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Some properties of bornological convergences ☆

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We study some basic properties of the so-called bornological convergences in the realm of quasi-uniform spaces. In particular, we revisit the results about when these convergences are topological by means of the use of pretopologies. This yields a presentation of the bornological convergences as a certain kind of hit-and-miss pretopologies. Furthermore, we characterize the precompactness and total boundedness of the natural quasi-uniformities associated to these convergences. We also obtain an extension of the classical result of Künzi and Ryser about the compactness of the topology generated by the Hausdorff quasi-uniformity to this framework.

1. Introduction

Through all the paper, we will mainly deal with quasi-uniform spaces due to its generality and the applications of the asymmetric topology to topological algebra, functional analysis and computer science [24,43]. Recall that a *quasi-uniformity* on a nonempty set X [23,25] is a filter $\mathcal{U}$ of reflexive relations such that if $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subseteq U$ where $V^2 = \{(x,z) \in X \times X\colon$ there exists $y \in X$ with $(x,y),(y,z) \in V\}$. By $\mathcal{U}^*$ we denote the uniformity which has as a base the elements of the form $U^* = U \cap U^{-1}$ where $U^{-1} = \{(x,y) \in X \times X\colon (y,x) \in U\}$.

Every quasi-uniformity $\mathcal{U}$ on X generates a quasi-proximity $\delta_{\mathcal{U}}$ on X such that $A\delta_{\mathcal{U}}B$ if $U(A) \cap B \neq \emptyset$ for all $U \in \mathcal{U}$.

In a quasi-uniform space $(X,\mathcal{U})$ we will denote by $\mathcal{P}_0(X)$ (resp. $\mathcal{CL}_0(X)$, $\mathcal{K}_0(X)$, $\mathcal{F}_0(X)$) the family of all nonempty (resp. nonempty closed, nonempty compact, nonempty finite) subsets of $(X,\mathcal{U})$. Our basic references for quasi-uniform spaces are [23,25].

Recall that a hypertopology is a topology defined over a certain family of sets. Our basic references for hypertopologies are [1,37].

Vietoris [45,36] defined the so-called *finite topology* on a topological space (X,τ) which is usually known as the *Vietoris topology*. On the family $\mathcal{P}_0(X)$ of all nonempty subsets of X, this topology τ_V has as a base all sets of the form

$$G^+ \cap V_1^- \cap \cdots \cap V_n^-$$

where $G, V_1, \ldots, V_n$ are open sets and

$$G^+ = \{A \in \mathcal{P}_0(X)\colon A \subseteq G\},$$
$$V_i^- = \{A \in \mathcal{P}_0(X)\colon A \cap V_i \neq \emptyset\}$$

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for all $i \in \{1, \dots, n\}$. Fell [22] considered a slight although very important modification of the above topology. In this way, the *Fell topology* τ_F has as a base all sets of the form $G^+ \cap V_1^- \cap \dots \cap V_n^-$ where $G, V_1, \dots, V_n$ are open sets and G^c is compact.

Notice that the only difference between τ_V and τ_F relies on the family to which the complement of G belongs: the closed sets in the case of the Vietoris topology and the closed and compact sets in the case of the Fell topology.

These two topologies follow a general pattern which was studied by Poppe [39]. Let Δ be a cobase, i.e. a family of closed sets containing the empty set, the singletons and closed under finite unions. Then the Δ-*hit-and-miss topology* has as a base all sets of the form $G^+ \cap V_1^- \cap \dots \cap V_n^-$ where $V_1, \dots, V_n$ are open sets and $G^c \in \Delta$.

In the literature about hypertopologies (see [1,37]), the most well-known is the so-called *topology of the Hausdorff distance*. Although this topology was first defined on a metric space, it was subsequently extended to a uniform space [14] and to a quasi-uniform space [13,30]. Given a quasi-uniform space $(X, \mathcal{U})$, for each $U \in \mathcal{U}$ define

$$U_H^+ = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X):\ B \subseteq U(A)\},$$
$$U_H^- = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X):\ A \subseteq U^{-1}(B)\}.$$

Then $\{U_H^+:\ U \in \mathcal{U}\}$ is a base for the *upper Hausdorff quasi-uniformity* $\mathcal{U}_H^+$ on $\mathcal{P}_0(X)$ and $\{U_H^-:\ U \in \mathcal{U}\}$ is a base for the *lower Hausdorff quasi-uniformity* $\mathcal{U}_H^-$ on $\mathcal{P}_0(X)$. The quasi-uniformity $\mathcal{U}_H = \mathcal{U}_H^+ \vee \mathcal{U}_H^-$ is the so-called *Hausdorff* (*or Bourbaki*) *quasi-uniformity* of $(X, \mathcal{U})$ on $\mathcal{P}_0(X)$.

We observe that a net $(A_\lambda)_{\lambda \in \Lambda}$ is convergent to A in the topology $\tau(\mathcal{U}_H)$ generated by the Hausdorff quasi-uniformity if and only if for all $U \in \mathcal{U}$

$$A_\lambda \subseteq U(A) \quad \text{and} \quad A \subseteq U^{-1}(A_\lambda) \text{ residually.}$$

The topology of the Hausdorff quasi-uniformity is also related to other hypertopology called the $\mathcal{CL}_0(X)$-*proximal miss topology* (or simply the upper proximal topology) and denoted by $\tau^{++}_{\mathcal{CL}_0(X)}$ [1]. This topology has as a base all the sets of the form $G^{++} = \{A \in \mathcal{P}_0(X):\ U(A) \subseteq G \text{ for some } U \in \mathcal{U}\}$ where G is an open set. Then it is easy to prove [1,42] that $\tau^{++}_{\mathcal{CL}_0(X)} = \tau(\mathcal{U}_H^+)$.

Nevertheless, in general, the topology generated by the Hausdorff quasi-uniformity is considered to be too strong. For example, let us consider $\mathbb{R}^2$ endowed with the usual uniformity. Then the graphs of the lines of slope $1/n$ passing through the origin form a sequence which is not convergent to the horizontal axis in the topology of the Hausdorff uniformity. This is due to the fact that this topology has not a good behavior with respect to unbounded sets.

A coarser topology is the so-called *Attouch–Wets topology* (see [2] for a survey). Traditionally, this topology is introduced as a topological convergence in a metric space [1]: given a metric space (X, d), a net $(A_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{P}_0(X)$ is said to be *Attouch–Wets* convergent to the nonempty set A if for every nonempty bounded subset $B \subseteq X$ and every $\varepsilon > 0$

$$A \cap B \subseteq B_d(A_\lambda, \varepsilon) \quad \text{and} \quad A_\lambda \cap B \subseteq B_d(A, \varepsilon) \text{ residually.}$$

The Attouch–Wets topology has been preferred for working in convex and set-valued analysis because it has a better behavior [1,31,41].

A uniform version of the Attouch–Wets topology was considered in [33, Section 6] by means of totally bounded sets, from where a quasi-uniform version can be naturally defined.

The two above topologies follow a pattern that can be generalized. Notice that if we consider the family $\mathcal{P}_0(X)$, then convergence of a net $(A_\lambda)_{\lambda \in \Lambda}$ to A in the topology of the Hausdorff quasi-uniformity is equivalent to ask that $A_\lambda \cap B \subseteq U(A)$ and $A \cap B \subseteq U(A_\lambda)$ residually for all $U \in \mathcal{U}$ and $B \in \mathcal{P}_0(X)$. So in both cases, the convergence is constructed by means of the truncation with a certain family of sets: the nonempty subsets in the case of the topology of the Hausdorff quasi-uniformity and the nonempty bounded subsets in the case of the Attouch–Wets topology.

Consequently, it is natural to study other convergences expressed in terms of truncations and enlargements with respect to an arbitrary family $\mathcal{S}$ of nonempty subsets of X. The filters which generate these convergences were perhaps first considered by Di Maio, Meccariello and Naimpally in [33,34,32] although the first deep study was made by Lechicki, Levi and Spakowski [29] (see [4] for a survey). We present here an asymmetric version of the so-called bornological convergences.

Definition 1.1. Let $(X, \mathcal{U})$ be a quasi-uniform space and $\mathcal{S}$ a family of nonempty subsets of X. We say that a net $(A_\lambda)_{\lambda \in \Lambda}$ of nonempty subsets:

1. $\mathcal{S}_{\mathcal{U}}^+$-converges to A if $A_\lambda \cap S \subseteq U(A)$ residually for each $S \in \mathcal{S}$ and $U \in \mathcal{U}$;
2. $\mathcal{S}_{\mathcal{U}}^-$-converges to A if $A \cap S \subseteq U^{-1}(A_\lambda)$ residually for each $S \in \mathcal{S}$ and $U \in \mathcal{U}$;
3. $\mathcal{S}_{\mathcal{U}}$-converges to A if $\mathcal{S}_{\mathcal{U}}^-$-converges to A and $\mathcal{S}_{\mathcal{U}}^+$-converges to A.

In the sequel, we will omit the subscript $\mathcal{U}$ if no confusion arises.

It is very easy to see that no different convergence appears if we replace $\mathcal{S}$ by the family of all subsets of finite unions of members of $\mathcal{S}$. Consequently, we will only consider *ideals*, i.e. families of nonempty subsets which are closed under nonempty subsets and finite unions. When an ideal $\mathcal{S}$ is also a cover then it is called a *bornology*. Since bornologies are more usual in applications, this kind of convergences is known as *bornological convergences*, whether or not the ideal is a bornology.

We will say that an ideal $\mathcal{S}$ has a *base* $\mathscr{B}$ if for all $S \in \mathcal{S}$ we can find $B \in \mathscr{B}$ such that $S \subseteq B$. If the elements of the base are closed, we say that $\mathscr{B}$ is a *closed base* for $\mathcal{S}$.

We will denote by $\bigcup \mathcal{S}$ the union of all the elements of the ideal $\mathcal{S}$.

Observe that if $\mathcal{S}$ is an ideal such that $X \in \mathcal{S}$ (like $\mathcal{P}_0(X)$ or $\mathcal{CL}_0(X)$) then $\mathcal{S}$-convergence is equivalent to convergence in the topology of the Hausdorff quasi-uniformity meanwhile the Attouch–Wets topology is obtained by means of the bornology of nonempty bounded subsets $\mathcal{B}_d(X)$.

Since the publication of [29], several papers have appeared studying this kind of convergences and bornologies [3–6, 8–12,46–48].

One of the main problems related to bornological convergences is to characterize when these convergences are topological [6,29]. The characterizations that have been already obtained are mainly based on constructing a (quasi-)uniformity compatible with the bornological convergence. In Section 2, we present a new approach to this problem by means of pretopological structures different of those considered in [29]. This allows us to present a pretopological structure whose aspect is similar to the base of a hit-and-far-miss topology [35]. From this presentation, we present new proofs about when bornological convergence is topological.

In Section 3, we characterize precompactness, total boundedness and compactness for bornological convergences. Our results extend well-known results in the asymmetric setting due to Künzi and Ryser [28].

We finish the paper showing a characterization of right K-completeness of the bornological convergence.

2. Topologicity of bornological convergences

In this section, we revisit some results about when bornological convergence is topological [6,29] by using certain pretopological structures which allow to show that bornological convergences are also, to some extent, hit-and-miss topologies.

Recall that a *pretopology* $\mathcal{N}$ on X is a collection of families of subsets of X $\{\mathcal{N}(x)\colon\ x \in X\}$ such that $\mathcal{N}(x)$ is a filter for all $x \in X$ and $x \in N$ for all $N \in \mathcal{N}(x)$. The pretopologies are nothing else but the neighborhood system of a closure space as defined by Čech [17].

