Harish-Chandra modules for the $q$-analog Virasoro-like algebra

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Abstract

Let $\hat{L}$ be the $q$-analog Virasoro-like algebra. In this paper we prove that any Harish-Chandra module of $\hat{L}$ with nontrivial center is isomorphic to a generalized highest weight module. Moreover, we show that a generalized highest weight Harish-Chandra module of $\hat{L}$ is induced from an irreducible module of a Heisenberg subalgebra of $\hat{L}$.

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1. Introduction

In the study of affine Kac–Moody algebras and the Virasoro algebra, one of the main ingredients is the classification of the highest weight modules of the Lie algebras. We know that both the affine Kac–Moody algebras and the Virasoro algebra are related to the rank one Laurent polynomial algebra $\mathbb{C}[t, t^{-1}]$. Recently there have been substantial activity in developing representation theories for the higher rank infinite Lie algebras, for example,
the extended affine Lie algebras (see [1]), and the toroidal Lie algebras (see [5]). The quantum torus \( C_q[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \) with quantum torus matrix \( q = (q_{ij})_{v \times v} \) is a noncommutative analogue of the Laurent polynomial algebra \( C[t, t^{-1}] \), which arises as the coordinate algebra of some extended affine Lie algebras. There are some papers devoted to the study of the structure and representations of the extended affine Lie algebras coordinated by quantum torus (see [4,8,9]).

The quantum torus \( C_q = C_q[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \) (see [14]) and its derivation Lie algebra have been studied by some authors. The authors of [11] studied the rank two case. They proved that when the quantum torus matrix \( q \) is generic, then the inner derivation Lie algebra of \( C_q \), called the (centerless) \( q \)-analog Virasoro-like algebra, is isomorphic to the derived Lie subalgebra of \( C_q \). The authors of [11,16], see also [9,10], investigated the universal central extension and automorphism group of the \( q \)-analog Virasoro-like algebra. In [12,17], the authors constructed two classes of modules for this Lie algebra with trivial center, while the authors of [6] constructed a class of Harish-Chandra modules for the algebra with nontrivial center. Recently, in the paper [2] the authors obtained a sufficient condition for a class of induced modules of the algebra with trivial center to be a Harish-Chandra module. In the present paper, we give a necessary condition for a module of the \( q \)-analog Virasoro-like algebra with nonzero center to be a Harish-Chandra module by using some techniques from [13,15].

In Section 2, we develop some notation and basic results about the \( q \)-analog Virasoro-like algebra and the Heisenberg algebra. In Section 3 we first prove that any Harish-Chandra module for the \( q \)-analog Virasoro-like algebra is a generalized highest weight module. Finally, in the last part of this section we prove that a generalized highest weight Harish-Chandra module of the \( q \)-analog Virasoro-like algebra is isomorphic to a module induced from an irreducible module of a Heisenberg subalgebra of the \( q \)-analog Virasoro-like algebra.

The complex number field, real number field, integer set, nonnegative integer set and positive integer set are respectively denoted by \( \mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{Z}_+, \) and \( \mathbb{N} \). Throughout this paper, we set \( e_1 = (1, 0), e_2 = (0, 1) \in \mathbb{C}^2 \), and \( \Gamma = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset \mathbb{C}^2 \). If \( u_1, \ldots, u_k \) are elements from a \( \mathbb{C} \)-vector space \( V \), we denote by \( \langle u_1, \ldots, u_k \rangle \) the subspace of \( V \) linearly spanned by these elements over \( \mathbb{C} \).

2. The \( q \)-analog Virasoro-like algebra

Let \( C_q[t_1^{\pm 1}, t_2^{\pm 1}] \) be the ring of Laurent polynomials in noncommutative variables \( t_1, t_2 \) subject to the relations

\[
 t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_2 t_1 = q t_1 t_2,
\]

where \( q \) is a nonzero complex number. Let \( \mathcal{L} \) be the Lie algebra of inner derivations of the quantum torus \( C_q[t_1^{\pm 1}, t_2^{\pm 1}] \). In this paper we require that the complex number \( q \) is not a root of unity. \( \mathcal{L} \) is then isomorphic to the Lie algebra spanned by elements of the form
$D(\vec{m})$, $\vec{m} = (m_1, m_2) \in \Gamma^* = \Gamma \setminus \{(0, 0)\}$, subject to the following relation

$$[D(\vec{m}), D(\vec{n})] = g(\vec{m}, \vec{n}) D(\vec{m} + \vec{n}),$$

where $g(\vec{m}, \vec{n}) = q^{m_2n_1} - q^{m_1n_2}$. It is clear that $\mathcal{L}$ is a perfect Lie algebra and the universal central extension $\tilde{\mathcal{L}}$ of $\mathcal{L}$ is a two-dimensional extension with the following Lie product (see [11])

$$[D(\vec{m}), D(\vec{n})] = g(\vec{m}, \vec{n}) D(\vec{m} + \vec{n}) + \delta_{\vec{m} + \vec{n}, \vec{0}} h(\vec{m}),$$

(2.1)

where

$$h(\vec{m}) = q^{-m_1m_2} (m_1c_1 + m_2c_2),$$

and $c_1, c_2$ are central elements.

Let $d_1, d_2$ be the degree derivations of the $\mathbb{Z} \times \mathbb{Z}$ graded Lie algebra $\tilde{\mathcal{L}}$. The $q$-analog of the Virasoro-like algebra is defined to be the set $\hat{\mathcal{L}} = \tilde{\mathcal{L}} \oplus \langle d_1, d_2 \rangle$ with the Lie product given by (2.1) above and

$$[d_i, D(\vec{m})] = m_i D(\vec{m}), \quad [d_2, D(\vec{m})] = m_2 D(\vec{m}), \quad [d_i, c_j] = 0$$

for $1 \leq i, j \leq 2$. One can easily show the following result (see [4,10]).

**Lemma 2.1.** For any $\vec{b}_1 = (b_{11}, b_{12}), \vec{b}_2 = (b_{21}, b_{22}) \in \Gamma^*$, let

$$\det(\vec{b}_1, \vec{b}_2) = \begin{vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{vmatrix}.$$

Then

1. $\mathcal{H}_{e_1} = \langle D(ke_1), c_1 \rangle | k \in \mathbb{Z} \rangle$ is a Heisenberg subalgebra of $\hat{\mathcal{L}}$.
2. $\vec{b}_1, \vec{b}_2$ form a $\mathbb{Z}$-basis of $\Gamma$ if $\det(\vec{b}_1, \vec{b}_2) = \pm 1$.
3. If $\vec{b}_1, \vec{b}_2$ form a $\mathbb{Z}$-basis of $\Gamma$, then there exists an automorphism $\sigma$ of $\hat{\mathcal{L}}$ such that $\sigma(\vec{b}_1) = e_1, \sigma(\vec{b}_2) = e_2$.

**Definition.** A module $V$ of the Lie algebra $\hat{\mathcal{L}}$ is called a weight module if it has a weight space decomposition:

$$V = \bigoplus_{\lambda \in \mathbb{C}^4} V_{\lambda}, \quad V_{\lambda} = \{v \in V | d_i v = \lambda_i v, \ c_j v = \mu_j v, \ 1 \leq i, j \leq 2\},$$

where $\lambda = (\lambda_1, \lambda_2, \mu_1, \mu_2) \in \mathbb{C}^4$. A weight module is called quasi-finite if all weight spaces $V_{\lambda}$ are finite-dimensional. An irreducible quasi-finite weight module is called a Harish-Chandra module of $\hat{\mathcal{L}}$.

**Remark.** Since the central elements $c_1, c_2$ of $\hat{\mathcal{L}}$ act on irreducible weight module $V$ as scalars, we shall use the same symbols to denote the scalars. The ordered pair of complex numbers $(c_1, c_2)$ will be called the level of the module $V$. For simplicity, we write
the weight space $V(\lambda_1, \lambda_2, c_1, c_2)$ of the irreducible weight module by $V(\lambda_1, \lambda_2)$. For a weight module $V$, we define
\[ \mathcal{P}(V) := \{ \lambda \in \mathbb{C}^2 | V_{\lambda} \neq 0 \} \]
which is called the weight set of $V$. One can easily see that, for an irreducible $\hat{\mathcal{L}}$-module $V$, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, such that $\mathcal{P}(V) \subset (\lambda_1, \lambda_2) + \Gamma$.

