operator $B(\cdot;t,T)$ to (3.5), we have

$$B(\Theta_1;T,t)+B((\lambda-l_1-\rho\rho')v;T,t)+B(h_1^{-1}v^2;T,t)\leqslant k(t,T)\rho(T)v(T). \tag{3.21}$$

Completing squares of v in (3.21) yields

$$\begin{aligned} &B(h_1^{-1}(v+\tfrac{1}{2}h_1(\lambda-l_1-\rho^{-1}\rho'))^2;T,t) \\ &\quad +B(\Theta_1-\tfrac{1}{4}h_1(\lambda-l_1-\rho^{-1}\rho')^2;T,t)\leqslant k(t,T)\rho(T)v(T). \end{aligned} \tag{3.22}$$

Note that the first term is nonnegative, so

$$B(\Theta_1-\tfrac{1}{4}h_1(\lambda-l_1-\rho^{-1}\rho')^2;T,t)\leqslant k(t,T)\rho(T)v(T). \tag{3.23}$$

Thus, we have, for $t \geqslant t_0$,

$$\begin{aligned} &B\left(\Theta_1-\frac{1}{4}h_1(\lambda-l_1-\rho^{-1}\rho')^2;t_0,t\right) \\ &\quad =B\left(\Theta_1-\frac{1}{4}h_1(\lambda-l_1-\rho^{-1}\rho')^2;t_0,T_0\right)+B\left(\Theta_1-\frac{1}{4}h_1(\lambda-l_1-\rho^{-1}\rho')^2;T_0,t\right) \\ &\quad \leqslant k(t,t_0)\left[\int_{t_0}^{T_0}|\Theta_1(s)|\rho(s)\,\mathrm{d}s+\rho(T_0)|v(T_0)|\right]. \end{aligned}$$

Dividing both sides of the above inequality and taking limsup in it as $t \to \infty$, we obtain a contradiction to condition (3.20). This completes the proof of Theorem 3.4. □

Theorem 3.5. *Let assumptions* (S2), (N1) *and* (N2) *hold. If there exist functions* $\varphi \in C^1(I, \mathbb{R}^+)$, $R \in C^1(I, \mathbb{R})$, $\rho \in C^1(I, \mathbb{R}^+)$ *and* $k \in \Im$ *such that*

$$\limsup_{t\to\infty} \frac{1}{k(t,t_0)} B\left(\Theta_2 - \frac{1}{4} h_2(\lambda - l_2 - \rho^{-1}\rho')^2; t_0, t\right) = \infty, \tag{3.24}$$

then Eq. (1.1) *is oscillatory.*

Proof. Starting with inequality (3.14), we proceed as in the proof of Theorem 3.4. □

Theorem 3.6. *Let assumption* (S3) *hold. If there exist functions* $\varphi \in C^1(I, \mathbb{R}^+)$, $R \in C^1(I, \mathbb{R})$, $\rho \in C^1(I, \mathbb{R}^+)$ *and* $k \in \Im$ *such that*

$$\limsup_{t\to\infty} \frac{1}{k(t,t_0)} B\left(\Theta_3 - \frac{1}{4} h_3(\lambda - l_3 - \rho^{-1}\rho')^2; t_0, t\right) = \infty, \tag{3.25}$$

then Eq. (1.1) *is oscillatory.*

Proof. The proof follows the same lines as that of Theorem 3.4 with the only difference that we start with inequality (3.19). □

Remark 3.1. For Eq. (1.3), Theorem 3.6 improves Theorem 1 in [13].

The following three oscillation criteria (Theorems 3.7–3.9) treat the cases when it is not possible to verify easily conditions (3.20), (3.24) and (3.25).