A pretopology $\mathcal{N}$ which also verifies:

$$\text{given } N \in \mathcal{N}(x) \text{ there exists } Q \in \mathcal{N}(x) \text{ such that } Q \subseteq N \text{ and } Q \in \mathcal{N}(y) \text{ for all } y \in Q,$$

is a neighborhood system for a topology. In this case we say that $\mathcal{N}$ is a topology.

Every pretopology $\mathcal{N}$ generates a topology $\tau(\mathcal{N})$ by considering a set G open if $G \in \mathcal{N}(x)$ for all $x \in G$.

2.1. Upper half

In [29], the authors introduce a natural pretopology to study $\mathcal{S}^+$-convergence. Here, we study a different one whose aspect is very similar to an (upper) miss topology. Recall [37] that given a topological space (X, τ) and Δ a cobase in X, the *upper miss topology* $\tau_{\mathcal{S}}^+$ has as a base all sets of the form $(D^c)^+ = \{A \in \mathcal{P}_0(X)\colon\ A \subseteq D^c\}$ where $D \in \Delta$.

Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X, \mathcal{U})$. For each $A \in \mathcal{P}_0(X)$ define $\mathcal{B}^+_{\mathcal{S},\mathcal{U}}(A) = \{(S^c)^+\colon\ A \in (S^c)^{++},\ S \in \mathcal{S}\}$. It is obvious that $\mathcal{B}^+_{\mathcal{S},\mathcal{U}} = \{\mathcal{B}^+_{\mathcal{S},\mathcal{U}}(A)\colon\ A \in \mathcal{P}_0(X)\}$ is a base for a pretopology $\mathcal{N}^+_{\mathcal{S},\mathcal{U}} = \{\mathcal{N}^+_{\mathcal{S},\mathcal{U}}(A)\colon\ A \in \mathcal{P}_0(X)\}$ on $\mathcal{P}_0(X)$.

A particular case of these pretopological structures was first studied in [38] in relation with the problem of obtaining a hit-and-miss topology equivalent to the Wijsman topology. The smallest topology which contains the pretopology $\mathcal{N}^+_{\mathcal{S},\mathcal{U}}$ was called the *upper Wijsman $\mathcal{S}$-topology* in [35]. This is due to the fact that when we consider a metric space (X, d) and the family $\overline{\mathcal{B}}_d(X)$ of all closed balls, then the (pre)topology $\mathcal{N}^+_{\overline{\mathcal{B}}_d(X),\mathcal{U}}$ coincides with the upper Wijsman topology [38]. We also observe that the above pretopology can also be obtained as an *upper Bombay pretopology* $\sigma(\gamma_1, \gamma_2; \mathcal{S})^+$ when $\gamma_1 = \delta_{\mathcal{U}}$ and γ_2 is the Wallman proximity [32].

The following example shows that $\mathcal{N}^+_{\mathcal{S},\mathcal{U}}$ is not always a topology.

Example 2.1. Let us consider in the real line $\mathbb{R}$ the usual uniformity $\mathcal{U}$ and the bornology $\mathcal{S}$ generated by the family $\mathcal{P}_0([0,1]) \cup \mathcal{F}_0([0,1]^c)$. Then $([0,1]^c)^+ \in \mathcal{N}^+_{\mathcal{S},\mathcal{U}}(\{-1\})$. However, given $S \in \mathcal{S}$ such that $\{-1\} \in (S^c)^{++}$ and $(S^c)^+ \subseteq ([0,1]^c)^+$ then $S = [0,1] \cup F$ where F is a finite subset verifying $F \cap [0,1] = \emptyset$. Therefore, $S^c \in (S^c)^+$ but $S^c \notin ([0,1]^c)^{++}$. This means that for every basic $\mathcal{N}^+_{\mathcal{S},\mathcal{U}}$-neighborhood $(S^c)^+$ of $\{-1\}$ contained in $([0,1]^c)^+$ we can find $A \in (S^c)^+$ such that $(S^c)^+ \notin \mathcal{N}^+_{\mathcal{S},\mathcal{U}}(A)$. Therefore, $\mathcal{N}^+_{\mathcal{S},\mathcal{U}}$ is not a topology.

It is also known that every pretopology is equivalent to a convergence satisfying certain conditions [19]. The next result proves that the convergence associated to $\mathcal{N}^+_{\mathcal{S},\mathcal{U}}$ is exactly the $\mathcal{S}^+$-convergence. This means that the upper half of a bornological convergence can be obtained as a generalization of a(n upper) miss topology.

Lemma 2.2. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$. Then the pretopology $\mathcal{N}^+_{\mathcal{S},\mathcal{U}}$ is compatible with $\mathcal{S}^+$-convergence.*

Proof. Suppose that $(A_\lambda)_{\lambda\in\Lambda}$ is $\mathcal{S}^+$-convergent to A. Let $S\in\mathcal{S}$ such that $A\in(S^c)^{++}$. Therefore, we can find $U\in\mathcal{U}$ such that $U(A)\subseteq S^c$. Suppose, to obtain a contradiction, that $A_\lambda\notin(S^c)^+$ cofinally, i.e. $A_\lambda\cap S\neq\emptyset$ cofinally. By assumption $A_\lambda\cap S\subseteq U(A)$ residually and since $U(A)\subseteq S^c$ this implies that $S\cap S^c\neq\emptyset$ which is not possible.

Now, suppose that $(A_\lambda)_{\lambda\in\Lambda}$ converges in the pretopology $\mathcal{N}^+_{\mathcal{S},\mathcal{U}}$ to A. Let $S\in\mathcal{S}$ and $U\in\mathcal{U}$. Suppose that $S\not\subseteq U(A)$ (otherwise, the proof is finished). Then $S_0=S\backslash U(A)\in\mathcal{S}$ and $A\in(S_0^c)^{++}$ so $A_\lambda\in(S_0^c)^+$ residually, i.e. $A_\lambda\cap S_0=\emptyset$ residually. Therefore, $A_\lambda\cap S\subseteq U(A)$ residually. □

Recall [18,20] that if $\mathcal{N}$ is a pretopology on a nonempty set X then the interior of a set A with respect to $\mathcal{N}$ is

$$\operatorname{int}_{\mathcal{N}}(A)=\{x\in X:\ A\in\mathcal{N}(x)\}.$$

Furthermore, we say that a set O is open if $O=\operatorname{int}_{\mathcal{N}}(O)$. The topology $\tau(\mathcal{N})$ generated by the open sets of the pretopology $\mathcal{N}$ is called the **topologization** of $\mathcal{N}$. Furthermore, a pretopology $\mathcal{N}$ is a topology if for every $N\in\mathcal{N}(x)$ then $\operatorname{int}_{\mathcal{N}}(N)\in\mathcal{N}(x)$ [20].

Lemma 2.3. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$. Given $S\in\mathcal{S}$ then $\operatorname{int}_{\mathcal{N}^+_{\mathcal{S},\mathcal{U}}}(S^c)^+=(S^c)^{++}$.*

Proof. Let $B\in\operatorname{int}_{\mathcal{N}^+_{\mathcal{S},\mathcal{U}}}(S^c)^+$. Since $(S^c)^+\in\mathcal{N}^+_{\mathcal{S},\mathcal{U}}(B)$ then there exists $S_0\in\mathcal{S}$ such that $B\in(S_0^c)^{++}$ and $(S_0^c)^+\subseteq(S^c)^+$. From this we deduce that $S_0^c\subseteq S^c$ so $B\in(S^c)^{++}$.

On the other hand, if $B\in(S^c)^{++}$ then $(S^c)^+\in\mathcal{N}^+_{\mathcal{S},\mathcal{U}}(B)$ so $B\in\operatorname{int}_{\mathcal{N}^+_{\mathcal{S},\mathcal{U}}}(S^c)^+$. □

The following concept was introduced in [6] in order to characterize when $\mathcal{S}^+$-convergence is topological on $\mathcal{CL}_0(X)$.

Definition 2.4. ([6, Definition 5.1]) Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$ and $\mathcal{M}\subseteq\mathcal{P}_0(X)$. We say that $S\in\mathcal{S}$ is **shielded from the family** $\mathcal{M}$ **by** $\mathcal{S}$ if there exists $S_0\in\mathcal{S}$ such that if $A\in\mathcal{M}$ and $A\cap S_0=\emptyset$ then $A\not\delta_{\mathcal{U}}S$. In this case, we say that S_0 is a *shield* for S.

Definition 2.5. Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$. The *upper $\mathcal{S}$-proximal topology* $\tau^{++}_{\mathcal{S}}$ on $\mathcal{P}_0(X)$ is generated by all sets of the form $(S^c)^{++}=\{A\in\mathcal{P}_0(X):\ U(A)\subseteq S^c \text{ for some } U\in\mathcal{U}\}=\{A\in\mathcal{P}_0(X):\ A\not\delta_{\mathcal{U}}S\}$ where $S\in\mathcal{S}$.

The following result characterizes when the pretopology $\mathcal{N}^+_{\mathcal{S},\mathcal{U}}$ is a topology. Of course, this yields the characterization of when $\mathcal{S}^+$-convergence is topological. Furthermore, in this case, the topology compatible with $\mathcal{S}^+$-convergence is nothing else but the upper $\mathcal{S}$-proximal topology $\tau^{++}_{\mathcal{S}}$ as was first observed in [11] (compare also with [32, Theorem 2.1] and [35, Theorem 3.5]).

Theorem 2.6. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$ and $\mathcal{F}_0(X)\subseteq\mathcal{M}\subseteq\mathcal{P}_0(X)$. The following statements are equivalent*:

1. *$\mathcal{S}^+$-convergence is topological on $\mathcal{M}$;*
2. *$\mathcal{N}^+_{\mathcal{S},\mathcal{U}}$ is a topology on $\mathcal{M}$;*
3. *$(S^c)^{++}$ is $\tau(\mathcal{N}^+_{\mathcal{S},\mathcal{U}})$-open for all $S\in\mathcal{S}$ (this implies that $\mathcal{S}^+=\tau^{++}_{\mathcal{S}}$);*
4. *S is shielded from the family $\mathcal{M}$ by $\mathcal{S}$ for all non-dense $S\in\mathcal{S}$.*

Proof. (1) $\Leftrightarrow$ (2). This is obvious since by Lemma 2.2, $\mathcal{S}^+$-convergence is compatible with $\mathcal{N}^+_{\mathcal{S},\mathcal{U}}$.

(2) $\Rightarrow$ (3). By Lemma 2.3, we know that $\operatorname{int}_{\mathcal{N}^+_{\mathcal{S},\mathcal{U}}}(S^c)^+=(S^c)^{++}$. Furthermore, it is well known [20] that a pretopology is a topology if the interior of the neighborhoods are open sets. This implies that $(S^c)^{++}$ is $\tau(\mathcal{N}^+_{\mathcal{S},\mathcal{U}})$-open.

It is clear that it is always true that $\tau(\mathcal{N}^+_{\mathcal{S},\mathcal{U}})\leqslant\mathcal{S}^+\leqslant\tau^{++}_{\mathcal{S}}$. To show the last inequality, let $(A_\lambda)_{\lambda\in\Lambda}$ be a net $\tau^{++}_{\mathcal{S}}$-convergent to A and let $S\in\mathcal{S}, U\in\mathcal{U}$. Suppose that $S_0=S\backslash U(A)\neq\emptyset$ (otherwise, the proof is finished). Then $A_\lambda\in(S_0^c)^{++}$ residually, so $A_\lambda\cap S_0=\emptyset$ residually. Hence $A_\lambda\cap S\subseteq A_\lambda\cap U(A)\subseteq U(A)$ residually.