**Definition.** If there exists a $\mathbb{Z}$-basis $B = \{ \vec{b}_1, \vec{b}_2 \}$ of $\Gamma$ and a nonzero vector $v_{\vec{\lambda}}$ of $V_{\vec{\lambda}}$, such that
\[ D(\vec{m})v_{\vec{\lambda}} = 0, \quad \forall \vec{m} \in \mathbb{Z}_{+}\vec{b}_1 + \mathbb{Z}_{+}\vec{b}_2, \]
we say that $V$ is a generalized highest weight module (GHW module for short) with generalized highest weight $\vec{\lambda}$ related to the $\mathbb{Z}$-basis $B$. The nonzero vector $v_{\vec{\lambda}}$ is called a generalized highest weight vector of $\vec{\lambda}$ related to the $\mathbb{Z}$-basis $B$, or simply a GHW vector.

Let
\[ \hat{\mathcal{L}}_0 = \mathcal{H}_{e_1} \oplus \langle c_2, d_1, d_2 \rangle, \]
\[ \hat{\mathcal{L}}_i = \langle D(me_1 + ie_2) | m \in \mathbb{Z} \rangle, \quad \text{for } i \neq 0, \]
\[ \hat{\mathcal{L}}_+ = \bigoplus_{i>0} \hat{\mathcal{L}}_i, \quad \hat{\mathcal{L}}_- = \bigoplus_{i<0} \hat{\mathcal{L}}_i. \]

Then $\hat{\mathcal{L}} = \hat{\mathcal{L}}_+ \oplus \hat{\mathcal{L}}_0 \oplus \hat{\mathcal{L}}_-$. For any weight $\hat{\mathcal{L}}_0$-module $V$, we define $V$ to be a $(\hat{\mathcal{L}}_+ \oplus \hat{\mathcal{L}}_0)$-module by defining $\hat{\mathcal{L}}_+ V = 0$. Then we have the induced $\hat{\mathcal{L}}$-module
\[ M(V) = M(e_1, e_2, V) = \text{Ind}_{\hat{\mathcal{L}}_+ \oplus \hat{\mathcal{L}}_0}^{\hat{\mathcal{L}}} V = U(\hat{\mathcal{L}}) \otimes_{U(\hat{\mathcal{L}}_+ \oplus \hat{\mathcal{L}}_0)} V, \]
where $U(\hat{\mathcal{L}})$ is the universal enveloping algebra of the Lie algebra $\hat{\mathcal{L}}$.

It is clear that, as vector spaces, $M(e_1, e_2, V) \cong U(\hat{\mathcal{L}}_-) \otimes_{\mathbb{C}} V$, and the $\hat{\mathcal{L}}$-module $M(e_1, e_2, V)$ has a unique maximal proper submodule $J(e_1, e_2, V)$ which intersects trivially with $V$. Then we have the quotient $\hat{\mathcal{L}}$-module
\[ \overline{M}(V) = \overline{M}(e_1, e_2, V) = M(e_1, e_2, V)/J(e_1, e_2, V) \tag{2.2} \]
which is uniquely determined by the $\hat{\mathcal{L}}_0$-module $V$, and $\overline{M}(V)$ is irreducible if and only if $V$ is an irreducible $\hat{\mathcal{L}}_0$-module (see [13]). If $V$ is irreducible then $\overline{M}(V)$ is the unique irreducible quotient of the module $M(V)$.

For the Heisenberg subalgebra $\mathcal{H}_{e_1}$ of $\hat{\mathcal{L}}$, we set
\[ \mathcal{H}_{e_1}^+ = \langle D(ke_1) | k \in \mathbb{N} \rangle, \quad \mathcal{H}_{e_1}^- = \langle D(ke_1) | k < 0, k \in \mathbb{Z} \rangle. \]
For any fixed $a \in \mathbb{C}^*$, let $Cv_a$ be a 1-dimensional $\mathcal{H}_{e_1}^\varepsilon \oplus \mathbb{C}c_1$ module such that

$$\mathcal{H}_{e_1}^\varepsilon v_a = 0, \quad c_1 v_a = a v_a, \quad \varepsilon \in \{+, -\}.$$ 

Then the induced $\mathcal{H}_{e_1}$-module

$$M^\varepsilon(a) = U(\mathcal{H}_{e_1}) \otimes U(\mathcal{H}_{e_1}^\varepsilon \oplus \mathbb{C}c_1) Cv_a \quad (2.3)$$

is irreducible. We call the constant $a$ the level of the module. The following result is from [7, Propositions 4.3(i), 4.5].

**Theorem 2.2.** If $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is a $\mathbb{Z}$-graded $\mathcal{H}_{e_1}$-module of level $a \in \mathbb{C}^*$ and $\dim V_i < \infty$ for at least one $i \in \mathbb{Z}$, then

1. if $V$ is an irreducible module, then $V \simeq M^\varepsilon(a)$, for some $\varepsilon \in \{+, -\}$;
2. $V$ is completely reducible.

Let $G = U(\mathcal{H}_{e_1})/U(\mathcal{H}_{e_1})c_1$ and $f : U(\mathcal{H}_{e_1}) \to G$ be the canonical mapping. For $r \in \mathbb{N}$, we consider a $\mathbb{Z}$-graded ring $A_r = \mathbb{C}[t^r, t^{-r}]$. We set deg $t = 1$, and denote by $P_r$ the set of all graded ring epimorphisms $\Lambda : G \to A_r$ such that $\Lambda(f(1)) = 1$.

Set $A_0 = \mathbb{C}$, $P_0 = \{\Lambda_0\}$, where $\Lambda_0 : G \to \mathbb{C}$ is the trivial homomorphism such that

$$\Lambda_0(1) = 1, \quad \Lambda_0(D(ke_1)) = 0$$

for $k \in \mathbb{Z} \setminus \{0\} = \mathbb{Z}^*$. For a given $\Lambda \in P_r$, $r \geq 0$, we make $A_r$ into an $\mathcal{H}_{e_1}$-module by setting

$$D(ke_1)t^rs = \Lambda(f(D(ke_1)))t^rs, \quad k \in \mathbb{Z}^*, \quad c_1t^rs = 0, \quad s \in \mathbb{Z},$$

and denote this $\mathcal{H}_{e_1}$-module by $A_{r, \Lambda}$. From [3, Lemma 3.6 and Proposition 3.8], we have the following result.

**Theorem 2.3.**

1. If $V$ is an irreducible $\mathbb{Z}$-graded $\mathcal{H}_{e_1}$-module of level zero, then $V \simeq A_{r, \Lambda}$, for some $r \in \mathbb{Z}^+$, $\Lambda \in P_r$.
2. $A_{r, \Lambda} \simeq A_{r', \Lambda'}$ if and only if $r = r'$ and there exists $b \in \mathbb{C}^*$ such that $\Lambda(f(D(ke_1))) = b^k\Lambda'(f(D(ke_1)))$, $k \in \mathbb{Z}^*$.

For any $\mathbb{Z}$-graded $\mathcal{H}_{e_1}$-module $V = \bigoplus_{i \in \mathbb{Z}} V_i$, and complex numbers $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, we extend $V$ to be an $\hat{\mathcal{L}}_0 = \mathcal{H}_{e_1} \oplus (c_2, d_1, d_2)$ module by defining

$$d_1v_j = (\lambda_1 + j)v_j, \quad d_2v_j = \lambda_2v_j, \quad c_2v = \lambda_3v$$

for $j \in \mathbb{Z}$, $v_j \in V_j$, $v \in V$. One can easily see that the vector space $V$ is a $\hat{\mathcal{L}}_0$-module and $\mathcal{P}(V) \subset (\lambda_1, \lambda_2) + \mathbb{Z}(1, 0)$. 

For any fixed $\varepsilon \in \{+, -, 0\}$, $a \in \mathbb{C}^*$, and $\mathcal{H}_{e_1}$-module $M^\varepsilon(a)$, we extend it to be an $\hat{\mathcal{L}}_0$-module in the above way by defining $d_1v_a = \lambda_1v_a$, $d_2v_a = \lambda_2v_a$. We denote this $\hat{\mathcal{L}}_0$-module by $M^\varepsilon(a, \lambda_1, \lambda_2)$. Theorems 2.2 and 2.3 then give us the following result.