Theorem 3.7. *Let assumptions* (S1) *and* (N1) *hold. If there exist functions* $\varphi \in C^1(I, \mathbb{R}^+)$, $R \in C^1(I, \mathbb{R})$, $\rho \in C^1(I, \mathbb{R}^+)$, $\varphi_1, \varphi_2 \in C(I, \mathbb{R})$ *and* $k \in \Im$ *such that for all* $T \geqslant t_0$,

$$\limsup_{t\to\infty} \frac{1}{k(t,T)} B(\Theta_1; T, t) \geqslant \varphi_1(T) \tag{3.26}$$

and

$$\limsup_{t\to\infty} \frac{1}{k(t,T)} B(h_1(\lambda - l_1 - \rho^{-1}\rho')^2; T, t) \leqslant \varphi_2(T), \tag{3.27}$$

where φ_1 *and* φ_2 *satisfy*

$$\liminf_{t\to\infty} \frac{1}{k(t,T)} B\left(h_1^{-1}\rho^{-2}\left(\varphi_1 - \frac{1}{4}\varphi_2\right)_+^2; T, t\right) = \infty, \tag{3.28}$$

where $\phi_+ = \max\{\phi, 0\}$, *then Eq.* (1.1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 3.4 and obtain (3.21) and (3.23) hold. Dividing (3.23) through $k(t,T)$, we obtain, by (3.26) and (3.27), that

$$\varphi_1(T) - \tfrac{1}{4}\varphi_2(T) \leqslant \rho(T)v(T), \quad T \geqslant T_0,$$

from which it follows that

$$\frac{1}{h_1(T)\rho^2(T)}\left(\varphi_1(T) - \frac{1}{4}\varphi_2(T)\right)_+^2 \leqslant \frac{1}{h_1(T)} v^2(T). \tag{3.29}$$

On the other hand, by (3.21),

$$\frac{1}{k(t,T)}B(h_1^{-1}v^2+(\lambda-l_1-\rho^{-1}\rho')v;T,t)\leqslant\rho(T)v(T)-\frac{1}{k(t,T)}B(\Theta_1;T,t),$$

this and (3.26) imply that

$$\liminf_{t\to\infty}\frac{1}{k(t,T)}B(h_1^{-1}v^2+(\lambda-l_1-\rho^{-1}\rho')v;T,t)$$
$$\leqslant\rho(T)v(T)-\varphi_1(T)\leqslant M_0,\quad t\geqslant T\geqslant T_0, \tag{3.30}$$

where M_0 is a constant.

Now, we claim that

$$\liminf_{t\to\infty}\frac{1}{k(t,T)}B(h_1^{-1}v^2;T,t)<\infty. \tag{3.31}$$

If (3.31) does not hold, there exists a sequence $\{t_n\}_{n=1}^{\infty}\subset[t_0,\infty)$ with $\lim_{n\to\infty}t_n=\infty$ such that

$$\lim_{n\to\infty}\frac{1}{k(t_n,T)}B(h_1^{-1}v^2;T,t_n)=\infty. \tag{3.32}$$

Note that, by (3.30), for n large enough,

$$\frac{1}{k(t_n,T)}B(h_1^{-1}v^2;T,t_n)+\frac{1}{k(t_n,T)}B((\lambda-l_1-\rho^{-1}\rho')v;T,t_n)\leqslant M_0+1.$$

This and (3.22) give, for n large enough, that

$$1+\frac{B((\lambda-l_1-\rho^{-1}\rho')v;T,t_n)}{B(h_1^{-1}v^2;T,t_n)}<\frac{1}{2},$$

that is,

$$\frac{|B((\lambda-l_1-\rho^{-1}\rho')v;T,t_n)|}{B(h_1^{-1}v^2;T,t_n)}>\frac{1}{2}. \tag{3.33}$$

The Schwarz inequality follows

$$[B((\lambda-l_1-\rho^{-1}\rho')v;T,t_n)]^2$$
$$\leqslant B(h_1^{-1}v^2;T,t_n)B(h_1(\lambda-l_1-\rho^{-1}\rho')^2;T,t_n). \tag{3.34}$$

From (3.33) and (3.34), we obtain

$$B(h_1^{-1}v^2;T,t_n)\leqslant 4B(h_1(\lambda-l_1-\rho^{-1}\rho')^2;T,t_n). \tag{3.35}$$

By (3.27), the right-hand side of (3.35) is bounded, which contradicts (3.32). Thus (3.31) holds. Hence, by (3.29), we obtain

$$\liminf_{t\to\infty}\frac{1}{k(t,T)}B\left(h_1^1\rho^{-2}\left(\varphi_1-\frac{1}{4}\varphi_2\right)_+^2;T,t\right)$$
$$\leqslant\liminf_{t\to\infty}\frac{1}{k(t,T)}B(h_1^{-1}v^2;T,t)<\infty,$$

which contradicts (3.28). This completes the proof. □

By using the same procedure of the proof of Theorem 3.7, we may obtain two analogous theorems to Theorem 3.7, which we state here for completeness.