Since $(S^c)^{++}$ is $\tau(\mathcal{N}^+_{\mathcal{S},\mathcal{U}})$-open then $\tau(\mathcal{N}^+_{\mathcal{S},\mathcal{U}})=\mathcal{S}^+=\tau^{++}_{\mathcal{S}}$.

(3) $\Rightarrow$ (4). Let $S\in\mathcal{S}$ which is not $\tau(\mathcal{U})$-dense. Then we can find $A\in(S^c)^{++}$. By hypothesis, $(S^c)^{++}\in\mathcal{N}^+_{\mathcal{S},\mathcal{U}}(A)$ so there exists $S_0\in\mathcal{S}$ such that $A\in(S_0^c)^{++}$ and $(S_0^c)^+\subseteq(S^c)^{++}$. Therefore, if $B\cap S_0=\emptyset$ then $B\not\delta_{\mathcal{U}}S$.

(4) $\Rightarrow$ (3). Let $S\in\mathcal{S}$ be non-dense and let $S_0\in\mathcal{S}$ be a shield for S. Since $\mathcal{F}_0(X)\subseteq\mathcal{M}$, then $S\subseteq S_0$. Let $A\in(S^c)^{++}$. If $A\in(S_0^c)^{++}$ then $(S^c)^{++}\in\mathcal{N}^+_{\mathcal{S},\mathcal{U}}(A)$ since $(S_0^c)^+\subseteq(S^c)^{++}$.

Otherwise, $U(A)\cap S_0\neq\emptyset$ for all $U\in\mathcal{U}$. Let $V\in\mathcal{U}$ such that $V^2(A)\cap S=\emptyset$. Then $S_1=S_0\backslash V(A)\in\mathcal{S}$. It is clear that $A\in(S_1^c)^{++}$. Furthermore, $S\subseteq S_1$ since $V(A)\cap S=\emptyset$ and $S\subseteq S_0$. Now we prove that $(S_1^c)^+\subseteq(S^c)^{++}$. Let $B\in(S_1^c)^+$

where $B \in \mathcal{M}$. If $B \cap S_0 = \emptyset$ there is nothing to prove because since S_0 is a shield for S then $B \not\delta_{\mathcal{U}} S$. If $B \cap S_0 \neq \emptyset$ then $B \cap S_0 \subseteq V(A) \cap S_0$ since $B \cap S_1 = \emptyset$. Furthermore $V(V(A) \cap S_0) \subseteq V^2(A)$ and $V^2(A) \cap S = \emptyset$ so $V(B \cap S_0) \cap S = \emptyset$, i.e. $B \cap S_0 \not\delta_{\mathcal{U}} S$. Furthermore $B \backslash S_0 \not\delta_{\mathcal{U}} S$ since $B \backslash S_0 \in (S_0^c)^+ \subseteq (S^c)^{++}$. Hence $[(B \cap S_0) \cup (B \backslash S_0)] \not\delta_{\mathcal{U}} S$, i.e. $B \not\delta_{\mathcal{U}} S$ so we have proved that $(S_1^c)^+ \subseteq (S^c)^{++}$ and since $A \in (S_1^c)^+$ this means that $(S^c)^{++} \in \mathcal{N}^+_{\mathcal{S},\mathcal{U}}(A)$.

Therefore, $(S^c)^{++} \in \mathcal{N}^+_{\mathcal{S},\mathcal{U}}(A)$ for all $A \in (S^c)^{++}$ so $(S^c)^{++}$ is $\tau(\mathcal{N}^+_{\mathcal{S},\mathcal{U}})$-open.

If S is dense then $(S^c)^{++} = \emptyset \in \tau(\mathcal{N}^+_{\mathcal{S},\mathcal{U}})$.

(3) $\Rightarrow$ (2) is obvious. □

In a quasi-pseudometric space, let us denote by $\mathcal{B}_{d^{-1}}(X)$ the set of all d^{-1}-bounded sets. Then we consider the asymmetric version τ^+_{AW} of the upper Attouch–Wets topology generated by the quasi-uniformity $\mathcal{U}^+_{d,\mathcal{B}_{d^{-1}}(X)}$ whose basic entourages are of the form $U^+_{\varepsilon,S} = \{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X)\colon\ S \cap B \subseteq B_d(A,\varepsilon)\}$ where S is d^{-1}-bounded and $\varepsilon > 0$.

Corollary 2.7. *Let (X,d) be a quasi-pseudometric space and let $\mathcal{F}_0(X) \subseteq \mathcal{M} \subseteq \mathcal{P}_0(X)$. Then $\tau^+_{AW} = \tau^{++}_{\mathcal{B}_{d^{-1}}(X)}$ on $\mathcal{M}$.*

Proof. This is obvious since every d^{-1}-bounded set is shielded from $\mathcal{M}$ by $\mathcal{B}_{d^{-1}}(X)$. □

The next example shows that, in general, condition (4) of the above theorem is not true for dense sets.

Example 2.8. Let us consider the following quasi-metric defined on $\mathbb{N}$:

$$d(n,m) = \begin{cases} \frac{1}{m} & \text{if } n < m, \\ 1 & \text{if } n > m, \\ 0 & \text{if } n = m. \end{cases}$$

Let $\mathcal{S} = \mathcal{P}_0(\{4,5,6,\ldots\})$ and let us consider the pretopology $\mathcal{N}^+_{\mathcal{U}_d,\mathcal{S}}$ on $\mathcal{P}_0(\mathbb{N})$. If $S \in \mathcal{S}$ is $\tau(d)$-dense then $(S^c)^{++} = \emptyset$ so $(S^c)^{++} \in \tau(\mathcal{N}^+_{\mathcal{U}_d,\mathcal{S}})$. If S is not $\tau(d)$-dense then S is finite so it is easy to see that $(S^c)^{++} = (S^c)^+ \in \tau(\mathcal{N}^+_{\mathcal{U}_d,\mathcal{S}})$. This shows that $\mathcal{N}^+_{\mathcal{U}_d,\mathcal{S}}$ is a topology.

However, taking $S = \{4,5,6,\ldots\} \in \mathcal{S}$, if $A \cap S = \emptyset$ then $d(A,S) = 0$.

We observe that the above example is T_1 but not Hausdorff. Under this assumption, we can prove the following.

Lemma 2.9. *([6, Lemma 4.1]) Let $\mathcal{S}$ be an ideal in a Hausdorff quasi-uniform space $(X,\mathcal{U})$ which contains a dense set. Let $\mathcal{F}_0(X) \subseteq \mathcal{M} \subseteq \mathcal{P}_0(X)$. Then $\mathcal{N}^+_{\mathcal{U},\mathcal{S}}$ is a topology on $\mathcal{M}$ if and only if $\mathcal{S} = \mathcal{P}_0(X)$.*

Proof. Suppose that $\mathcal{N}^+_{\mathcal{U},\mathcal{S}}$ is a topology on $\mathcal{M}$ so by Theorem 2.6 every non-dense $S \in \mathcal{S}$ is shielded from $\mathcal{M}$ by $\mathcal{S}$. Let $S \in \mathcal{S}$ be a dense set. If S is a singleton then $X = S$ since the space is Hausdorff so $\mathcal{S} = \mathcal{P}_0(X)$ trivially. Suppose that we can find two different points s_1, s_2 in S. Since $(X,\mathcal{U})$ is Hausdorff there exists $U \in \mathcal{U}$ such that $U(s_1) \cap U(s_2) = \emptyset$. Then $S = S_1 \cup S_2$ where $S_1 = (S \backslash U(s_1)) \in \mathcal{S}$ and $S_2 = (S \backslash U(s_2)) \in \mathcal{S}$. Furthermore, neither S_1 nor S_2 are dense sets so by assumption they are shielded from $\mathcal{M}$ by $\mathcal{S}$. This immediately implies that S is shielded from $\mathcal{M}$ by $\mathcal{S}$. Let $S_0 \in \mathcal{S}$ such that if $A \cap S_0 = \emptyset$ then $A \not\delta_{\mathcal{U}} S$, where $A \in \mathcal{M}$. Since S is dense the only possibility is that $S_0 = X$.

The converse is obvious because we obtain a pretopology compatible with the Hausdorff quasi-uniform topology. □

Definition 2.10. Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$. We say that:

- $\mathcal{S}$ is **(almost) closed under $\mathcal{U}^{-1}$-small enlargements** if for each (non-dense) $S \in \mathcal{S}$ there exists $U \in \mathcal{U}$ such that $U^{-1}(S) \in \mathcal{S}$;
- $\mathcal{S}$ is **(almost) closed under $\mathcal{U}$-small enlargements** if for each (non-dense) $S \in \mathcal{S}$ there exists $U \in \mathcal{U}$ such that $U(S) \in \mathcal{S}$;
- $\mathcal{S}$ is an **E-ideal** if $\mathcal{S}$ is closed under $\mathcal{U}$-small enlargements and under $\mathcal{U}^{-1}$-small enlargements.

Note that if $\mathcal{S}$ is an E-ideal, then $\overline{S} \in \mathcal{S}$ for each $S \in \mathcal{S}$ (where the closure can be taken with respect to $\tau(\mathcal{U})$ and also with respect to $\tau(\mathcal{U}^{-1})$).

Corollary 2.11. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$. Then $\mathcal{S}^+$-convergence is topological on $\mathcal{P}_0(X)$ if and only if $\mathcal{S}$ is almost closed under $\mathcal{U}^{-1}$-small enlargements.*

Proof. It is obvious that if $\mathcal{S}$ is almost closed under $\mathcal{U}^{-1}$-small enlargements the condition (4) of Theorem 2.6 holds.

Now, suppose that condition (4) is true. Let $S \in \mathcal{S}$ non-dense, then there exists $U \in \mathcal{U}$ such that $X \backslash U^{-1}(S) \neq \emptyset$. Let $A \subseteq X \backslash U^{-1}(S)$. Then $A \in (S^c)^{++}$ and since $\mathcal{N}^+_{\mathcal{S},\mathcal{U}}$ is a topology there exists $S_0 \in \mathcal{S}$ such that $A \in (S_0^c)^{++}$, $(S_0^c)^+ \subseteq (S^c)^+$

and $(S_0^c)^+ \in \mathcal{N}^+_{\mathcal{S},\mathcal{U}}(B)$ for all $B \subseteq S_0^c$. If $V^{-1}(S) \not\subseteq S_0$ for all $V \in \mathcal{U}$ let $x_V \in V^{-1}(S) \cap S_0^c$. Then $C = \{x_V\colon\ V \in \mathcal{U}\} \subseteq S_0^c$ so $(S_0^c)^+ \in \mathcal{N}^+_{\mathcal{S},\mathcal{U}}(C)$. Hence $C \in (S_0^c)^{++}$ so $W(C) \cap S_0 = \emptyset$ for some $W \in \mathcal{U}$. However, $W(x_W) \cap S \neq \emptyset$ so $W(x_W) \cap S_0 \neq \emptyset$ since $S \subseteq S_0$. Of course this is a contradiction, hence we can find $V \in \mathcal{U}$ with $V^{-1}(S) \subseteq S_0$ so $V^{-1}(S) \in \mathcal{S}$. □

Corollary 2.12. *Let $\mathcal{S}$ be an ideal with a closed base in a quasi-uniform space $(X,\mathcal{U})$. Then $\mathcal{S}^+$-convergence is topological on $\mathcal{K}_0(X)$, so on $\mathcal{F}_0(X)$.*