**Corollary 2.4.** Let $V = \bigoplus_{i \in \mathbb{Z}} V_i$ be any weight module of level $a$ where $V_i = V_{(\lambda_1, \lambda_2) + ie_1}$. Assume $\dim V_i < \infty$ for at least one $i \in \mathbb{Z}$. Then

1. if $V$ is irreducible and $a \neq 0$, then
   
   $$V \cong M^\varepsilon(a, \lambda_1, \lambda_2),$$

   for some $\lambda_1, \lambda_2 \in \mathbb{C}$, $\varepsilon \in \{+, -, 0\}$;
2. if $V$ is irreducible and $a = 0$, then $V \cong A_r, \Lambda$, for some $r \in \mathbb{Z}^+$, $\Lambda \in P_r$;
3. if $a \neq 0$, then
   
   $$V \cong \left( \bigoplus_{i \in I} \left( \bigoplus_{s \in S_i} M^+_s(a, (\lambda_1, \lambda_2) + ie_1) \right) \right) \oplus \left( \bigoplus_{j \in J} \left( \bigoplus_{t \in T_j} M^-_t(a, (\lambda_1, \lambda_2) + je_1) \right) \right),$$

   where $I, J, S_i, T_j \subset \mathbb{Z}$, and
   
   $$M^+_s(a, (\lambda_1, \lambda_2) + ie_1) \cong M^+(a, (\lambda_1, \lambda_2) + ie_1),$$

   $$M^-_t(a, (\lambda_1, \lambda_2) + je_1) \cong M^-(a, (\lambda_1, \lambda_2) + je_1)$$

   for $s \in S_i, t \in T_j, S_i, T_j$ are finite sets, and set $I$ has a upper bound, set $J$ has a lower bound.

The following lemmas will be used later on.

**Lemma 2.5.** For any irreducible weight module $V$ of $\hat{\mathcal{L}}$, if there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $V_{(\lambda_1, \lambda_2)} \neq 0$ and $\mathcal{P}(V) \cap ((\lambda_1, \lambda_2) + \mathbb{Z}e_1 + \mathbb{N}e_2) = \emptyset$, then either $V \cong \overline{M}(e_1, e_2, A_r, \Lambda)$, for some $r \in \mathbb{Z}^+$, $\Lambda \in P_r$, or $V \cong \overline{M}(e_1, e_2, M^\varepsilon(a, \alpha_1, \alpha_2))$, for some $a \in \mathbb{C}^*$, $\alpha_1, \alpha_2 \in \mathbb{C}$, $\varepsilon \in \{+, -, 0\}$.

**Proof.** It is clear that $W = \bigoplus_{i \in \mathbb{Z}} V_{(\lambda_1, \lambda_2) + ie_1}$ is an $\hat{\mathcal{L}}_0$-module, and $\hat{\mathcal{L}}_0 W = 0$ as $\mathcal{P}(V) \cap ((\lambda_1, \lambda_2) + \mathbb{Z}e_1 + \mathbb{N}e_2) = \emptyset$. By the PBW theorem and the construction of $M(e_1, e_2, W)$, we see that there exists an epimorphism $\varphi$ from $M(e_1, e_2, W)$ to $V$ such that $\varphi|_W = \text{id}_W$. Therefore, the result follows from Corollary 2.4 and the irreducibility of the module. \[\square\]

**Lemma 2.6.** $V = \overline{M}(e_1, e_2, M^\varepsilon(a, \alpha_1, \alpha_2))$, for $a \in \mathbb{C}^*$, $\alpha_1, \alpha_2 \in \mathbb{C}$ and $\varepsilon \in \{+, -, 0\}$, is not a Harish-Chandra $\hat{\mathcal{L}}$-module.
**Proof.** From the construction of $\overline{M}(e_1, e_2, M^n(a, \alpha_1, \alpha_2))$, we see that $P(V) \cap ((\alpha_1, \alpha_2) + Z e_1 + N e_2) = \emptyset$, and there exists $v_a \in V(\alpha_1, \alpha_2)$ such that

$$H^e_{e_1} v_a = 0, \quad c_1 v_a = a v_a \neq 0, \quad d_1 v_a = \alpha_1 v_1, \quad d_2 v_a = \alpha_2 v_a.$$ 

Without loss of generality, we assume that $\varepsilon = +$. For $n \in N$, we choose $i_j \in Z$, so that $0 < i_1 < i_2 < \cdots < i_n$ and $h(-i_j e_1 + e_2) v_a \neq 0$, for $1 \leq j \leq n$. We then prove that

$$\{D(i_j e_1 - e_2) D(-i_j e_1) v_a \mid 1 \leq j \leq n\} \subset V(\alpha_1, \alpha_2) - e_2$$

is a set of linearly independent vectors. Indeed, if

$$\sum_{j=1}^{n} \lambda_j D(i_j e_1 - e_2) D(-i_j e_1) v_a = 0$$

then

$$0 = D(-i_1 e_1 + e_2) \left( \sum_{j=1}^{n} \lambda_j D(i_j e_1 - e_2) D(-i_j e_1) v_a \right)$$

$$= \lambda_1 h(-i_1 e_1 + e_2) D(-i_1 e_1) v_a + \sum_{j=2}^{n} \lambda_j (q^{i_j} - q^{i_1}) D((i_j - i_1) e_1) D(-i_j e_1) v_a$$

$$= \lambda_1 D(-i_1 e_1) h(-i_1 e_1 + e_2) v_a.$$ 

We then obtain $\lambda_1 D(-i_1 e_1) v_a = 0$ since $h(-i_1 e_1 + e_2) v_a \neq 0$. This implies that

$$0 = D(i_1 e_1) (\lambda_1 D(-i_1 e_1) v_a) = \lambda_1 i_1 c_1 v_a = \lambda_1 i_1 a v_a.$$ 

Thus $\lambda_1 = 0$. By a similar argument, we have $\lambda_2 = \lambda_3 = \cdots = \lambda_n = 0$. Therefore, $V$ is not a Harish-Chandra module because $\dim V(\alpha_1, \alpha_2) - e_2 = \infty$. 

For a representation $(\pi, V)$ of the Lie algebra $\widehat{L}$ and Lie algebra automorphism $\sigma$ of $\widehat{L}$, we set $W$ to be a linear copy of the vector space $V$, and make $W$ to be a module of $\widehat{L}$ as follows:

$$\rho(x).w = \pi(\sigma(x)) . w, \quad \forall x \in \widehat{L}, \ w \in W.$$ 

Then one can easily see that $(\rho, W)$ is also a representation of $\widehat{L}$. In this case, we say that the corresponding module $W$ is isomorphic to the module $V$ up to the automorphism $\sigma$ of $\widehat{L}$. Therefore, applying Lemmas 2.1, 2.5 and 2.6, we get the following result.

**Proposition 2.7.** Let $\{b_1, b_2\}$ be a $Z$-basis of $\Gamma$, and $V$ be a Harish-Chandra module of $\widehat{L}$. If there exist $\lambda_1, \lambda_2 \in C$ such that $V(\lambda_1, \lambda_2) \neq 0$, and $P(V) \cap ((\lambda_1, \lambda_2) + Z b_1 + N b_2) = \emptyset$, then $V$ is isomorphic to $\overline{M}(e_1, e_2, A_r, \Lambda)$ up to an automorphism of $\widehat{L}$ for some $r \in Z_+, \Lambda \in P_r$. 


3. Harish-Chandra modules

We define relations on \( \Gamma \) by setting \((x_1, x_2) > (y_1, y_2)\) if \( x_1 > y_1 \) and \( x_2 > y_2 \), and \((x_1, x_2) \geq (y_1, y_2)\) if \( x_1 \geq y_1 \) and \( x_2 \geq y_2 \).

For convenience, in this section we denote the set \( \{x \mid x \in \mathbb{Z}, p \leq x \leq r\} \) by \([p, r]\), and similarly for \((−∞, p], [r, ∞)\) and \((−∞, +∞)\).

**Proposition 3.1.** If \( V \) is a Harish-Chandra module of \( \widehat{L} \) with center acting nontrivial, then \( V \) is a generalized highest weight module.