Theorem 3.8. *Let assumptions* (S2), (N1) *and* (N2) *hold. If there exist functions* $\varphi \in C^1(I, \mathbb{R}^+)$, $R \in C^1(I, \mathbb{R})$, $\rho \in C^1(I, \mathbb{R}^+)$, $\varphi_1, \varphi_2 \in C(I, \mathbb{R})$ *and* $k \in \Im$ *such that for all* $T \geqslant t_0$,

$$\limsup_{t\to\infty} \frac{1}{k(t,T)} B(\Theta_2; T, t) \geqslant \varphi_1(T) \tag{3.36}$$

and

$$\limsup_{t\to\infty} \frac{1}{k(t,T)} B(h_2(\lambda - l_2 - \rho^{-1}\rho')^2; T, t) \leqslant \varphi_2(T), \tag{3.37}$$

where φ_1 *and* φ_2 *satisfy*

$$\liminf_{t\to\infty} \frac{1}{k(t,T)} B\left(h_2^{-1}\rho^{-2}\left(\varphi_1 - \frac{1}{4}\varphi_2\right)_+^2; T, t\right) = \infty, \tag{3.38}$$

then Eq. (1.1) *is oscillatory.*

Theorem 3.9. *Let assumption* (S3) *hold. If there exist functions* $\varphi \in C^1(I, \mathbb{R}^+)$, $R \in C^1(I, \mathbb{R})$, $\rho \in C^1(I, \mathbb{R}^+)$, φ_1, $\varphi_2 \in C(I, \mathbb{R}^+)$ *and* $k \in \Im$ *such that for all* $T \geqslant t_0$,

$$\limsup_{t\to\infty} \frac{1}{k(t,T)} B(\Theta_3; T, t) \geqslant \varphi_1(T) \tag{3.39}$$

and

$$\limsup_{t\to\infty} \frac{1}{k(t,T)} B(h_3(\lambda - l_3 - \rho^{-1}\rho')^2; T, t) \leqslant \varphi_2(T), \tag{3.40}$$

where φ_1 *and* φ_2 *satisfy*

$$\liminf_{t\to\infty} \frac{1}{k(t,T)} B\left(h_3^{-1}\rho^{-2}\left(\varphi_1 - \frac{1}{4}\varphi_2\right)_+^2; T, t\right) = \infty, \tag{3.41}$$

then Eq. (1.1) *is oscillatory.*

Remark 3.2. For Eq. (1.3), Theorem 3.9 improves and unifies Theorems 2 and 3 of Wang [13].

4. Corollaries and examples

As Theorems 3.1–3.9 are rather general, it is convenient for applications to derive a number of oscillation criteria with the appropriate choice of the functions ϕ, φ, R, ρ and k.

Corollary 4.1. *Let assumptions* (S1) *and* (N1) *hold. If there exist functions* $\phi \in C^1(I, \mathbb{R}^+)$ *and* $R \in C^1(I, \mathbb{R})$ *such that* (3.1) *holds, and*

$$\int_{t_0}^{\infty} \frac{1}{h_1(s)}\,\mathrm{d}s = \int_{t_0}^{\infty} \left[\Theta_1(s) - \frac{1}{4}h_1(s)l_1^2(s)\right]\mathrm{d}s = \infty, \tag{4.1}$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $\phi(s) = 1/h_1(s)$. Then, for $0 < \mu < 1$ and $t \geqslant T \geqslant t_0$, we have

$$\begin{aligned}\lim_{t\to\infty} \int_T^t \frac{\alpha^v(T,s)}{\beta(h_1; T, s)}\,\mathrm{d}s &= \lim_{t\to\infty} \int_T^t \frac{1}{h_1(s)}\left(\int_T^s \frac{1}{h_1(u)}\,\mathrm{d}u\right)^{v-1}\mathrm{d}s \\ &= \frac{1}{v}\lim_{t\to\infty}\left(\int_T^t \frac{1}{h_1(s)}\,\mathrm{d}s\right)^v = \infty\end{aligned}$$