Proof. Given $S \in \mathcal{S}$, choose a closed set $S_0 \in \mathcal{S}$ such that $S \subseteq S_0$. Pick up $A \in \mathcal{K}_0(X)$ verifying $A \cap S_0 = \emptyset$ and for each $a \in A$ let $U_a \in \mathcal{U}$ such that $U_a(a) \cap S_0 = \emptyset$. Since A is compact then $A \subseteq \bigcup_{i=1}^n U_{a_i}(a_i)$ for a finite subset $\{a_1,\ldots,a_n\} \subseteq A$. It is obvious that $A \,\delta\!\!\!/_{\mathcal{U}} S_0$ so $A \,\delta\!\!\!/_{\mathcal{U}} S$. The proof follows from Theorem 2.6. □

Corollary 2.13. *([6, Theorem 5.9]) Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$. Then $\mathcal{S}^+$-convergence is topological on $\mathcal{CL}_0(X)$ if and only if S is shielded from closed sets by $\mathcal{S}$ for every non-dense $S \in \mathcal{S}$.*

Corollary 2.14. *Let $(X,\mathcal{U})$ be a uniform space. Then the upper Fell topology coincides with $\mathcal{R}_0(X)^+$-convergence on $\mathcal{CL}_0(X)$, where $\mathcal{R}_0(X)$ denotes the bornology of all the relatively compact sets.*

Corollary 2.15. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\mathcal{F}_0(X)^+$-convergence is topological on $\mathcal{CL}_0^{-1}(X)$, the family of all nonempty $\tau(\mathcal{U}^{-1})$-closed subsets.*

2.2. Lower half

Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$. For each $A \in \mathcal{P}_0(X)$ define $\mathcal{B}^-_{\mathcal{S},\mathcal{U}}(A) = \{\bigcap_{x\in S} U(x)^-\colon\ S \in \mathcal{S},\ S \subseteq A$ and $U \in \mathcal{U}\}$ where $U(x)^- = \{A \in \mathcal{P}_0(X)\colon\ A \cap U(x) \neq \emptyset\}$. It is obvious that $\mathcal{B}^-_{\mathcal{S},\mathcal{U}} = \{\mathcal{B}^-_{\mathcal{S},\mathcal{U}}(A)\colon\ A \in \mathcal{P}_0(X)\}$ is a base for a pretopology (see [29, Theorem 2.11]) $\mathcal{N}^-_{\mathcal{S},\mathcal{U}} = \{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}(A)\colon\ A \in \mathcal{P}_0(X)\}$ on $\mathcal{P}_0(X)$.

Observe that $\bigcap_{x\in A\cap S} U(x)^- = \{B \in \mathcal{P}_0(X)\colon\ A \cap S \subseteq U^{-1}(B)\}$ whenever $A \cap S \neq \emptyset$. Consequently, this is a different presentation of the neighborhood system of the pretopology $\lambda(\mathcal{S}^-)$ introduced in [29]. We have chosen this aspect of the neighborhoods in order to present $\mathcal{S}^-$-convergence as a certain kind of hit topology [1,37]. In fact, when $\mathcal{S} = \mathcal{F}_0(X)$ then $\mathcal{N}^-_{\mathcal{S},\mathcal{U}}$ is nothing else but the neighborhood system for the lower Vietoris topology.

We also remark that the above pretopology is a generalization of the *lower locally finite topology* as defined in [33,38].

The following result, whose easy proof is omitted, reconciles the pretopology $\mathcal{N}^-_{\mathcal{S},\mathcal{U}}$ with $\mathcal{S}^-$-convergence.

Lemma 2.16. *([29, Lemma 2.10]) Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$ and $\mathcal{F}_0(X) \subseteq \mathcal{M} \subseteq \mathcal{P}_0(X)$. Then the pretopology $\mathcal{N}^-_{\mathcal{S},\mathcal{U}}$ is compatible with $\mathcal{S}^-$-convergence on $\mathcal{M}$.*

Lemma 2.17. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$. Given $U \in \mathcal{U}$ and $S \in \mathcal{S}$*

$$\operatorname{int}_{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}}\left(\bigcap_{x\in S} U(x)^-\right) = \left\{B \in \mathcal{P}_0(X)\colon \text{ there exist } S_0 \in \mathcal{S} \text{ contained in } B \text{ and } V \in \mathcal{U} \text{ such that if } S_0 \subseteq V^{-1}(A) \text{ then } S \subseteq U^{-1}(A)\right\}.$$

Proof. Let $B \in \operatorname{int}_{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}}(\bigcap_{x\in S} U(x)^-)$. Therefore, $\bigcap_{x\in S} U(x)^- \in \mathcal{N}^-_{\mathcal{S},\mathcal{U}}(B)$ so there exist $V \in \mathcal{U}$ and $S_0 \in \mathcal{S}$ such that $S_0 \subseteq B$ and $B \in \bigcap_{x\in S_0} V(x)^- \subseteq \bigcap_{x\in S} U(x)^-$. Suppose that $S_0 \subseteq V^{-1}(A)$. Then $A \in \bigcap_{x\in S_0} V(x)^-$ so $A \in \bigcap_{x\in S} U(x)^-$, i.e. $S \subseteq U^{-1}(A)$.

Conversely, let $B \in \mathcal{P}_0(X)$ such that there exist $S_0 \in \mathcal{S}$ and $V \in \mathcal{U}$ verifying that $S_0 \subseteq B$ and if $S_0 \subseteq V^{-1}(A)$ then $S \subseteq U^{-1}(A)$. Hence $B \in \bigcap_{x\in S_0} V(x)^- \subseteq \bigcap_{x\in S} U(x)^-$. Therefore, $B \in \operatorname{int}_{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}}(\bigcap_{x\in S} U(x)^-)$. □

Definition 2.18. ([11, Definition 25]) Let $(X,\mathcal{U})$ be a quasi-uniform space and $A \subseteq X$. Given $U \in \mathcal{U}$, the *U-approximate projection of A* is the multifunction $U - \mathrm{Proj}_A : X \rightrightarrows \mathcal{P}(X)$ given by

$$U - \mathrm{Proj}_A(x) = U(x) \cap A.$$

In the next theorem, we will also use the following notation:

$$\bigcap_{x\in S} U(x)^{-\mathcal{S}} = \left\{B \in \mathcal{P}_0(X)\colon \text{ there exists } S_0 \in \mathcal{S} \text{ such that } S_0 \subseteq B \text{ and } S \subseteq U^{-1}(S_0)\right\}.$$

Theorem 2.19. *([6, Theorem 3.3], [29, Corollary 2.12]) Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$ and $\mathcal{F}_0(X)\subseteq\mathcal{M}\subseteq\mathcal{P}_0(X)$. Suppose that $\mathcal{S}\subseteq\mathcal{M}$. The following statements are equivalent*:

1. *$\mathcal{S}^-$-convergence is topological on $\mathcal{M}$;*
2. *$\mathcal{N}^-_{\mathcal{S},\mathcal{U}}$ is a topology on $\mathcal{M}$;*
3. *$\operatorname{int}_{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}}(\bigcap_{x\in S}U(x)^-)$ is $\tau(\mathcal{N}^-_{\mathcal{S},\mathcal{U}})$-open whenever $S\in\mathcal{S}$ and $U\in\mathcal{U}$;*
4. *given $S\in\mathcal{S}$ and $U\in\mathcal{U}$ there exists $V\in\mathcal{U}$ such that, if $A\in\mathcal{M}$ and $S\subseteq V^{-1}(A)$, there exists $S_0\in\mathcal{S}$ with $S_0\subseteq A$ and $S\subseteq U^{-1}(S_0)$;*
5. *given $S\in\mathcal{S}$ and $U\in\mathcal{U}$ there exists $V\in\mathcal{U}$ such that if $V-\operatorname{Proj}_A(s)$ is nonempty for every $s\in S$ where $A\in\mathcal{M}$ then $U-\operatorname{Proj}_A$ has a selection f such that $f(S)\in\mathcal{S}$;*
6. *$\mathcal{N}^-_{\mathcal{S},\mathcal{U}}(A)$ is generated by $\{\bigcap_{x\in S}U(x)^{-\mathcal{S}}\colon S\in\mathcal{S},\ S\subseteq A,\ U\in\mathcal{U}\}$.*

Proof. (1) $\Leftrightarrow$ (2). This is obvious since by Lemma 2.16, $\mathcal{S}^-$-convergence is compatible with $\mathcal{N}^-_{\mathcal{S},\mathcal{U}}$.

(2) $\Leftrightarrow$ (3). As we have commented before, a pretopology forms a neighborhood system for a topology if the interior of the neighborhoods are open sets. Therefore $\operatorname{int}_{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}}(\bigcap_{x\in S}U(x)^-)$ is $\tau(\mathcal{N}^-_{\mathcal{S},\mathcal{U}})$-open.

(3) $\Rightarrow$ (4). Given $U\in\mathcal{U}$ and $S\in\mathcal{S}$, it is clear that $S\in\operatorname{int}_{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}}(\bigcap_{x\in S}U(x)^-)$. Since this set is $\tau(\mathcal{N}^-_{\mathcal{S},\mathcal{U}})$-open we can find $V\in\mathcal{U}$ and $S'\in\mathcal{S}$ such that $S'\subseteq S$ and $\bigcap_{x\in S'}V(x)^-\subseteq\operatorname{int}_{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}}(\bigcap_{x\in S}U(x)^-)$. If $A\in\mathcal{M}$ and $S'\subseteq S\subseteq V^{-1}(A)$ then $A\in\bigcap_{x\in S'}V(x)^-\subseteq\operatorname{int}_{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}}(\bigcap_{x\in S}U(x)^-)$ so by Lemma 2.17 there exists $S_0\in\mathcal{S}$ with $S_0\subseteq A$ and $S\subseteq U^{-1}(S_0)$.

(4) $\Rightarrow$ (3). Let $B\in\operatorname{int}_{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}}(\bigcap_{x\in S}U(x)^-)$. Then there exist $S_0\in\mathcal{S}$ and $V\in\mathcal{U}$ such that $S_0\subseteq B$ and $\bigcap_{x\in S_0}V(x)^-\subseteq\bigcap_{x\in S}U(x)^-$. Let $W'\in\mathcal{U}$ such that $W'^2\subseteq V$. By assumption we can find $W\in\mathcal{U}$ such that $W\subseteq W'$ and if $S_0\subseteq W^{-1}(A)$ there exists $S'\in\mathcal{S}$ with $S'\subseteq A$ and $S_0\subseteq W'^{-1}(S')$.

We show that $\bigcap_{x\in S_0}W(x)^-\subseteq\operatorname{int}_{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}}(\bigcap_{x\in S}U(x)^-)$ which implies that $\operatorname{int}_{\mathcal{N}^-_{\mathcal{S},\mathcal{U}}}(\bigcap_{x\in S}U(x)^-)\in\mathcal{N}^-_{\mathcal{S},\mathcal{U}}(B)$.

Let $A\in\bigcap_{x\in S_0}W(x)^-$. Then $S_0\subseteq W^{-1}(A)$ so there exists $S'\in\mathcal{S}$ with $S'\subseteq A$ and $S_0\subseteq W'^{-1}(S')$. If $B\in\bigcap_{x\in S'}W(x)^-$ then $S'\subseteq W^{-1}(B)$ so $S_0\subseteq W'^{-1}(S')\subseteq(W'\circ W)^{-1}(B)\subseteq V^{-1}(B)$, i.e. $B\in\bigcap_{x\in S_0}V(x)^-\subseteq\bigcap_{x\in S}U(x)^-$. Consequently $\bigcap_{x\in S'}W(x)^-\subseteq\bigcap_{x\in S}U(x)^-$ so $\bigcap_{x\in S}U(x)^-\in\mathcal{N}^-_{\mathcal{S},\mathcal{U}}(A)$, which finishes the proof.