**Proof.** Without loss of generality, we may assume the central element \( c_1 \) acts as \( a \neq 0 \), and set

\[
V = \bigoplus_{b \in \Gamma} U_{(\lambda_1, \lambda_2) + b}.
\]

For convenience, we set \( V_{\overline{b}} := U_{(\lambda_1, \lambda_2) + b} \). It is clear that \( W_0 := \bigoplus_{i \in \mathbb{Z}} V_{i e_1} \neq 0 \). Let \( W_1 = \bigoplus_{i \in \mathbb{Z}} V_{i e_1 + e_2} \). Then one can easily see that \( W_0, W_1 \) are \( \widehat{L}_0 = \mathcal{H}_{e_1} \oplus (e_2, d_1, d_2) \) modules with nonzero level. Hence by Corollary 2.4,

\[
W_1 = \left( \bigoplus_{i \in I} \left( \bigoplus_{s \in S_i} M^+_s(a, i e_1 + e_2) \right) \right) \oplus \left( \bigoplus_{j \in J} \left( \bigoplus_{t \in T_j} M^-_t(a, j e_1 + e_2) \right) \right),
\]

where \( I, J, S_i, T_j \subset \mathbb{Z} \), and

\[
M^+_s(a, i e_1 + e_2) \simeq M^+_s(a, i e_1 + e_2), \quad M^-_t(a, j e_1 + e_2) \simeq M^-_t(a, j e_1 + e_2).
\]

Set

\[
m = \max\{i \mid i \in I\} \in \mathbb{Z}, \quad n = \min\{j \mid j \in J\} \in \mathbb{Z}.
\]

Then there exists \( 0 \neq v_a \in V_{i_0 e_1} \) such that \( \mathcal{H}_{e_1}^\varepsilon v_a = 0 \) and \( c_1 v_a = a v_a \), for some \( \varepsilon \in \{+, −\} \).

We may assume \( \varepsilon = + \) and \( i_0 = 0 \). Hence

\[
D((n - 1)e_1 + e_2) v_a \in V_{(n-1)e_1+e_2} \subset W_1,
\]

and

\[
D((n - 1)e_1 + e_2) v_a \in \bigoplus_{i \in I} \left( \bigoplus_{s \in S_i} M^+_s(a, i e_1 + e_2) \right).
\]
For \( p = \max\{1, m - n + 2\} \), we have

\[
0 = D(pe_1)D((n - 1)e_1 + e_2)v_a
= (1 - q^p)D((n + p - 1)e_1 + e_2)v_a + D((n - 1)e_1 + e_2)D(pe_1)v_a
= (1 - q^p)D((n + p - 1)e_1 + e_2)v_a.
\]

Therefore, there exists \( t \in \mathbb{Z} \) such that \( D(te_1 + e_2)v_a = 0 \). We denote \( te_1 + e_2 \) by \( \vec{b}_1 \), and deduce that \( D(x\vec{b}_1 + ye_1)v_a = 0 \), for all \( x \in \mathbb{Z}_+, y \in \mathbb{N} \).

Let \( \vec{b}_2 = \vec{b}_1 + e_1 \), then

\[
\det(e_1, \vec{b}_2) = \begin{vmatrix} 1 & t + 1 \\ 0 & 1 \end{vmatrix} = 1,
\]

and \( D(xe_1 + y\vec{b}_2)v_a = 0 \), for \( x, y \in \mathbb{Z}_+ \). These imply that \( \{e_1, \vec{b}_2\} \) is a \( \mathbb{Z} \)-basis of \( \Gamma \), and \( V \) is a GHW module with generalized highest weight \( \vec{0} \) related to the \( \mathbb{Z} \)-basis \( \{e_1, \vec{b}_2\} \). \( \Box \)

From the above proposition, we see that the classification of Harish-Chandra module of \( \widehat{\mathcal{L}} \) with center acting nontrivial is reduced to the classification of generalized highest weight Harish-Chandra module. Therefore, from now on, we assume that the \( \widehat{\mathcal{L}} \)-module \( V = \bigoplus_{\vec{b} \in \Gamma} V_{\vec{b} \Phi_0 + \vec{b}} \), and \( V \) is a nontrivial GHW Harish-Chandra module with generalized highest weight \( \Phi_0 = (\lambda_1, \lambda_2) \) related to a fixed \( \mathbb{Z} \)-basis \( B = \{\vec{b}_1, \vec{b}_2\} \) of \( \Gamma \). Moreover, by Lemma 2.1(3) one can easily see that we may assume \( B = \{e_1, e_2\} \), and \( \Phi_0 = \vec{0} \).

**Lemma 3.2.**

1. For any vector \( v \in V \), there exists \( p > 0 \), such that \( D(i_1e_1 + i_2e_2)v = 0 \), for \( (i_1, i_2) \geq (p, p) \).
2. If \( 0 \neq v \in V \) and \( (m_1, m_2) > 0 \), then \( D(-m_1, -m_2)v \neq 0 \).
3. If \( \vec{b} := (i_1, i_2) \in P(V) \), then for \( \overrightarrow{c} = (m_1, m_2) > 0 \), there exists \( m \geq 0 \) such that \( \{x \in \mathbb{Z} | \vec{b} + x\overrightarrow{c} \in P(V)\} = (-\infty, m] \).

**Proof.** Choose a GHW vector \( v_0 \) of \( V \) related to the \( \mathbb{Z} \)-basis \( B = \{e_1, e_2\} \).

1. Since \( v = u.v_0 \) for some \( u \in U(\widehat{\mathcal{L}}) \), \( u \) is a linear combination of elements of the form \( u_n = D(i_1e_1 + j_1e_2) \cdots D(i_ne_1 + j_ne_2) \). Without loss of generality, we may assume \( u = u_n \). Take

\[
p_1 = -\sum_{i_s < 0} i_s + 1, \quad p_2 = -\sum_{j_s < 0} j_s + 1,
\]

by induction on \( n \) one gets \( D(i_1e_1 + j_1e_2)v = 0 \), for \( (i, j) \geq (p_1, p_2) \), and this gives the result with \( p = \max\{p_1, p_2\} \).

2. Suppose \( D(-m_1e_1 - m_2e_2)v = 0 \), for some nonzero vector \( v \in V \). Let \( p \) be the integer given in part (1). Then \( D(-m_1e_1 - m_2e_2), D(e_1 + p(m_1e_1 + m_2e_2)) \) and \( D(e_2 +
$p(m_1e_1 + m_2e_2)$ act trivially on $v$. Since the Lie algebra $\hat{L}$ is generated by such elements, we obtain that $\hat{L}.v = 0$, which contradicts the fact that $V$ is a nontrivial irreducible module.

(3) Let

$$J = \{ x \in \mathbb{Z} \mid \vec{b} + x\vec{c} \in \mathcal{P}(V) \}, \quad V_x := V_{\vec{b} + x\vec{c}}.$$ 

By using (2), we have $J = (-\infty, m]$, for some integer $m \geq 0$, or $J = \mathbb{Z}$.

If $J = \mathbb{Z}$, we want to get a contradiction. For any $i \in \mathbb{N}$, there are $0 \neq v_i \in V_i$, $p_i > 0$ such that $D(\vec{k})v_i = 0$, for $\vec{k} = k_1e_1 + k_2e_2$ and $(k_1, k_2) \geq (p_i, p_i)$. Choose a set $\{x_i \in \mathbb{N} \mid i \in \mathbb{N} \}$ so that $x_{i+1} > x_i + p_i + 2$. We then prove that $\{D(-x_i\vec{c})v_{x_i} \mid 1 \leq i \}$ is a set of linearly independent vectors in $V_{\vec{b}}$.

Indeed for fixed $r \in \mathbb{N}$, we deduce from (1) that there exists $t_r \in \mathbb{N}$ such that $D(y\vec{c} + e_1)v_{x_r} = 0$, for $y > t_r$. On the other hand, by (2) we have $D(y\vec{c} + e_1)v_{x_r} \neq 0$, for $y < 1$. Hence there exists $y_r \geq 2$ such that

$$D(y_r\vec{c} + e_1)v_{x_r} \neq 0, \quad D(y\vec{c} + e_1)v_{x_r} = 0$$

for $y > y_r$. Suppose

$$\sum_{i=1}^{r} \lambda_i D(-x_i\vec{c})v_{x_i} = 0,$$

then, for $1 \leq j < r$,

$$D((x_r + y_r)\vec{c} + e_1)D(-x_j\vec{c})v_{x_j} = [D((x_r + y_r)\vec{c} + e_1), D(-x_j\vec{c})]v_{x_j}$$

$$= q^{x_r m_1 m_2 (x_r + y_r)}(1 - q^{-x_j m_2})D((x_r - x_j + y_r)\vec{c} + e_1)v_{x_j} = 0,$$

$$D((x_r + y_r)\vec{c} + e_1)D(-x_r\vec{c})v_{x_r} = [D((x_r + y_r)\vec{c} + e_1), D(-x_r\vec{c})]v_{x_r}$$

$$= q^{-x_r m_1 m_2 (x_r + y_r)}(1 - q^{-x_r m_2})D(y_r\vec{c} + e_1)v_{x_r} \neq 0,$$

where we have used the fact that $x_r + y_r - x_j > p_{x_j}$. Therefore,

$$0 = D((x_r + y_r)\vec{c} + e_1)\left(\sum_{i=1}^{r} \lambda_i D(-x_i\vec{c})v_{x_i}\right)$$

$$= \lambda_r q^{-x_r m_1 m_2 (x_r + y_r)}(1 - q^{-x_r m_2})D(y_r\vec{c} + e_1)v_{x_r}.$$