and

$$\lim_{t\to\infty} A\left(\Theta_1 - \frac{1}{4}h_1 l_1^2; T, t\right)$$
$$= \lim_{t\to\infty}\left(\int_T^t \frac{1}{h_1(s)}\,\mathrm{d}s\right)^{-1}\int_T^t \frac{1}{h_1(s)}\int_T^s \left[\Theta_1(u) - \frac{1}{4}h_1(u)l_1^2(u)\right]\mathrm{d}u\,\mathrm{d}s$$
$$= \lim_{t\to\infty}\int_{t_0}^t \left[\Theta_1(s) - \frac{1}{4}h_1(s)l_1^2(s)\right]\mathrm{d}s = \infty.$$

By Theorem 3.1, Eq. (1.1) is oscillatory. □

Remark 4.1. If $r(t)=1$, $\psi(x)=1$ and $f(x)=x$, then Corollary 4.1 improves the results of Yu and Fu [15].

Corollary 4.2. *Let assumptions* (S1) *and* (N1) *hold. If there exist functions* $\phi \in C^1(I,\mathbb{R}^+)$ *and* $R \in C^1(I,\mathbb{R})$ *such that* (3.1) *holds*, *and*

$$\lim_{t\to\infty} \frac{1}{t^2}\int_{T_0}^t h_1(s)\,\mathrm{d}s = 0 \tag{4.2}$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^t \int_{t_0}^s \left[\Theta_1(u) - \frac{1}{4}h_1(u)l_1^2(u)\right]\mathrm{d}u\,\mathrm{d}s = \infty, \tag{4.3}$$

then Eq. (1.1) *is oscillatory*.

Proof. Let $\phi(s)=1$. It follows from Theorem 3.1 that Eq. (1.1) is oscillatory. □

Corollary 4.3. *Let assumptions* (S1) *and* (N1) *hold. Suppose that* $\lim_{t\to\infty}\varphi(t)=\infty$ *and*

$$\liminf_{t\to\infty}\ \varphi(t)\int_t^\infty \left\{\int_a^b q(s,\xi)f[1-p(g(s,\xi))]\,\mathrm{d}\sigma(\xi)\right\}\mathrm{d}s \geqslant \frac{1}{4M}, \tag{4.4}$$

where

$$\varphi(t) = \int_{t_0}^t \frac{k_1 L g'(s,a)}{r[g(s,a)]}\,\mathrm{d}s.$$

Then Eq. (1.1) *is oscillatory*.

Proof. By (4.4), there exist two numbers $T \geqslant t_0$ and $\varepsilon > 1/(4M)$ such that

$$\varphi(t)\int_t^\infty \left\{\int_a^b q(s,\xi)f[1-p(g(s,\xi))]\,\mathrm{d}\sigma(\xi)\right\}\mathrm{d}s \geqslant \varepsilon, \quad t \geqslant T.$$

Let

$$k(t,s) = [\varphi(t)-\varphi(s)]^2 \quad \text{and} \quad \rho(t)=1, \quad R(t) = -\frac{1}{2\varphi(t)}.$$

Then

$$\lambda(t,s) = \frac{2\varphi'(t)}{\varphi(t)-\varphi(s)} \quad \text{and} \quad h_1(t) = \frac{\varphi(t)}{\varphi'(t)}, \quad l_1(t)=0.$$

Then, for all $t \geqslant T$,

$$B\left(\Theta_1 - \frac{1}{4} h_1(\lambda - l_1 - \rho^{-1}\rho')^2; T, t\right)$$
$$= \int_T^t [\varphi(t) - \varphi(s)]^2 \varphi(s) \left\{ M \int_a^b q(s, \xi) f[1 - p(g(s, \xi))] \, d\sigma(\xi) - \frac{\varphi'(s)}{4\varphi^2(s)} \right\} ds - \frac{1}{2} [\varphi^2(t) - \varphi^2(T)].$$

Define

$$w(t) = M \int_t^\infty \int_a^b q(s, \xi) f[1 - p(g(s, \xi))] \, d\sigma(\xi) \, ds.$$