(4) $\Leftrightarrow$ (5). This equivalence follows from the following facts: $S\subseteq V^{-1}(A)$ is equivalent to assert that $V-\operatorname{Proj}_A(s)$ is nonempty for every $s\in S$; the existence of $S_0\in\mathcal{S}$ verifying $S_0\subseteq A$ and $S\subseteq U^{-1}(S_0)$ is equivalent to the existence of a selection f of $U-\operatorname{Proj}_A$ such that $f(S)\in\mathcal{S}$.

(4) $\Leftrightarrow$ (6). It is clear that given $A\in\mathcal{M}$, $U\in\mathcal{U}$ and $S\in\mathcal{S}$ with $S\subseteq A$ then $\bigcap_{x\in S}U(x)^{-\mathcal{S}}\subseteq\bigcap_{x\in S}U(x)^-$. On the other hand, by assumption, there exists $V\in\mathcal{U}$ such that if $B\in\mathcal{M}$ and $S\subseteq V^{-1}(B)$ there exists $S_0\in\mathcal{S}$ verifying $S_0\subseteq B$ and $S\subseteq U^{-1}(S_0)$. Therefore, $\bigcap_{x\in S}V(x)^-\subseteq\bigcap_{x\in S}U(x)^{-\mathcal{S}}$.

The converse follows also easily. □

Remark 2.20. We observe that the fact that $\mathcal{S}\subseteq\mathcal{M}$ is only used in the implication (3) $\Rightarrow$ (4). This implication is also valid if $\overline{S}\in\mathcal{M}$ for all $S\in\mathcal{S}$.

Example 2.21. Let $(X,\mathcal{U})$ be a quasi-uniform space. A subset A of X is called *$\mathcal{U}^{-1}$-separated* [16] if there exist $U\in\mathcal{U}$ and an ordinal γ such that $A=\{a_\alpha\colon\alpha<\gamma\}$ and $a_\beta\notin U^{-1}(a_\alpha)$ whenever $\alpha<\beta<\gamma$. Let $\mathfrak{D}$ be the family of finite unions of $\mathcal{U}^{-1}$-separated sets. It is easy to see that $\mathfrak{D}$ is a bornology.

Let $U\in\mathcal{U}$ and $D\in\mathfrak{D}$. Let $V\in\mathcal{U}$ verifying that $V^2\subseteq U$ and suppose that $D\subseteq V^{-1}(A)$. Suppose that $A=\{x_\alpha\colon\alpha<\gamma\}$ where γ is an ordinal. Then define $y_1=x_1$ and for each $\beta<\gamma$ define by transfinite recursion $y_\beta=x_{\beta_0}$ where $\beta_0=\min\{\alpha<\gamma\colon x_\alpha\in A\setminus\bigcup_{\lambda<\beta}V^{-1}(y_\lambda)\}$. Then it is easy to see that $D_0=\{y_\beta\colon\beta<\gamma\}$ is a V^{-1}-separated subset of A (so $D_0\in\mathfrak{D}$) and that $A\subseteq V^{-1}(D_0)$. Then $D\subseteq V^{-1}(A)\subseteq V^{-2}(D_0)\subseteq U^{-1}(D_0)$.

We have shown that $\mathfrak{D}$ verifies condition (4) of the above theorem so $\mathcal{N}^-_{\mathfrak{D},\mathcal{U}}$ is the neighborhood system for a topology.

In the following, we prove that $\tau(\mathcal{N}^-_{\mathfrak{D},\mathcal{U}})=\tau(\mathcal{U}^-_H)$. Naimpally [38, Lemma 3.4] was the first to prove this equality for uniformities. Let $U\in\mathcal{U}$ and $A\in\mathcal{P}_0(X)$. Let $V\in\mathcal{U}$ with $V^2\subseteq U$ and let D be a maximal V^{-1}-separated subset of A. Then $A\in\bigcap_{x\in D}V(x)^-\subseteq U^-_H(A)$. In fact, if $B\in\bigcap_{x\in D}V(x)^-$ then $D\subseteq V^{-1}(B)$. Furthermore, $A\subseteq V^{-1}(D)$ so $A\subseteq V^{-2}(B)\subseteq U^{-1}(B)$.

Now, let D be a V-separated subset of A and $U\in\mathcal{U}$. Then if $A\subseteq U^{-1}(B)$ we deduce that $D\subseteq U^{-1}(B)$, i.e. $B\in\bigcap_{x\in D}U(x)^-$. Therefore, $U^-_H(A)\subseteq\bigcap_{x\in D}U(x)^-$.

Corollary 2.22. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\mathcal{F}_0(X)\subseteq\mathcal{M}\subseteq\mathcal{P}_0(X)$. Then $\mathcal{F}_0(X)^-$-convergence is topological on $\mathcal{M}$ and coincides with the lower Vietoris topology.*

Corollary 2.23. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$ closed under $\mathcal{U}$-enlargements. Then $\mathcal{S}^-$-convergence is topological on $\mathcal{P}_0(X)$.*

Proof. Let $U \in \mathcal{U}$ and $S \in \mathcal{S}$. By assumption, there exists $V \in \mathcal{U}$ such that $V \subseteq U$ and $V(S) \in \mathcal{S}$. Then if $S \subseteq V^{-1}(A)$, the set $S_0 = V(S) \cap A$ belongs to $\mathcal{S}$ and it is obvious that $S \subseteq V^{-1}(S_0) \subseteq U^{-1}(S_0)$. Therefore, condition (4) of Theorem 2.19 holds so $\mathcal{S}^-$-convergence is topological. □

Observe that, in general, the reverse implication is not true. It is enough to consider the real line endowed with the usual metric and with the bornology of all finite subsets. Then by Corollary 2.22, $\mathcal{S}^-$-convergence is topological on $\mathcal{P}_0(\mathbb{R})$ but the bornology is not closed under small enlargements.

Corollary 2.24. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$ which contains the singletons. Then $\mathcal{S}^-$-convergence is topological on $\mathcal{K}_0^*(X)$, the family of all nonempty compact subsets of $(X, \tau(\mathcal{U}^*))$.*

Proof. Let $S \in \mathcal{S}$ and $U \in \mathcal{U}$. Let $V \in \mathcal{U}$ such that $V^2 \subseteq U$ and suppose that $S \subseteq V^{-1}(A)$ where A is compact. Let $F = \{a_1, \ldots, a_n\} \subseteq A$ such that $A \subseteq V^s(F)$. Then $S \subseteq V^{-2}(F) \subseteq U^{-1}(F)$. This implies that condition (4) of Theorem 2.19 is verified so the convergence is topological. □

Corollary 2.25. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$ such that $\mathcal{F}_0(X) \subseteq \mathcal{S}$. Then $\mathcal{S}^-$-convergence is topological on $\mathcal{F}_0(X)$ and coincides with the lower Vietoris topology.*

2.3. Bilateral results

Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$. For each $A \in \mathcal{P}_0(X)$ define $\mathcal{B}_{\mathcal{S},\mathcal{U}}(A) = \{(S^c)^+ \cap \bigcap_{x\in S'} U(x)^-\colon\ A \in (S^c)^{++}, S' \subseteq A \text{ and } S, S' \in \mathcal{S}\}$. It is obvious that $\mathcal{B}_{\mathcal{S},\mathcal{U}} = \{\mathcal{B}_{\mathcal{S},\mathcal{U}}(A)\colon\ A \in \mathcal{P}_0(X)\}$ is a base for a pretopology $\mathcal{N}_{\mathcal{S},\mathcal{U}} = \{\mathcal{N}_{\mathcal{S},\mathcal{U}}(A)\colon\ A \in \mathcal{P}_0(X)\}$ on $\mathcal{P}_0(X)$.

Putting together Lemmas 2.2 and 2.16 we obtain the following.

Lemma 2.26. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$. Then the pretopology $\mathcal{N}_{\mathcal{S},\mathcal{U}}$ is compatible with $\mathcal{S}$-convergence.*

Theorem 2.27. *([6, Theorem 5.16]) Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$ and $\mathcal{F}_0(X) \subseteq \mathcal{M} \subseteq \mathcal{P}_0(X)$. The following statements are equivalent:*

1. *$\mathcal{S}$-convergence is topological on $\mathcal{M}$;*
2. *$\mathcal{N}_{\mathcal{S},\mathcal{U}}$ is a topology on $\mathcal{M}$;*
3. - *S is shielded from the family $\mathcal{M}$ by $\mathcal{S}$ for all non-dense $S \in \mathcal{S}$;*
   - *given $S \in \mathcal{S}$ and $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that if $A \in \mathcal{M}$ and $S \subseteq V^{-1}(A)$ there exists $S_0 \in \mathcal{S}$ with $S_0 \subseteq A$ and $S \subseteq U^{-1}(S_0)$.*

From the above results, we can obtain a lot of consequences. We only present here two of them.

Corollary 2.28. *Let $\mathcal{S}$ be an E-ideal in a quasi-uniform space $(X,\mathcal{U})$. Then $\mathcal{S}$-convergence is topological on $\mathcal{P}_0(X)$.*

Corollary 2.29. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\mathcal{F}_0(X)$-convergence is topological on $\mathcal{CL}_0^{-1}(X)$, the family of all nonempty $\tau(\mathcal{U}^{-1})$-closed subsets.*

2.4. Quasi-uniformities compatible with bornological convergences

In [29], the authors introduce a natural family of sets which under some assumptions is the base for a uniform structure compatible with the bornological convergence. We provide an asymmetric version of those results.

Let $\mathcal{S}$ be an ideal in a quasi-uniform space. For each $U \in \mathcal{U}$ and for each $S \in \mathcal{S}$ let us define:

$$U_S^+ = \{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X)\colon\ B \cap S \subseteq U(A)\};$$
$$U_S^- = \{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X)\colon\ A \cap S \subseteq U^{-1}(B)\};$$
$$U_S = U_S^+ \cap U_S^-.$$

We set $\mathcal{U}_{\mathcal{S}}^+$, $\mathcal{U}_{\mathcal{S}}^-$ and $\mathcal{U}_{\mathcal{S}}$ the filters which have as a base $\{U_S^+\colon\ U \in \mathcal{U},\ S \in \mathcal{S}\}$, $\{U_S^-\colon\ U \in \mathcal{U},\ S \in \mathcal{S}\}$ and $\{U_S\colon\ U \in \mathcal{U},\ S \in \mathcal{S}\}$, respectively.

In the following, we characterize when the three above structures are quasi-uniformities.