This implies $\lambda_r = 0$. Similarly, one can deduce that $\lambda_{r-1} = \cdots = \lambda_1 = 0$. Therefore, $\dim V_{\vec{b}} = \infty$, and which contradicts the fact that $V$ is a Harish-Chandra module. \hfill $\Box$

**Lemma 3.3.** There exists a $\mathbb{Z}$-basis $B' = \{\vec{b}'_1, \vec{b}'_2\}$ of $\Gamma$ such that:

1. $V$ is a GHW module with generalized highest weight $\vec{0}$ related to the $\mathbb{Z}$-basis $B'$;
2. $(\mathbb{Z}_+\vec{b}'_1 + \mathbb{Z}_+\vec{b}'_2) \cap \mathcal{P}(V) = \{(0,0)\};
(3) \(-\mathbb{Z}_+b_1 + \mathbb{Z}_+b_2 \subset \mathcal{P}(V)\);  
(4) if \(i_1 \vec{b}_1 + i_2 \vec{b}_2 \notin \mathcal{P}(V)\), then \(k_1 \vec{b}_1 + k_2 \vec{b}_2 \notin \mathcal{P}(V)\), for \((k_1, k_2) \geq (i_1, i_2)\);  
(5) if \(i_1 \vec{b}_1 + i_2 \vec{b}_2 \in \mathcal{P}(V)\), then \(k_1 \vec{b}_1 + k_2 \vec{b}_2 \in \mathcal{P}(V)\), for \((k_1, k_2) \leq (i_1, i_2)\);  
(6) for \(\vec{0} \neq (k_1, k_2) \geq \vec{0}\), \((i_1, i_2) \in \Gamma\), we have  
\[ \{ x \in \mathbb{Z} | i_1 \vec{b}_1 + i_2 \vec{b}_2 + x(k_1 \vec{b}_1 + k_2 \vec{b}_2) \in \mathcal{P}(V) \} = (-\infty, m) \]  
for some \(m \in \mathbb{Z}\).

**Proof.** By Lemma 3.2(3) we may assume  
\[ \{ x \in \mathbb{Z} | x(e_1 + e_2) \in \mathcal{P}(V) \} = (-\infty, p - 2) \]  
for some \(p \geq 2\). Let  
\[ \vec{b}_1 = (p + 1)e_1 + (p + 2)e_2, \quad \vec{b}_2 = pe_1 + (p + 1)e_2 \]  
then  
\[ \det(\vec{b}_1, \vec{b}_2) = \begin{vmatrix} p + 1 & p \\ p + 2 & p + 1 \end{vmatrix} = 1. \]  
This implies that \(B' = \{ \vec{b}_1, \vec{b}_2 \}\) is a \(\mathbb{Z}\)-basis of \(\Gamma\). Thus statement (1) follows from this and the fact that \(\mathbb{Z}_+b_1 + \mathbb{Z}_+b_2 \subset \mathbb{Z}_+e_1 + \mathbb{Z}_+e_2\).

To prove (2), we denote \(e_1 + e_2\) by \(\vec{b}\). If \(r_1 \vec{b}_1 + r_2 \vec{b}_2 \in \mathcal{P}(V)\), for some \(r_1, r_2 \in \mathbb{Z}_+\), then there is a nonzero element \(v \in V_{r_1 \vec{b}_1 + r_2 \vec{b}_2}\). But  
\[ r_1 \vec{b}_1 + r_2 \vec{b}_2 = (r_1(p + 1) + r_2 p)e_1 + (r_1(p + 2) + r_2(p + 1)e_2 \]  
\[ = (r_1 + r_2)(p - 1)\vec{b} + r_1(2e_1 + 3e_2) + r_2(e_1 + 2e_2). \]  
(3.1)  
This implies \((r_1 + r_2)(p - 1)\vec{b} \in \mathcal{P}(V)\) by using Lemma 3.2(2). Hence \(r_1 = r_2 = 0\) by the choice of \(p\). This proves statement (2).

Statements (3) through (5) follow from the fact that \((0, 0)\) is a weight and Lemma 3.2(2). Finally, we prove statement (6). Set  
\[ \vec{m} = i_1 \vec{b}_1 + i_2 \vec{b}_2, \quad \vec{n} = k_1 \vec{b}_1 + k_2 \vec{b}_2, \quad A = \{ x \in \mathbb{Z} | \vec{m} + x \vec{n} \in \mathcal{P}(V) \}. \]  
Since \(\vec{n} = k_1 \vec{b}_1 + k_2 \vec{b}_2 = (pk_1 + pk_2 + k_1, pk_1 + pk_2 + 2k_1 + k_2) > (0, 0)\), we can choose \(x_0 \in \mathbb{Z}\) so that \(\vec{m} + x_0 \vec{n} := ie_1 + je_2\) with \((i, j) < (0, 0)\). Therefore, statement (6) follows from the facts that \((0, 0) \in \mathcal{P}(V)\), \(\vec{m} + x_0 \vec{n} \in \mathcal{P}(V)\) and Lemma 3.2(3). \(\Box\)

From Lemma 2.1(3), we can assume that \(V\) is a GHW Harish-Chandra module with generalized highest weight \((0, 0)\) related to the \(\mathbb{Z}\)-basis \(B = \{ e_1, e_2 \}\), and the basis \(B\) satisfies the properties given in Lemma 3.3.
Lemma 3.4. If there exist an integer \( s > 0 \) and \((i_1, i_2), (k_1, k_2) \in \Gamma\) such that \( k_1, k_2 \) are coprime, and

\[
\{(i_1 + x_1s, i_2 + x_2s) \mid (x_1, x_2) \in \Gamma, k_1x_1 + k_2x_2 = 0\} \cap \mathcal{P}(V) = \emptyset, \tag{3.2}
\]

then the \( \widehat{L} \)-module \( V \) is isomorphic to the module \( \overline{M}(e_1, e_2, A_r, \Lambda) \), for some \( r \in Z_+ \), \( \Lambda \in \mathcal{P}_r \), up to an automorphism of \( \widehat{L} \).

**Proof.** If \( k_1 > 0, k_2 < 0 \), or \( k_1k_2 = 0 \), then there exists \((0, 0) \neq (x_1, x_2) \geq 0 \) such that \( k_1x_1 + k_2x_2 = 0 \), which is a contradiction to Lemma 3.3(6). Hence we have \((k_1, k_2) > 0 \) or \((k_1, k_2) < 0 \).

We may assume \((k_1, k_2) > 0 \). First, we prove that there exists \( m \in Z \) such that

\[
\{x_1e_1 + x_2e_2 \mid (x_1, x_2) \in \Gamma, x_1k_1 + x_2k_2 \geq m\} \cap \mathcal{P}(V) = \emptyset. \tag{3.3}
\]

Indeed, let \( N = k_1s^2, m = k_1(N + i_1) + k_2(N + i_2) \), and \((x_1, x_2) \in \Gamma\) satisfying \( k_1x_1 + k_2x_2 \geq m \), and let \((y_1, y_2) = (x_1, x_2) - (i_1 + N, i_2 + N) \). Then we have \( k_1y_1 + k_2y_2 \geq 0 \).

Choosing \( l \in Z \) such that \( -y_2/(s^2k_1) \geq l > -y_2/(s^2k_1) - 1 \), then \( 0 \geq y_2 + ls^2k_1 > -s^2k_1 \), which gives \( y_1 - ls^2k_2 \geq 0 \) as \( k_1(y_1 - ls^2k_2) \geq -k_2(y_2 + ls^2k_1) \geq 0 \). Notice that

\[
(x_1, x_2) = (y_1, y_2) + (i_1 + N, i_2 + N)
= (i_1, i_2) + (l(s^2k_2 - s^2k_1)) + (N + y_1 - ls^2k_2, N + y_2 + ls^2k_1)
\geq (i_1, i_2) + (l(s^2k_2, -s^2k_1))
\]

so we get, from (3.2), that

\[ (i_1 + ls^2k_2, i_2 - ls^2k_1) \notin \mathcal{P}(V). \]

Hence \( x_1e_1 + x_2e_2 \notin \mathcal{P}(V) \) by Lemma 3.3(4). This completes the proof of (3.3).