Then, for all $t \geqslant T$,

$$B\left(\Theta_1 - \frac{1}{4} h_1(\lambda - l_1 - \rho^{-1}\rho')^2; T, t\right)$$
$$= \int_T^t [\varphi(t) - \varphi(s)]^2 \varphi(s) \, d\left(-w(s) + \frac{1}{4\varphi(s)}\right) - \frac{1}{2} [\varphi^2(t) - \varphi^2(T)]$$
$$= [\varphi(t) - \varphi(T)]^2 \varphi(T) \left(w(T) - \frac{1}{4\varphi(T)}\right) - \frac{1}{2} [\varphi^2(t) - \varphi^2(T)]$$
$$+ \int_T^t \left[\varphi(s) w(s) - \frac{1}{4}\right] \left[-4\varphi(t) + 3\varphi(s) + \frac{\varphi^2(t)}{\varphi(s)}\right] \varphi'(s) \, ds$$
$$\geqslant \left(\varepsilon - \frac{1}{4}\right) \int_T^t \left[-4\varphi(t) + 3\varphi(s) + \frac{\varphi^2(t)}{\varphi(s)}\right] \varphi'(s) \, ds - \frac{1}{2} [\varphi^2(t) - \varphi^2(T)]$$
$$\geqslant \left(\varepsilon - \frac{1}{4}\right) \left[\ln \frac{\varphi(t)}{\varphi(T)} - \frac{5}{2}\right] \varphi^2(t) - \frac{1}{2} [\varphi^2(t) - \varphi^2(T)].$$

This implies that

$$\lim_{t \to \infty} \frac{1}{k(t, T)} B\left(\Theta_1 - \frac{1}{4} h_1(\lambda - l_1 - \rho^{-1}\rho')^2; T, t\right) = \infty,$$

which is equivalent to (3.20). It follows from Theorem 3.4 that Eq. (1.1) is oscillatory. □

Corollary 4.4. *Let assumptions* (S1) *and* (N1) *hold. If there exist a function* $\varphi \in C^1(I, \mathbb{R}^+)$ *and an integer* $n > 1$ *such that*

$$\limsup_{t \to \infty} \frac{1}{H_1^n(t)} \int_{t_0}^t [H_1(t) - H(s)]^n \Theta_1(s) \, ds = \infty, \tag{4.5}$$

where

$$R(t) = -\frac{r[g(t, a)]}{2k_1 L g'(t, a)} \frac{\varphi'(t)}{\varphi(t)} \quad \text{and} \quad H_1(t) = \int_{t_0}^t \frac{ds}{h_1(s)}, \quad t \geqslant t_0,$$

then Eq. (1.1) *is oscillatory.*

Proof. Let

$$k(t, s) = [H_1(t) - H_1(s)]^n \quad \text{and} \quad \rho(t) = 1.$$

Note that

$$\lambda(t,s)=\frac{n}{h_1(s)[H_1(t)-H_1(s)]}\quad\text{and}\quad l_1(t)=0.$$

Then

$$B(h_1(\lambda-l_1-\rho^{-1}\rho')^2;t_0,t)=\frac{n^2}{n-1}H_1^{n-1}(t),\qquad t\geqslant t_0.$$

This implies that

$$\limsup_{t\to\infty}\frac{1}{k(t,t_0)}B\left(\Theta_1-\frac{1}{4}(\lambda-l_1-\rho^{-1}\rho')^2;t_0,t\right)$$
$$=\limsup_{t\to\infty}\frac{1}{H_1^n(t)}\left\{\int_{t_0}^t[H_1(t)-H_1(s)]^n\Theta_1(s)-\frac{n^2}{4(n-1)H_1(t)}\right\}=\infty.$$

It follows from Theorem 3.4 that Eq. (1.1) is oscillatory. □

Remark 4.2. Similar to the proof of Corollaries 4.1–4.4, by Theorems 3.2, 3.3, 3.5 and 3.6, we can establish analogues oscillation criteria for Eq. (1.1), here we omit the details.

Let the function $k(t,s)$ be defined by

$$k(t,s)=(t-s)^n,\qquad n>1, \tag{4.6}$$

we can easily check $k\in\Im$. Further, the function

$$\lambda(t,s)=\frac{n}{t-s} \tag{4.7}$$

is continuous and satisfies (H2). Therefore, as the consequences of Theorems 3.4 and 3.7, we obtain the following oscillation criteria.