Proposition 2.30. *Let $\mathcal{S}$ be an ideal and $(X,\mathcal{U})$ a quasi-uniform space. Suppose that for each $S \in \mathcal{S}$ there exist $U \in \mathcal{U}$ and $S' \in \mathcal{S}$ with $S \subseteq S'$ such that $\bigcap_{x\in S'} U(x) = \emptyset$.*

1. *If $\mathcal{U}_{\mathcal{S}}^+$ is a quasi-uniformity then $\mathcal{S}$ is closed under $\mathcal{U}^{-1}$-small enlargements.*
2. *If $\mathcal{U}_{\mathcal{S}}^-$ is a quasi-uniformity then $\mathcal{S}$ is closed under $\mathcal{U}$-small enlargements.*
3. *If $\mathcal{U}_{\mathcal{S}}$ is a quasi-uniformity then $\mathcal{S}$ is an E-ideal.*

Proof. Suppose that $\mathcal{U}_{\mathcal{S}}^+$ is a quasi-uniformity and that there exists $S_0 \in \mathcal{S}$ with $U^{-1}(S_0) \notin \mathcal{S}$ for each $U \in \mathcal{U}$. Let $U \in \mathcal{U}$ and $S \in \mathcal{S}$ with $S_0 \subseteq S$ and such that $\bigcap_{x \in S} U(x) = \emptyset$. Since $\mathcal{U}_{\mathcal{S}}^+$ is a quasi-uniformity, there exist $W \in \mathcal{U}$ and $S_1 \in \mathcal{S}$ such that $(W_{S_1}^+)^2 \subseteq U_S^+$. Since $W^{-1}(S) \nsubseteq S_1$, there exists $z \in W^{-1}(S) \setminus S_1$, so we can find $x \in S$ such that $x \in W(z)$. By hypothesis, there exists $y \in S$ such that $x \notin U(y)$. Let $A = \{x\}$, $B = \{z\}$ and $C = \{y\}$. Then $A \cap S_1 \subseteq \{x\} \subseteq W(B)$ and $B \cap S_1 = \emptyset \subseteq W(C)$. It follows that $(C, A) \in (W_{S_1}^+)^2 \subseteq U_S^+$ and hence $A \in U_S^+(C)$, that is, $x \in U(y)$, a contradiction. Therefore there exists $U \in \mathcal{U}$ with $U^{-1}(S_0) \in \mathcal{S}$ whence $\mathcal{S}$ is closed under $\mathcal{U}^{-1}$-small enlargements.

The second item follows similarly and the third item is a consequence of the first and second ones. □

Proposition 2.31. *([29]) Let $\mathcal{S}$ be an ideal and $(X, \mathcal{U})$ a quasi-uniform space.*

1. *If $\mathcal{S}$ is closed under $\mathcal{U}$-small enlargements then $\mathcal{U}_{\mathcal{S}}^-$ is a quasi-uniformity.*
2. *If $\mathcal{S}$ is closed under $\mathcal{U}^{-1}$-small enlargements then $\mathcal{U}_{\mathcal{S}}^+$ is a quasi-uniformity.*
3. *If $\mathcal{S}$ is an E-ideal then $\mathcal{U}_{\mathcal{S}}$ is a quasi-uniformity.*
4. *If $(X, \mathcal{U})$ is Hausdorff and $\mathcal{U}_{\mathcal{S}}^-$ is a quasi-uniformity then $\mathcal{S}$ is closed under $\mathcal{U}$-small enlargements.*
5. *If $(X, \mathcal{U})$ is Hausdorff and $\mathcal{U}_{\mathcal{S}}^+$ is a quasi-uniformity then $\mathcal{S}$ is closed under $\mathcal{U}^{-1}$-small enlargements.*
6. *If $(X, \mathcal{U})$ is Hausdorff and $\mathcal{U}_{\mathcal{S}}$ is a quasi-uniformity then $\mathcal{S}$ is an E-ideal.*

Proof. Suppose that $\mathcal{S}$ is closed under $\mathcal{U}$-small enlargements.

Let $U \in \mathcal{U}$ and $S \in \mathcal{S}$. Take $V \in \mathcal{U}$ with $V^2 \subseteq U$ and $V(S) \in \mathcal{S}$. Let us prove that $(V_{V(S)}^-)^2 \subseteq U_S^-$. Let $A, B, C \subseteq X$ with $C \in V_{V(S)}^-(B)$ and $B \in V_{V(S)}^-(A)$. Then $B \cap V(S) \subseteq V^{-1}(C)$ and $A \cap V(S) \subseteq V^{-1}(B)$. Let $x \in A \cap S$, then $x \in V^{-1}(B)$ so there exists $b \in B$ with $x \in V^{-1}(b)$. Now $b \in V(x) \cap B \subseteq V(S) \cap B \subseteq V^{-1}(C)$, and hence $x \in V^{-2}(C) \subseteq U^{-1}(C)$. It follows that $A \cap S \subseteq U^{-1}(C)$, that is, $C \in U_S^-(A)$. Therefore $\mathcal{U}_{\mathcal{S}}^-$ is a quasi-uniformity.

The second item follows similarly and the third item is a consequence of the first and second ones.

The rest of the items follows from the previous proposition. □

3. Precompactness, total boundedness and compactness of bornological structures

This section is devoted to study precompactness, total boundedness and compactness of the filter $\mathcal{U}_{\mathcal{S}}$. Although this filter is not always a quasi-uniformity, the aforementioned notions can be extended to this setting.

Definition 3.1. Let $(X, \mathcal{U})$ be a quasi-uniform space and $\mathcal{B} \subseteq \mathcal{P}_0(X)$. We say that $A \subseteq X$ is ***$\mathcal{B}$*-weakly precompact** if for every $U \in \mathcal{U}$ we can find $\{B_1, \dots, B_n\} \subseteq \mathcal{B}$ such that $A \subseteq \bigcup_{i=1}^n U(B_i)$.

Definition 3.2. Let $(X, \mathcal{U})$ be a quasi-uniform space and $B \subseteq X$. We say that $A \subseteq X$ is ***B*-precompact** if for every $U \in \mathcal{U}$ we can find $\{b_1, \dots, b_n\} \subseteq B$ such that $A \subseteq \bigcup_{i=1}^n U(b_i)$.

Proposition 3.3. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X, \mathcal{U})$ and $\mathcal{M}$ a nonempty subset of $\mathcal{P}_0(X)$ which covers every element of $\mathcal{S}$. Let $\mathcal{B} \subseteq \mathcal{M}$ be closed under finite unions. Then $(\mathcal{M}, \mathcal{U}_{\mathcal{S}}^+)$ is $\mathcal{B}$-precompact if and only if S is $\mathcal{B}$-weakly precompact for all $S \in \mathcal{S}$.*

Proof. Let $S \in \mathcal{S}$ and $U \in \mathcal{U}$. Then there exists $\{B_1, \dots, B_n\} \subseteq \mathcal{B}$ such that $\mathcal{M} = \bigcup_{i=1}^n U_S^+(B_i)$. Since S can be covered by elements of $\mathcal{M}$ given $s \in S$ there exists $A_s \in \mathcal{M}$ such that $s \in A_s \in U_S^+(B_j)$ for some $j \in \{1, \dots, n\}$. Therefore, $s \in A_s \cap S \subseteq U(B_j)$ so $S \subseteq \bigcup_{i=1}^n U(B_j)$.

Conversely, if $\mathcal{S} \subseteq \mathcal{B}$, given $U \in \mathcal{U}$ and $S \in \mathcal{S}$ then $\mathcal{M} = U_S^+(S)$. Otherwise, suppose that there exists $S \in \mathcal{S} \backslash \mathcal{B}$. Then $S \subseteq \bigcup_{i=1}^n U(B_i)$ where $B_i \in \mathcal{B}$ for all $i \in \{1, \dots, n\}$. It is easy to see that $\mathcal{M} = U_S^+(\bigcup_{i=1}^n B_i)$. □

Corollary 3.4. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X, \mathcal{U})$. Then $(\mathcal{P}_0(X), \mathcal{U}_{\mathcal{S}}^+)$ is $\mathcal{S}$-precompact so precompact.*

Corollary 3.5. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X, \mathcal{U})$. Then $(\mathcal{P}_0(X), \mathcal{U}_{\mathcal{S}}^+)$ is $\mathcal{F}_0(X)$-precompact if and only if S is X-precompact for all $S \in \mathcal{S}$.*

Recall that given a family $\mathcal{F}$ of subsets of X, the *grill* of $\mathcal{F}$ is $\mathcal{F}^\sharp = \{A \subseteq X\colon\ A \cap F \neq \emptyset \text{ for all } F \in \mathcal{F}\}$.

Proposition 3.6. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X, \mathcal{U})$ and $\mathcal{M}$ a nonempty subset of $\mathcal{P}_0(X)$ such that $\mathcal{S} \subseteq \mathcal{M}$. Let $\mathcal{F}_0(X) \subseteq \mathcal{B} \subseteq \mathcal{M}$. Then $(\mathcal{M}, \mathcal{U}_{\mathcal{S}}^-)$ is $\mathcal{B}$-precompact if and only if S is precompact for every $S \in \mathcal{S} \cap \mathcal{M}^\sharp$.*

Proof. Suppose that $(\mathcal{M}, \mathcal{U}_{\mathcal{S}}^{-})$ is $\mathcal{B}$-precompact. Let $S \in \mathcal{S}$ such that $A \cap S \neq \emptyset$ for all $A \in \mathcal{M}$. Given $U \in \mathcal{U}$ there exists $\{B_1, \ldots, B_n\} \subseteq \mathcal{B}$ such that $\mathcal{M} \subseteq \bigcup_{i=1}^{n} U_S^{-}(B_i)$. Since $\mathcal{B} \subseteq \mathcal{M}$ then $B_i \cap S \neq \emptyset$ so let $b_i \in B_i \cap S$ for all $i \in \{1, \ldots, n\}$. Given $s \in S$, since $\{s\} \in \mathcal{M}$, there exists $j \in \{1, \ldots, n\}$ with $\{s\} \in U_S^{-}(B_j)$, i.e. $B_j \cap S \subseteq U^{-1}(\{s\})$. In particular $b_j \in U^{-1}(s)$ so $S \subseteq \bigcup_{i=1}^{n} U(b_i)$.

Conversely, let $S \in \mathcal{S}$ and $U \in \mathcal{U}$. If there exists $A \in \mathcal{M}$ such that $A \cap S = \emptyset$ then $\mathcal{M} = U_S^{-}(A)$. Otherwise, $S \cap A \neq \emptyset$ for all $A \in \mathcal{M}$. Since S is precompact we can find a finite subset S_0 of S such that $S \subseteq \bigcup_{s \in S_0} U(s)$. We show that $\mathcal{M} = \bigcup_{F \in \mathcal{P}_0(S_0)} U_S^{-}(F)$. Given $A \in \mathcal{M}$ then $A \cap S \neq \emptyset$. Since $A \cap S \subseteq S$ there exists $F \in \mathcal{P}_0(S_0)$ such that $U(x) \cap (A \cap S) \neq \emptyset$ for all $x \in F$, i.e. $F \cap S \subseteq U^{-1}(A \cap S) \subseteq U^{-1}(A)$ so $A \in U_S^{-}(F)$. $\square$

Corollary 3.7. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X, \mathcal{U})$.*

1. *If $X \notin \mathcal{S}$ then $(\mathcal{P}_0(X), \mathcal{U}_{\mathcal{S}}^{-})$ is precompact.*
2. *If $X \in \mathcal{S}$ then $(\mathcal{P}_0(X), \mathcal{U}_{\mathcal{S}}^{-}) = (\mathcal{P}_0(X), \mathcal{U}_{H}^{-})$ is precompact if and only if X is precompact.*

In the following, we prove some bilateral results.

Theorem 3.8. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X, \mathcal{U})$. Then $(\mathcal{P}_0(X), \mathcal{U}_{\mathcal{S}})$ is precompact if and only if for every $S \in \mathcal{S}$ and $U \in \mathcal{U}$ there exists a finite set $S_0 \subseteq S$ such that $S \subseteq U(S_0) \cup U(X \backslash S)$.*

If $\mathcal{S}$ is closed under $\mathcal{U}^{-1}$-small enlargements, the above condition reduces to S is X-precompact for all $S \in \mathcal{S}$.