From (3.3), we see that there is a unique integer \( m_0 \) with the following properties:

1. \( \{x_1e_1 + x_2e_2 \in \mathcal{P}(V) \mid (x_1, x_2) \in \Gamma, k_1x_1 + k_2x_2 \geq m_0\} = \emptyset; \)

2. \( A := \{x_1e_1 + x_2e_2 \in \mathcal{P}(V) \mid (x_1, x_2) \in \Gamma, k_1x_1 + k_2x_2 = m_0 - 1\} \neq \emptyset. \)

Since \( k_1, k_2 \) are coprime, we can choose \( n_1, n_2 \in Z \) so that \( k_1n_1 + k_2n_2 = 1 \). Let

\[
\vec{b}_1 = k_2e_1 - k_1e_2, \quad \vec{b}_2 = n_1e_1 + n_2e_2,
\]

then \( \det(\vec{b}_1, \vec{b}_2) = 1 \), and \( \{\vec{b}_1, \vec{b}_2\} \) is a \( Z \)-basis of \( \Gamma \). Moreover, for any \( \lambda_1e_1 + \lambda_2e_2 \in A \), we have \( V_{\lambda_1e_1 + \lambda_2e_2} \neq 0 \), and \( \mathcal{P}(V) \cap ((\lambda_1, \lambda_2) + Z\vec{b}_1 + N\vec{b}_2) = \emptyset. \) This then gives the result of the lemma by applying Lemma 2.1(3) and Proposition 2.7. \( \square \)
Corollary 3.5. If there exist \((i_1, i_2), (0, 0) \neq (k_1, k_2) \in \Gamma\) such that
\[
\begin{bmatrix}
i_1e_1 + i_2e_2 + x(k_1e_1 + k_2e_2) | x \in \mathbb{Z}\n\end{bmatrix} \cap \mathcal{P}(V) = \emptyset,
\]
then the \(\hat{L}\)-module \(V\) is isomorphic to \(\overline{M}(e_1, e_2, A_r, \Lambda)\), for some \(r \in \mathbb{Z}_+\), \(\Lambda \in \mathcal{P}_r\), up to an automorphism of \(\hat{L}\).

Proof. By Lemma 3.3(6), we can assume \(k_1k_2 < 0\). Let \((k_1, k_2) = s(k_1', k_2')\) with \(k_1', k_2'\) coprime and \(s > 0\). Then
\[
\begin{bmatrix}
i_1e_1 + i_2e_2 + x(k_1e_1 + k_2e_2) | x \in \mathbb{Z}\n\end{bmatrix} = \begin{bmatrix}
i_1e_1 + i_2e_2 + x_1se_1 + x_2se_2 | (x_1, x_2) \in \Gamma, k_2'x_1 - k_1'x_2 = 0\n\end{bmatrix}.
\]
Then the result of the lemma follows from Lemma 3.4. \(\square\)

By Lemmas 2.1(3), 3.2, 3.3 and Corollary 3.5, one gets the following result (see [13, Lemma 3.5], or [15, p. 545]).

Lemma 3.6. If there exist \(0 \neq (m, n) \in \Gamma, (i, j) \in \Gamma,\) and \(p, t \in \mathbb{Z}\), such that
\[
\begin{bmatrix}
x \in \mathbb{Z} | ie_1 + je_2 + x(me_1 + ne_2) \in \mathcal{P}(V)\n\end{bmatrix} \supset (-\infty, p] \cup [t, +\infty),
\]
then the \(\hat{L}\)-module \(V\) is isomorphic to \(\overline{M}(e_1, e_2, A_r, \Lambda)\), for some \(r \in \mathbb{Z}_+\), \(\Lambda \in \mathcal{P}_r\), up to an automorphism of \(\hat{L}\).

Lemma 3.7. If there exist \((i, j), (k, l) \in \Gamma\), and \(x_1, x_2, x_3 \in \mathbb{Z}\) such that \(x_1 < x_2 < x_3\), and
\[
\begin{align*}
ie_1 + je_2 + x_1(ke_1 + le_2) \notin \mathcal{P}(V), \\
ie_1 + je_2 + x_2(ke_1 + le_2) \in \mathcal{P}(V), \\
ie_1 + je_2 + x_3(ke_1 + le_2) \notin \mathcal{P}(V),
\end{align*}
\]
then the \(\hat{L}\)-module \(V\) is isomorphic to \(\overline{M}(e_1, e_2, A_r, \Lambda)\), for some \(r \in \mathbb{Z}_+\), \(\Lambda \in \mathcal{P}_r\), up to an automorphism of \(\hat{L}\).

Proof. Without loss of generality, we may assume \(k, l\) are coprime. Thus we can choose \((m, n) \in \Gamma\) such that \(kn - lm = 1\). Let
\[
\begin{bmatrix}
\rightarrow b_1 = ke_1 + le_2, \\
\rightarrow b_2 = me_1 + ne_2,
\end{bmatrix}
\]
then \(\{\rightarrow b_1, \rightarrow b_2\}\) is a \(\mathbb{Z}\)-basis of \(\Gamma\). Replacing \(x_2\) by the largest integer so that \(x < x_3\) and \(ie_1 + je_2 + x(ke_1 + le_2) \in \mathcal{P}(V)\), then we substitute \(x_3\) by \(x_2 + 1\), and \((i, j)\) by \((i, j) + x_2(k, l)\). Therefore, we can assume
\[
x_1 < x_2 = 0, \quad x_3 = 1.
\]  
(3.4)
Let
\[ X := \{ x \in \mathbb{Z} \mid ie_1 + je_2 + \overrightarrow{b_2} + x\overrightarrow{b_1} \}. \]

If \( X \subset \mathcal{P}(V) \), then the result of the lemma follows from Lemma 3.6. Hence we may assume that there exists \( m \in \mathbb{Z} \) such that \( ie_1 + je_2 + \overrightarrow{b_2} + mb_1 \notin \mathcal{P}(V) \). From the assumption and (3.4), we have
\[ D(x\overrightarrow{b_1})v_{ie_1+je_2} = D(\overrightarrow{b_1})v_{ie_1+je_2} = D(mb_1 + \overrightarrow{b_2})v_{ie_1+je_2} = 0, \]
where \( 0 \neq v_{ie_1+je_2} \in V_{ie_1+je_2} \). Moreover, note that \( \{D(p\overrightarrow{b_1} + t\overrightarrow{b_2}) \mid p \in \mathbb{Z}, t \in \mathbb{N} \} \) belongs to a subalgebra of \( \hat{L} \) generated by elements of the form \( \{D(x\overrightarrow{b_1}), D(\overrightarrow{b_1}), D(mb_1 + \overrightarrow{b_2})\} \), so we obtain \( D(p\overrightarrow{b_1} + t\overrightarrow{b_2})v_{ie_1+je_2} = 0 \) for \( p \in \mathbb{Z}, t \in \mathbb{N} \).

Since \( \{\overrightarrow{b_1}, \overrightarrow{b_2}\} \) is a \( \mathbb{Z} \)-basis of \( \Gamma \), by using the PBW theorem and the irreducibility of the \( \hat{L} \)-module \( V \), we have \( V = U(\hat{L})v_{ie_1+je_2} \) and
\[ \{ie_1 + je_2 + \mathbb{Z}\overrightarrow{b_1} + \mathbb{N}\overrightarrow{b_2} \} \cap \mathcal{P}(V) = \emptyset. \]
Therefore, we get the result of the lemma by applying Proposition 2.7. \( \square \)

**Lemma 3.8.** If there exist integers \( i > 0, j < 0 \) and \( 0 \neq v_\overrightarrow{c} \in V_\overrightarrow{c}, \overrightarrow{b} = me_1 + ne_2 \neq 0, \) such that \( D(i\overrightarrow{b})v_\overrightarrow{c} = 0 = D(j\overrightarrow{b})v_\overrightarrow{c} \), then the \( \hat{L} \)-module \( V \) is isomorphic to \( \overline{M}(e_1, e_2, A_r, \Lambda) \), for some \( r \in \mathbb{Z}_+, \Lambda \in P_r \), up to an automorphism of \( \hat{L} \).