Corollary 4.5. *Let assumptions* (S1) *and* (N1) *hold. If there exist a function* $\varphi\in C^1(I,\mathbb{R}^+)$ *and an integer* $n>1$ *such that*

$$\limsup_{t\to\infty}\frac{1}{(t-T)^n}\int_{t_0}^t\left[(t-s)^n\Theta_1(s)-\frac{n^2}{4}(t-s)^{n-2}h_1(s)\right]\mathrm{d}s=\infty, \tag{4.8}$$

where $R(t)$ *is defined as in Corollary* 4.4, *then Eq.* (1.1) *is oscillatory*.

Corollary 4.6. *Let assumptions* (S1) *and* (N1) *hold. If there exist functions* $\varphi\in C^1(I,\mathbb{R}^+)$, $\varphi_1,\varphi_1\in C(I,\mathbb{R})$ *and an integer* $n>1$ *such that for all* $T\geqslant t_0$,

$$\limsup_{t\to\infty}\frac{1}{(t-T)^n}\int_T^t(t-s)^n\Theta_1(s)\,\mathrm{d}s\geqslant\varphi_1(T) \tag{4.9}$$

and

$$\limsup_{t\to\infty}\frac{1}{(t-T)^n}\int_T^t(t-s)^{n-2}h_1(s)\,\mathrm{d}s\leqslant\varphi_2(T), \tag{4.10}$$

where φ_1 *and* φ_2 *satisfy*

$$\liminf_{t\to\infty}\frac{1}{(t-T)^n}\int_T^t\frac{(t-s)^n}{h_1(s)}\left(\varphi_1(s)-\frac{1}{4}\varphi_2(s)\right)_+^2\mathrm{d}s=\infty, \tag{4.11}$$

where $R(t)$ *is defined as in Corollary* 4.4. *Then Eq.* (1.1) *is oscillatory*.

Remark 4.3. We point out that we can deduce corollaries similar to Corollaries 4.5 and 4.6 from Theorems 3.5, 3.6, 3.8 and 3.9 as well. Of course, we are not limited only to the choice of function k and λ defined, respectively, by (4.6) and (4.7), which has become standard and goes back to the well-known paper by Kamenev [9]. With a different choice of these functions it is possible to derive from Theorems 3.4–3.9 other sets of oscillation criteria. In fact, another possibility is to choose the functions k and λ as follows:

$$k(t,s)=\left(\ln\frac{t}{s}\right)^n \quad\text{and}\quad \lambda(t,s)=\frac{n}{s}\left(\ln\frac{t}{s}\right)^{-1}. \tag{4.12}$$

One may also choose the more general forms for the function k and λ:

$$k(t,s)=\left(\int_s^t\frac{\mathrm{d}u}{\xi(u)}\right)^n \quad\text{and}\quad \lambda(t,s)=\frac{n}{\xi(s)}\left(\int_s^t\frac{\mathrm{d}u}{\xi(u)}\right)^{-1}, \tag{4.13}$$

where $n>1$ is an integer, and $\xi\in C(I,\mathbb{R}^+)$ satisfying condition $\lim_{t\to\infty}\int_{t_0}^t 1/\xi(u)\,\mathrm{d}u=\infty$. It is a simple matter to check that in both cases assumptions (H1) and (H2) are verified.

Finally, we will give several examples to illustrate our results. To the best of our knowledge, no previous criteria for oscillation can be applied to these examples.

Example 4.1. Consider the equation

$$\left(\frac{1}{1+x^2(t)}\left(x(t)+\frac{1}{t+1}x(t-1)\right)'\right)'+\int_0^1\frac{\gamma(t+\xi+1)}{t^2(t+\xi)}x(t+\xi)\,\mathrm{d}\xi=0,\quad t\geqslant 1, \tag{4.14}$$

where

$$r(t)=1,\quad \psi(x)=\frac{1}{1+x^2},\quad p(t)=\frac{1}{t+1},$$

$$q(t,\xi)=\frac{\gamma(t+\xi+1)}{t^2(t+\xi)},\quad \gamma>\frac14,\quad g(t,\xi)=t+\xi,\quad f(x)=x.$$

If we take $L=k_1=M=1$, $\phi(t)=t$, $R(t)=-1/(2t)$, then

$$l_1(t)=0,\quad h_1(t)=t,\quad \Theta_1(t)=\frac{4\gamma-1}{4t}.$$

Hence, by Corollary 4.1, Eq. (4.14) is oscillatory if $\gamma>\frac14$.