Proof. Suppose that $(\mathcal{P}_0(X), \mathcal{U}_{\mathcal{S}})$ is precompact and let $S \in \mathcal{S}$ and $U \in \mathcal{U}$. Then, there exists $\{A_1, \ldots, A_n\} \subseteq \mathcal{P}_0(X)$ such that $\mathcal{P}_0(X) = \bigcup_{i=1}^{n} U_S(A_i)$. Let $J \subseteq \{1, \ldots, n\}$ such that there exists $s_j \in A_j \cap S$ if and only if $j \in J$. Define $S_0 = \bigcup_{j \in J} s_j$. We see that $S \subseteq U(S_0) \cup U(X \backslash S)$. Given $s \in S$ we can find $j \in \{1, \ldots, n\}$ such that $\{s\} \in U_S(A_j)$. If $j \in J$ then $s \in U(s_j)$ and if $j \notin J$ then $s \in U(A_j) \subseteq U(X \backslash S)$.

Conversely, let $S \in \mathcal{S}$ and $U \in \mathcal{U}$. By assumption, there exists a finite subset S_0 of S such that $S \subseteq U(S_0) \cup U(X \backslash S)$. Let $\mathcal{F} = \{B \cup (X \backslash S)\colon\ B \in \mathcal{P}_0(S_0)\}$. We show that $\mathcal{P}_0(X) = \bigcup_{F \in \mathcal{F}} U_S(F)$. In fact, given $A \in \mathcal{P}_0(X)$ we can find $B \in \mathcal{F}_0(S_0)$ (maybe empty) such that $A \cap S \subseteq U(B) \cup U(X \backslash S)$ and $U(b) \cap A \cap S \neq \emptyset$ for all $b \in B$. Then $A \in U_S(B \cup (X \backslash S))$.

Now suppose that S is closed under $\mathcal{U}^{-1}$-small enlargements. It is clear that if S is X-precompact for all $S \in \mathcal{S}$ then the above condition holds. Given $S \in \mathcal{S}$ and $U \in \mathcal{U}$ we can find $V \in \mathcal{U}$ with $V \subseteq U$ and $V^{-1}(S) \in \mathcal{S}$. By assumption, there exits a finite subset S_0 of $V^{-1}(S)$ such that $S \subseteq V^{-1}(S) \subseteq V(S_0) \cup V(X \backslash V^{-1}(S))$. Since $S \cap V(X \backslash V^{-1}(S)) = \emptyset$ then $S \subseteq V(S_0) \subseteq U(S_0)$. $\square$

Our next example shows that, without the assumption of $\mathcal{S}$ been closed under $\mathcal{U}^{-1}$-small enlargements, the above condition is not equivalent to X-precompactness.

Example 3.9. Let us consider the real line with the usual uniformity $\mathcal{U}$. Let $\mathcal{S} = \mathcal{P}_0(\mathbb{Q})$. It is easy to see that for each $S \in \mathcal{S}$ and $U \in \mathcal{U}$, $\mathcal{P}_0(\mathbb{R}) = U_S(\mathbb{R} \backslash \mathbb{Q})$. Therefore $(\mathcal{P}_0(\mathbb{R}), \mathcal{U}_{\mathcal{S}})$ is precompact. Nevertheless, $\mathbb{Q}$ is not $\mathbb{R}$-precompact.

Definition 3.10. ([21,23]) Let $(X, \mathcal{U})$ be a quasi-uniform space.

- We say that $(X, \mathcal{U})$ is *point-symmetric* if for each $x \in X$ and $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^{-1}(x) \subseteq U(x)$.
- If $\mathcal{S}$ is an E-ideal in $(X, \mathcal{U})$, we say that $(X, \mathcal{U})$ is *closed-symmetric for* $\mathcal{S}$ if for each closed subset A of X, $S \in \mathcal{S}$ and $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V^{-1}(A) \cap S \subseteq U(A)$.
  When $\mathcal{S} = \mathcal{P}_0(X)$, we simply say that $(X, \mathcal{U})$ is *closed-symmetric*.

We observe that every uniform space is point-symmetric and closed-symmetric.

Proposition 3.11. *Let $\mathcal{S}$ be an E-ideal and $(X, \mathcal{U})$ a quasi-uniform space.* (1) *implies* (2) *and* (2) *and* (3) *are equivalent.*

1. *$(\mathcal{C}_0(X), \mathcal{U}_{\mathcal{S}})$ is point-symmetric.*
2. *$(X, \mathcal{U})$ is closed-symmetric for $\mathcal{S}$.*
3. *For each closed $S \in \mathcal{S}$ and each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V^{-1}(S) \subseteq U(S)$.*

Proof. (1) $\Rightarrow$ (2). Let $A \in \mathcal{C}_0(X)$, $S \in \mathcal{S}$, $U \in \mathcal{U}$. By (1) there exist $V \in \mathcal{U}$ and $S_1 \in \mathcal{S}$ with $V_{S_1}^{-1}(A) \subseteq U_S(A)$. Let $W \in \mathcal{U}$ with $W^2 \subseteq V$. Then $\overline{W^{-1}(A)} \subseteq V^{-1}(A)$ so $\overline{W^{-1}(A)} \in V_{S_1}^{-1}(A) \subseteq U_S(A)$ and hence $W^{-1}(A) \cap S \subseteq U(A)$.

(2) $\Rightarrow$ (3). Let $S \in \mathcal{C}_0(X) \cap \mathcal{S}$, $U \in \mathcal{U}$ and $V \in \mathcal{U}$ with $V^{-1}(S) \in \mathcal{S}$. By (2), there exists $W \in \mathcal{U}$ with $W \subseteq V$ and $W^{-1}(S) \cap V^{-1}(S) \subseteq U(S)$. It follows that $W^{-1}(S) \subseteq U(S)$.

(3) ⇒ (2). Let $A \in \mathcal{C}_0(X)$, $S \in \mathcal{S}$, $U \in \mathcal{U}$ and $V \in \mathcal{U}$ with $V \subseteq U$ and $V(S) \in \mathcal{S}$. By (3) there exists $W \in \mathcal{U}$ with $W \subseteq V$ and such that $W^{-1}(A \cap \overline{V(S)}) \subseteq U(A \cap \overline{V(S)})$ (note that $\overline{V(S)} \in \mathcal{S}$ and hence $A \cap \overline{V(S)} \in \mathcal{S}$). Then $W^{-1}(A \cap V(S)) \subseteq U(A)$.

Let $x \in W^{-1}(A) \cap S$, then there exists $a \in A$ with $x \in W^{-1}(a)$ and hence $a \in W(x) \subseteq V(S)$. It follows that $x \in W^{-1}(A \cap V(S)) \subseteq U(A)$. Therefore $W^{-1}(A) \cap S \subseteq U(A)$. □

Corollary 3.12. *Let $\mathcal{S}$ be an ideal closed under $\mathcal{U}^{-1}$-small enlargements in a closed-symmetric quasi-uniform space $(X,\mathcal{U})$. Then $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ is precompact if and only if $\overline{S}$ is precompact for all $S \in \mathcal{S}$.*

Proof. Suppose that $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ is precompact. Let $S \in \mathcal{S}$ and $U, V \in \mathcal{U}$ with $V^2 \subseteq U$. Then $\overline{S} \in \mathcal{S}$ and by assumption we can find $W \in \mathcal{U}$ such that $W \subseteq V$ and $W^{-1}(\overline{S}) \subseteq V(\overline{S})$. Also, by the above corollary, there exists a finite subset S_0 of X such that $\overline{S} \subseteq W(S_0)$. Then $S_0 \subseteq W^{-1}(\overline{S}) \subseteq V(\overline{S})$ so there exists a finite subset S_0' of $\overline{S}$ such that $S_0 \subseteq V(S_0')$. Consequently, $\overline{S} \subseteq W(S_0) \subseteq V(V(S_0')) \subseteq U(S_0')$.

The converse follows from the above results. □

The following example shows that if the space is not closed-symmetric, the above result could fail.

Example 3.13. Let us consider two countable families $\{A_n\colon n \in \mathbb{N}\}$ and $\{B_m\colon m \in \mathbb{N}\}$ of countable disjoint sets where $A_n = \{a_k^n\colon k \in \mathbb{N}\}$ and $B_m = \{b_q^m\colon q \in \mathbb{N}\}$ for all $n, m \in \mathbb{N}$. Let $X = \bigcup_{n\in\mathbb{N}}(A_n \cup B_n)$ and endow this set with the following quasi-metric

$$\begin{cases} d(a_n^k, a_m^q) = d(a_n^k, b_m^q) = 1 & \text{if } k \leqslant q \text{ and } n \neq m, \\ d(a_n^k, a_m^q) = d(a_n^k, b_m^q) = d(b_n^k, b_m^q) = k - q & \text{if } k > q, \\ d(b_n^k, a_m^q) = d(b_n^k, b_m^q) = 1 & \text{if } k < q, \\ d(b_n^k, b_m^k) = \dfrac{1}{n} & \text{if } n \neq m, \\ d(b_n^k, a_m^q) = \dfrac{1}{n} + k - q & \text{if } k \geqslant q, \\ d(x, y) = 0 & \text{if } x = y \end{cases}$$

where $m, n, k, q \in \mathbb{N}$. Then A_1 is a closed set and for all $\varepsilon > 0$, $B_{d^{-1}}(A_1, \varepsilon) \nsubseteq B_d(A_1, 1/2) = A$ since if $1/n < \varepsilon$ then $b_n^1 \in B_{d^{-1}}(A_1, \varepsilon)$. Therefore, $(X, \mathcal{U}_d)$ is not closed-symmetric.

Now, let us define $\mathcal{S} = \{A \subseteq X\colon A \text{ only intersects finitely many } A_n'\text{s and } B_n'\text{s}\}$ which is an ideal closed under d^{-1}-small enlargements. It is easy to see that every $S \in \mathcal{S}$ is X-precompact so $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ is precompact by Theorem 3.8. However, A_n is not precompact for all $n \in \mathbb{N}$.

Corollary 3.14. *Let $(X,\mathcal{U})$ be a uniform space and $\mathcal{S}$ an ideal closed under $\mathcal{U}$-small enlargements. Then $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ is precompact if and only if $\overline{S}$ is precompact for all $S \in \mathcal{S}$.*

The following result characterizes total boundedness of $\mathcal{U}_{\mathcal{S}}$.

Theorem 3.15. *Let $\mathcal{S}$ be an ideal in a quasi-uniform space $(X,\mathcal{U})$. The following statements are equivalent*:

1. $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ *is totally bounded*;
2. $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}}^+)$ *is totally bounded*;
3. $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}}^-)$ *is totally bounded*;
4. *S is totally bounded for all $S \in \mathcal{S}$.*

Proof. (1) ⇒ (4). Suppose that $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ is totally bounded and let $S \in \mathcal{S}$ and $U \in \mathcal{U}$. Then we can find a finite number $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ of families of subsets of X such that $\mathcal{P}_0(X) = \bigcup_{i=1}^n \mathcal{A}_i$ and $\mathcal{A}_i \times \mathcal{A}_i \subseteq U_S$ for all $i \in \{1, \ldots, n\}$. Let us define $S_i = \{s \in S\colon \{s\} \in \mathcal{A}_i\}$. It is clear that $S = \bigcup_{i=1}^n S_i$ and $S_i \times S_i \subseteq U$ since given $x, y \in S_i$ then $(\{x\}, \{y\}) \in \mathcal{A}_i \times \mathcal{A}_i \subseteq U_S$ so $\{y\} \cap S = \{y\} \subseteq U(\{x\})$, i.e. $(x, y) \in U$.

The implications (2) ⇒ (4) and (3) ⇒ (4) follow similarly.