**Proof.** Suppose the statement is false, we want to get a contradiction. By Lemma 2.1(3), we may assume that
\[ V \not\cong \overline{M}(\overrightarrow{b_1}, \overrightarrow{b_2}, A_r, \Lambda) \]
for any \( \mathbb{Z} \)-basis \( \{\overrightarrow{b_1}, \overrightarrow{b_2}\} \) of \( \Gamma \) and \( r \in \mathbb{Z}_+, \Lambda \in P_r \). Let \( (m, n) = s(m', n') \) with \( m', n' \) co-prime, and \( s \geq 1 \). Then we can choose \( (m_2, n_2) \in \Gamma' \) so that \( n'm_2 - m'n_2 = 1 \). Let
\[ \overrightarrow{b_1}' = m'e_1 + n'e_2, \quad \overrightarrow{b_2}' = m_2e_1 + n_2e_2, \]
then \( \{\overrightarrow{b_1}', \overrightarrow{b_2}'\} \) is a \( \mathbb{Z} \)-basis of \( \Gamma' \). By Corollary 3.5, Lemmas 3.6 and 3.7, we see that, for any \( 0 \neq t \in \mathbb{Z} \), there exists an integer \( p_t \) such that
\[ A_t := \{ x \in \mathbb{Z} \mid \overrightarrow{c} + t\overrightarrow{b_2}' + xb_1' \in \mathcal{P}(V) \} = (-\infty, p_t] \text{ or } [p_t, +\infty). \quad (3.5) \]
First, we assume \( A_t = (-\infty, p_t] \). Then
\[ D(t\overrightarrow{b_2}' - jxs\overrightarrow{b_1}' \pm \overrightarrow{b_1})v_\overrightarrow{c} = 0, \]
for a sufficiently large integer \( x > 0 \). Hence we obtain \( D(t\overrightarrow{b_2}' \pm \overrightarrow{b_1})v_\overrightarrow{c} = 0 \) as \( D(js\overrightarrow{b_1}')v_\overrightarrow{c} = D(j\overrightarrow{b})v_\overrightarrow{c} = 0 \). Therefore, we have
\[ D(\pm(\overrightarrow{b_1}' + \overrightarrow{b_2}'))v_\overrightarrow{c} = 0 = D(\pm(\overrightarrow{b_1} + 2\overrightarrow{b_2}))v_\overrightarrow{c}. \]
But, since \( \{ \vec{b}_1 + \vec{b}_2, \vec{b}_1' + 2\vec{b}_2' \} \) is a \( \mathbb{Z} \)-basis of \( \Gamma \), \( \hat{L} \) is generated by elements of the form
\[
\{ D(\pm(\vec{b}_1 + \vec{b}_2)), D(\pm(\vec{b}_1' + 2\vec{b}_2')) \}.
\]
Thus \( V = U(\hat{L})v_\gamma \) is a trivial \( \hat{L} \)-module, which is a contradiction. The argument in the other case is similar. \( \square \)

Now we deal with the classification problem of the GHW Harish-Chandra modules of the \( q \)-analog Virasoro-like algebra \( \hat{L} \).

**Proposition 3.9.** If \( V \) is a nontrivial GHW Harish-Chandra \( \hat{L} \)-module with generalized highest weight \((\lambda_1, \lambda_2)\) related to a \( \mathbb{Z} \)-basis \( B = \{\vec{b}_1, \vec{b}_2\} \) of \( \Gamma \), then \( V \) is isomorphic to \( M(e_1, e_2, A_r, \Lambda) \), for some \( r \in \mathbb{Z}_+ \), \( \Lambda \in P_r \), up to an automorphism of \( \hat{L} \).

**Proof.** Without loss of generality, we let \( (\lambda_1, \lambda_2) = (0, 0) \). From Lemmas 2.1(3) and 3.3, we can assume that the \( \mathbb{Z} \)-basis \( B = \{e_1, e_2\} \), which satisfies the properties of Lemma 3.3. If the statement of the proposition is false, we want to get a contradiction.

By Lemma 2.1(3), we may also assume that
\[
V \not\cong M(\vec{b}_1, \vec{b}_2, A_r, \Lambda)
\]
for any \( \mathbb{Z} \)-basis \( \{\vec{b}_1, \vec{b}_2\} \) of \( \Gamma \), and \( r \in \mathbb{Z}_+ \), \( \Lambda \in P_r \). From Corollary 3.5, Lemmas 3.6 and 3.7, we see that, for any \((i, j)\), and \( 0 \neq (k, l) \in \Gamma \), there exists \( p \in \mathbb{Z} \), such that
\[
\{ x \in \mathbb{Z} \mid i e_1 + j e_2 + x(k e_1 + l e_2) \in \mathcal{P}(V) \} = (-\infty, p] \text{ or } [p, +\infty).
\]
(3.6)

Let
\[
y_i = \max \{ y \in \mathbb{Z} \mid -i e_1 + ye_2 \in \mathcal{P}(V) \}, \quad \forall i \in \mathbb{N},
\]
\[
x_i = \max \{ x \in \mathbb{Z} \mid x e_1 - i e_2 \in \mathcal{P}(V) \}, \quad \forall i \in \mathbb{N}.
\]
We obtain the following results (see [13, Theorem 3.7], or [15, pp. 547 to 549], for detail):

(A1) The following limits exist
\[
\alpha = \lim_{i \to \infty} \frac{y_i}{i}, \quad \beta = \lim_{i \to \infty} \frac{x_i}{i}.
\]
(A2) \( \alpha = \beta^{-1} \) is a positive irrational number.
(A3) We have a total order \( >_\alpha \) define on \( \Gamma \) as follows:
\[
(i, j) >_\alpha (k, l) \iff i \alpha + j > k \alpha + l.
\]
(3.7)
The order \( \succ \) is dense, that is, for any \( \lambda \in \mathbb{R}_+ \setminus \{0\} \), there exist \( p, q \in \mathbb{Z} \), such that \( 0 < p\alpha + q < \lambda \).

Let 
\[
G^+ = \{ ie_1 + je_2 | (i, j) \succ (0, 0) \}, \quad G^- = \{ ie_1 + je_2 | (i, j) \prec (0, 0) \}.
\]

Then we have

(A4) If \( ie_1 + je_2 \in \mathcal{P}(V) \), then \( ke_1 + le_2 \in \mathcal{P}(V) \), for any \( (k, l) \prec (i, j) \), that is,
\[
(ie_1 + je_2) - (ke_1 + le_2) \in G^+.
\] (3.8)

From (3.8), we get \( G^- \subset \mathcal{P}(V) \) as \( (0, 0) \in \mathcal{P}(V) \). By (3.6) and the definition of the ordering \( \succ \), we have, for any \( (i, j) \in \Gamma \), that there exists an integer \( p \), such that
\[
\{ x \in \mathbb{Z} | ie_1 + je_2 + x(ke_1 + le_2) \in \mathcal{P}(V) \} = (-\infty, p],
\] (3.9) for \( ke_1 + le_2 \in G^+ \).

**Claim 1.** For any \((\lambda_1, \lambda_2) \in \mathcal{P}(V)\), we have \((\lambda_1, \lambda_2) + G^+ \cap \mathcal{P}(V) \neq \emptyset\).