Example 4.2. Consider the equation

$$\left(\frac{1}{e^t(1+x^2(t))}\left(x(t)+(1-e^{-(1/2)t})x(t-1)\right)'\right)'+\int_0^1 e^{-(1/2)(t+\xi)}x(t+\xi)\,\mathrm{d}\xi=0,\quad t\geqslant 1, \tag{4.15}$$

where

$$r(t)=e^{-t},\quad \psi(x)=\frac{1}{1+x^2},\quad p(t)=1-e^{-(1/2)t},$$

$$q(t,\xi)=e^{-(1/2)(t+\xi)},\quad g(t,\xi)=t+\xi,\quad f(x)=x.$$

For Corollary 4.3, we take $L=k_1=M=1$, then

$$\varphi(t)=e^t-e \quad\text{and}\quad \int_0^1 q(t,\xi)f[1-p(g(t,\xi))]\,\mathrm{d}\sigma(\xi)=e^{-t}(1-e^{-1}).$$

Then

$$\liminf_{t\to\infty} \varphi(t)\int_t^{\infty}\int_0^1 q(s,\xi)f[1-p(g(s,\xi))]\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s$$
$$=\left(1-\frac{1}{e}\right)\liminf_{t\to\infty}\frac{e^t-e}{e^t}=1-\frac{1}{e}>\frac{1}{4}.$$

Therefore, Eq. (4.15) is oscillatory by Corollary 4.3.

Example 4.3. Consider the equation

$$\left(\frac{1}{t^2(1+x^2(t))}\left(x(t)+\frac{1}{2}x(t-1)\right)'\right)'+\int_0^1\frac{x(t+\xi)}{(t+\xi)^2}\,\mathrm{d}\xi=0,\quad t\geqslant 1, \tag{4.16}$$

where

$$r(t)=\frac{1}{t^2},\quad \psi(x)=\frac{1}{1+x^2},\quad p(t)=\frac{1}{2},$$
$$q(t,\xi)=\frac{1}{(t+\xi)^2},\quad g(t,\xi)=t+\xi,\quad f(x)=x.$$

For Corollary 4.6, we take $L=k_1=M=1$, and $\varphi(t)=1$, then

$$R(t)=0,\quad h_1(t)=\frac{1}{t^2},\quad \Theta_1(t)=\frac{1}{t(t+1)}>\frac{1}{(t+1)^2}.$$

Now, for all $t\geqslant T\geqslant 1$,

$$\limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_T^t (t-s)^2\Theta_1(s)\,\mathrm{d}s$$
$$\geqslant\frac{1}{2}\limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_T^t\frac{(t-s)^2}{(s+1)^2}\,\mathrm{d}s=\frac{1}{2(T+1)}$$

and

$$\limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_T^t (t-s)^2 h_1(s)\,\mathrm{d}s$$
$$=\limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_T^t\frac{(t-s)^2}{s^2}\,\mathrm{d}s=\frac{1}{T}.$$

Set

$$\varphi_1(T)=\frac{1}{2(T+1)},\quad \varphi_2(T)=\frac{1}{T}.$$

It is clear that

$$\liminf_{t\to\infty}\frac{1}{(t-T)^2}\int_T^t\frac{(t-s)^2}{h_1(s)}\left(\varphi_1(s)-\frac{1}{4}\varphi_2(s)\right)_+^2\mathrm{d}s$$
$$=\frac{1}{16}\liminf_{t\to\infty}\frac{1}{(t-T)^2}\int_T^t (t-s)^2\left(\frac{s-1}{s+1}\right)^2\mathrm{d}s=\infty.$$

Hence, Eq. (4.16) is oscillatory by Corollary 4.6.

Remark 4.4. Additional examples may readily be given to illustrate the oscillation criteria of the other results. We leave this to the interested reader.

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