**Proof.** Suppose
\[
((\lambda_1, \lambda_2) + G^+) \cap \mathcal{P}(V) = \emptyset,
\]
for some \((\lambda_1, \lambda_2) \in \mathcal{P}(V)\). Since the order \( \succ \) is dense, we see that, for any \( n \in \mathbb{N} \), there is \( (p, t) \in \Gamma \) such that \( 0 > p\alpha + t > -1/(4n) \) with \( p \neq 0 \). Thus
\[
(0, 0) \succ i(p, t) \succ (0, -1),
\]
for \( 1 \leq i \leq 4n \). Set \( \overline{e} = pe_1 + te_2 \). For \( 0 \neq v \in V(\lambda_1, \lambda_2) \), \( 1 \leq i \leq n \), then \( D(-e_2 - (2i - 1)\overline{e})D((2i - 1)\overline{e})v \) is a vector with weight \((\lambda_1, \lambda_2) - e_2\). We prove that these vectors are linearly independent. In fact, if
\[
\sum_{i=1}^{n} \mu_i D(-e_2 - (2i - 1)\overline{e})D((2i - 1)\overline{e})v = 0,
\]
for some \( \mu_i \in \mathbb{C} \), then
\[
0 = D(e_2 + 2\overline{e}) \left( \sum_{i=1}^{n} \mu_i D(-e_2 - (2i - 1)\overline{e})D((2i - 1)\overline{e})v \right) \\
= q^{-2tp(2i-1)} \left( (q^{-p} - q^{-2p})\mu_1 D(\overline{e})D(\overline{e})v \right. \right. \\
+ \left. \left. \sum_{i=2}^{n} (q^{-(2i-1)p} - q^{-2p})\mu_i D((3-2i)\overline{e})D((2i - 1)\overline{e})v \right) \right) 
\]
= q^{-2lp(2i-1)}(q^{-p} - q^{-2p})\mu_1 D(\overrightarrow{c})D(\overrightarrow{c})v,$

where we have used the fact $e_2 + (2i + 1)\overrightarrow{c} \in G^+$, for $1 \leq i \leq n$, and $((\lambda_1, \lambda_2) + G^+) \cap \mathcal{P}(V) = \emptyset$. Moreover, by using the assumption of the proposition and the fact that $(0, 0) \succ_a \overrightarrow{c}$, we obtain

$$D(-2\overrightarrow{c})D(\overrightarrow{c})v = 0 = D(-\overrightarrow{c})v.$$ 

This and Lemma 3.8 imply that $D(\overrightarrow{c})D(\overrightarrow{c})v \neq 0$. Thus we obtain $\mu_1 = 0$. Similarly, we can prove $\mu_2 = \cdots = \mu_n = 0$. Therefore, $\dim V_{(\lambda_1, \lambda_2)e_2} \geq n$ for any $n \in \mathbb{N}$, which contradicts to the assumption that $V$ is a Harish-Chandra$_2$ module. This completes the proof of Claim 1. □

Claim 2. If $\overrightarrow{b} = ie_1 + je_2 \in G^+$ and $0 \neq v \in V_{(\lambda_1, \lambda_2)}$, then $D(-\overrightarrow{b})v \neq 0$.

Proof. Suppose $D(-\overrightarrow{b})v = 0$, for some $\overrightarrow{b} = ie_1 + je_2 \in G^+$ and $0 \neq v \in V_{(\lambda_1, \lambda_2)}$. We note, from (3.9), that $D(s\overrightarrow{b})v = 0$, for some $s > 0$, which, by Lemma 3.8, implies that $V \simeq \overline{M}(e_1, e_2, A_r, \Lambda)$, for some $r \in \mathbb{Z}_+$, $\Lambda \in \mathcal{P}_r$, up to an automorphism of $\overline{L}$. This contradicts the assumption, and thus completes the proof of Claim 2. □

Claim 3. For any $n \in \mathbb{N}$, there exists

$$\overrightarrow{a} = ie_1 + je_2 \in (G^+ \cup \{(0, 0)\}) \cap \mathcal{P}(V)$$

and $\dim V_{\overrightarrow{a}} \geq n$.

Proof. From Claim 1, we see that there exists $\overrightarrow{b} = ie_1 + je_2 \in G^+ \cap \mathcal{P}(V)$. By Lemma 3.3(2), this implies that $ij \neq 0$. Let $(i, j) = (i', j')$ with $i', j'$ coprime and $s \geq 1$. Then (3.9) gives $i'e_1 + j'e_2 \in G^+ \cap \mathcal{P}(V)$. Therefore, we may assume $i, j$ are coprime. Since the order $\succ_a$ is dense and $\overrightarrow{b} \in G^+$, one can choose $(p, t) \in \Gamma$ so that $0 < p\alpha + t < (i\alpha + j)/(4n)$. Then

$$(0, 0) <_a k(p, t) <_a (i, j)$$

for $1 \leq k \leq 4n$. Denote $pe_1 + te_2$ by $\overrightarrow{c}$. We prove $\det(\overrightarrow{b}, \overrightarrow{c}) \neq 0$.

Indeed, if $\det(\overrightarrow{b}, \overrightarrow{c}) = 0$, that is $it - pj = 0$, then $(p, t) = s'(i, j)$, for some integer $s'$. This contradicts to the choice of $(p, t)$.

Let $m_1 = \max\{x \in \mathbb{Z} | x\overrightarrow{b} \in \mathcal{P}(V)\}, \quad m_2 = \max\{x \in \mathbb{Z} | m_1\overrightarrow{b} + x\overrightarrow{c} \in \mathcal{P}(V)\}.$

Then, by (3.9) and the fact that $\overrightarrow{b} \in \mathcal{P}(V)$, we have $m_1 \geq 1$ and $m_2 \geq 0$. For $0 \neq v \in V_{m_1\overrightarrow{b} + m_2\overrightarrow{c}}$, and $1 \leq k \leq n$, we see that

$$D(-\overrightarrow{b} + (2k - 1)\overrightarrow{c})D(-(2k - 1)\overrightarrow{c})v$$
is a vector of weight \( \overline{\alpha} := (m_1 - 1) \overline{b} + m_2 \overline{c} \in G^+ \cup \{(0, 0)\} \). We prove that they are linearly independent vectors.

In fact, if

\[
\sum_{k=1}^{n} \mu_k D(-\overline{b} + (2k - 1) \overline{c}) D(-(2k - 1) \overline{c}) v = 0,
\]

for some \( \mu_k \in \mathbb{C} \), we have, by (3.8), that

\[
D(\overline{b} - 2 \overline{c}) D(-(2k - 1) \overline{c}) v \in V_{m_1 \overline{b} + m_2 \overline{c} + \overline{b} - (2k + 1) \overline{c}} = \{0\},
\]

for \( 1 \leq k \leq n \), where the identity is due to the fact that \( m_1 \overline{b} + m_2 \overline{c} + \overline{c} \not\in \mathcal{P}(V) \) (recall the choice of \( m_1, m_2 \)), and

\[
(m_1 \overline{b} + m_2 \overline{c} + \overline{b} - (2k + 1) \overline{c}) - (m_1 \overline{b} + m_2 \overline{c} + \overline{c}) \in G^+,
\]

for \( 1 \leq k \leq n \). Thus

\[
0 = D(\overline{b} - 2 \overline{c}) \left( \sum_{k=1}^{n} \mu_k D(-\overline{b} + (2k - 1) \overline{c}) D(-(2k - 1) \overline{c}) v \right)
\]

\[
= \sum_{k=1}^{n} \mu_k q^{-2pt(2k-1)-ij+2pj+2it} (q^{pj(2k-3)} - q^{-it(2k-3)}) D((2k - 3) \overline{c}) D(-(2k - 1) \overline{c}) v
\]

\[
= \mu_1 q^{-2pt-it+2pj+2it} (q^{-pj} - q^{-it}) D(-\overline{c}) D(-\overline{c}) v,
\]

and therefore, \( \mu_1 D(-\overline{c}) D(-\overline{c}) v = 0 \) as \( it - pj \neq 0 \). On the other hand, recall the choice of \( v \), we have

\[
D(2 \overline{c}) D(-\overline{c}) v = 0 = D(\overline{c}) v,
\]

which and Lemma 3.8 imply that \( D(-\overline{c}) D(-\overline{c}) v \neq 0 \). Hence \( \mu_1 = 0 \). Similarly, one can prove \( \mu_2 = \cdots = \mu_n = 0 \). Therefore, \( \dim V_{\overline{\alpha}} \geq n \). This completes the proof of Claim 3. \( \square \)

Now we finish the proof of the proposition. For \( n \in \mathbb{N} \), we know from Claim 3 that \( \dim V_{\overline{\alpha}} \geq n \), for some \( \overline{\alpha} \in G^+ \cup \{(0, 0)\} \). Hence, with \( \overline{b} \) and \( (\lambda_1, \lambda_2) \) replaced by \( \overline{\alpha} \) in Claim 2, we obtain \( \dim V_{(0,0)} \geq n \). Therefore, \( \dim V_{(0,0)} = \infty \), which contradicts the fact that \( V \) is a Harish-Chandra module. \( \square \)

Propositions 3.1 and 3.9 then give us the main result of the paper.
Theorem 3.10. If \( V \) is a Harish-Chandra weight \( \hat{L} \)-module with center acting nontrivial, then \( V \) is isomorphic to \( M(e_1, e_2, A_r, \Lambda) \), for some \( r \in \mathbb{Z}_+, \Lambda \in P_r \), up to an automorphism of \( \hat{L} \).

References