(4) ⇒ (1). Let $U \in \mathcal{U}$ and $S \in \mathcal{S}$. Since S is totally bounded we can find a finite number $S_1, \ldots, S_n$ of subsets of S such that $S = \bigcup_{i=1}^n S_i$ and $S_i \times S_i \subseteq U$. Let $\mathcal{F} = \mathcal{F}_0(\{1, \ldots, n\})$ and $\mathcal{A}_F = \{A \in \mathcal{P}_0(X)\colon A \cap S_j \neq \emptyset \Leftrightarrow j \in F\}$ for all $F \in \mathcal{F}$. Then it is straightforward to see that $\mathcal{P}_0(X) = \bigcup_{F\in\mathcal{F}} \mathcal{A}_F \cup \mathcal{P}_0(X\backslash S)$. Furthermore, given $(A, B) \in \mathcal{A}_F \times \mathcal{A}_F$ for some $F \in \mathcal{F}$ then if $b \in B \cap S$ there exists $j \in F$ such that $b \in B \cap S_j$. Since $A \in \mathcal{A}_F$ we can find $a \in A \cap S_j$. From $S_j \times S_j \subseteq U$ we obtain $(a, b) \in U$ so $b \in U(A)$ which proves $(A, B) \in U_S^+$. A similar reasoning shows $(A, B) \in U_S^-$ so $(A, B) \in U_S$.

On the other hand, if $(A, B) \in \mathcal{P}_0(X\backslash S) \times \mathcal{P}_0(X\backslash S)$ then $A \cap S = \emptyset$ and $B \cap S = \emptyset$ which trivially implies that $(A, B) \in U_S$. □

Remark 3.16. We observe that the above result is also true if we substitute $\mathcal{P}_0(X)$ for an arbitrary subset of $\mathcal{P}_0(X)$ which contains the singletons.

In the following, we study the compactness of $\mathcal{S}$-convergence beginning with the case when $\mathcal{S}$ is a bornology.

Proposition 3.17. *Let $(X,\mathcal{U})$ be a compact quasi-uniform space and $\mathcal{S}$ an E-bornology. Then $\mathcal{S}=\mathcal{P}_0(X)$.*

Proof. Since $\mathcal{S}$ is an E-bornology, for each $S\in\mathcal{S}$ there exists $U^S\in\mathcal{U}$ such that $U^S(S)\in\mathcal{S}$ and $U^S(S)$ is open. Then $X=\bigcup\{U^S(S)\colon S\in\mathcal{S}\}$, and since X is compact there exists a finite subcovering. Since $\mathcal{S}$ is an ideal, $X\in\mathcal{S}$ and hence $\mathcal{S}=\mathcal{P}_0(X)$. $\square$

Corollary 3.18. *Let $\mathcal{S}$ be an E-bornology in a quasi-uniform space $(X,\mathcal{U})$. If $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ is compact then $\mathcal{S}=\mathcal{P}_0(X)$.*

Proof. By the previous proposition, it is enough to prove that X is compact. Let $(x_\alpha)_{\alpha\in\Lambda}$ be a net in X, then $(\{x_\lambda\})_{\lambda\in\Lambda}$ is a net in $\mathcal{P}_0(X)$ so it clusters to some $A\in\mathcal{P}_0(X)$. Since $\mathcal{S}$ is a bornology, let $a\in A\cap S$ for some $S\in\mathcal{S}$. It easily follows that a is a cluster point of $(x_\lambda)_{\lambda\in\Lambda}$ in X, and hence X is compact. $\square$

The following corollary follows from the previous ones and the corresponding result for the Hausdorff quasi-uniformity [26, Corollary 2].

Corollary 3.19. *Let $\mathcal{S}$ be an E-bornology and $(X,\mathcal{U})$ a T_1 quasi-uniform space. The following statements are equivalent*:

1. *$(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ is compact*;
2. *$(X,\mathcal{U})$ is compact and $\mathcal{U}^{-1}$ is hereditarily precompact*;
3. *$(X,\mathcal{U})$ is compact, $\mathcal{U}^{-1}$ is hereditarily precompact and $\mathcal{S}=\mathcal{P}_0(X)$ (and hence $\mathcal{U}_{\mathcal{S}}$ is the Hausdorff quasi-uniformity $\mathcal{U}_H$).*

Next, we study the compactness of $\mathcal{S}$-convergence for ideals. We also note that if $S_1,S_2\in\mathcal{S}$ and $S_1\subseteq S_2$, then $U_{S_2}\subseteq U_{S_1}$ for each $U\in\mathcal{U}$.

Lemma 3.20. *Let $\mathcal{S}$ be an E-ideal in a quasi-uniform space $(X,\mathcal{U})$ and let $\mathcal{F}_0(X)\subseteq\mathcal{M}\subseteq\mathcal{P}_0(X)$. If $(\mathcal{M},\mathcal{U}_{\mathcal{S}})$ is compact then $\bar{S}$ is compact for each $S\in\mathcal{S}$.*

Proof. Let $S\in\mathcal{S}$ and suppose that S is closed. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in S. Then $(\{x_\lambda\})_{\lambda\in\Lambda}$ is a net in $\mathcal{M}$ so there exists $A\in\mathcal{M}$ such that $(\{x_\lambda\})_{\lambda\in\Lambda}$ clusters to A. Let $U\in\mathcal{U}$ such that $U^{-1}(S)\in\mathcal{S}$. For each λ_0 there exists $\lambda\geqslant\lambda_0$ such that $\{x_\lambda\}\in U_S(A)$. It follows that $x_\lambda\in U(A)\cap S$ and hence $U^{-1}(S)\cap A\neq\emptyset$. Let $a\in U^{-1}(S)\cap A$.

Let us prove that a is a cluster point of $(x_\lambda)_{\lambda\in\Lambda}$. Given $V\in\mathcal{U}$ and β_0, there exists $\beta\geqslant\beta_0$ such that $\{x_\beta\}\in V_{U^{-1}(S)}(A)$. Then $a\in A\cap U^{-1}(S)\subseteq V^{-1}(x_\beta)$, so $x_\beta\in V(a)$. Therefore a is a cluster point of $(x_\lambda)_{\lambda\in\Lambda}$ and hence S is compact. $\square$

Lemma 3.21. *Let $\mathcal{S}$ be an E-ideal in a T_1 quasi-uniform space $(X,\mathcal{U})$ and let $\mathcal{F}_0(X)\subseteq\mathcal{M}\subseteq\mathcal{P}_0(X)$. If $(\mathcal{M},\mathcal{U}_{\mathcal{S}})$ is compact then $(S,\mathcal{U}^{-1})$ is precompact for each $S\in\mathcal{S}$.*

Proof. Suppose that there exist $S\in\mathcal{S}$, $U_0\in\mathcal{U}$ and points $a_n\in S$ such that $a_{n+1}\notin U_0^{-1}(\{a_1,\dots,a_n\})$ for each $n\in\mathbb{N}$. Let $A_n=\{a_i\colon i\leqslant n\}$. Since $A_n\in\mathcal{M}$ and $\mathcal{M}$ is compact, the sequence $(A_n)_{n\in\mathbb{N}}$ clusters to some $A\in\mathcal{M}$.

Let $U\in\mathcal{U}$ with $U^2\subseteq U_0$ and $U^{-1}(S)\in\mathcal{S}$. Let $k\in\mathbb{N}$ be such that $A_k\in U_{U^{-1}(S)}(A)$. By Lemma 3.20, $\overline{U^{-1}(S)}$ is compact and hence point-symmetric, so there exists $W\in\mathcal{U}$ with $W\subseteq U$ and such that $W^{-1}(a_{k+1})\cap\overline{U^{-1}(S)}\subseteq U(a_{k+1})$. Since $W^{-1}(a_{k+1})\subseteq U^{-1}(S)$, then $W^{-1}(a_{k+1})\subseteq U(a_{k+1})$. Let $n\geqslant k+1$ be such that $A_n\in W_{U^{-1}(S)}(A)$. Then $a_{k+1}\in A_n\cap S\subseteq W(A)$, so there exists $a\in A$ such that $a_{k+1}\in W(a)$. Then $a\in W^{-1}(a_{k+1})\subseteq U(a_{k+1})$, that is, $a_{k+1}\in U^{-1}(a)$. On the other hand, $a\in A\cap W^{-1}(a_{k+1})\subseteq A\cap U^{-1}(S)\subseteq U^{-1}(A_k)$. It follows that $a_{k+1}\in U^{-2}(A_k)\subseteq U_0^{-1}(A_k)$, a contradiction. Therefore $(S,\mathcal{U}^{-1})$ is precompact for each $S\in\mathcal{S}$. $\square$

To prove the following results, we will make use of the following concepts.

Definition 3.22. ([40,44]) A net $(x_\lambda)_{\lambda\in\Lambda}$ is said to be *left K-Cauchy* if for each $U\in\mathcal{U}$ there exists $\lambda_0\in\Lambda$ such that $x_{\lambda_2}\in U(x_{\lambda_1})$ whenever $\lambda_2\geqslant\lambda_1\geqslant\lambda$.

The quasi-uniformity $\mathcal{U}$ is called *left K-complete* provided that each left K-Cauchy net converges.

Lemma 3.23. *Let $\mathcal{S}$ be an E-ideal which is not a bornology in a quasi-uniform space $(X,\mathcal{U})$. Suppose that $\bar{S}$ is compact and $(S,\mathcal{U}^{-1})$ is precompact for each $S\in\mathcal{S}$. Then $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ is compact.*

Proof. Recall that a quasi-uniform space is compact if and only if it is precompact and left K-complete [25, Remark 2.6.16].

First, note that $(\mathcal{P}_0(X), \mathcal{U}_\mathcal{S})$ is precompact by Theorem 3.8.

Let $(A_\lambda)_{\lambda\in\Lambda}$ be a left K-Cauchy net in $\mathcal{P}_0(X)$. We consider two cases:

1. For each closed $S \in \mathcal{S}$ and λ_0 there exists $\lambda \geqslant \lambda_0$ with $A_\lambda \cap S = \emptyset$.
   Since $\mathcal{S}$ is not a bornology, there exists $x \notin \bigcup \mathcal{S}$. Let $C = \{x\}$, then $(A_\lambda)_{\lambda\in\Lambda}$ $\mathcal{S}$-converges to C. Indeed, let $U \in \mathcal{U}$, $S \in \mathcal{S}$, λ_0 and $\lambda \geqslant \lambda_0$ such that $A_\lambda \cap S = \emptyset$. Since $C \cap S = \emptyset$, it follows that $A_\lambda \in U_S(C)$. Since $(A_\lambda)_{\lambda\in\Lambda}$ is left K-Cauchy, let λ_0 be such that $A_{\lambda_2} \in U_S(A_{\lambda_1})$ for $\lambda_2 \geqslant \lambda_1 \geqslant \lambda_0$. Let $\lambda \geqslant \lambda_0$ with $A_\lambda \in U_S(C)$. If $\beta \geqslant \lambda$, $A_\beta \in U_S(A_\lambda) \subseteq U_S^2(C)$. It follows that $(A_\lambda)_{\lambda\in\Lambda}$ converges to C.
2. $\mathcal{S}_1 \neq \emptyset$, with $\mathcal{S}_1 = \{S \in \mathcal{S}\colon\ S$ is closed and there exists λ_0 with $A_\lambda \cap S \neq \emptyset$ for each $\lambda \geqslant \lambda_0\}$.

Claim. Let $S \in \mathcal{S}$ be such that for each λ_0 there exists $\lambda \geqslant \lambda_0$ with $A_\lambda \cap S \neq \emptyset$. Then there exists $S' \in \mathcal{S}_1$ such that $S \subseteq S'$.

In order to prove that claim, let $S \in \mathcal{S}$ be such that for each λ_0 there exists $\lambda \geqslant \lambda_0$ with $A_\lambda \cap S \neq \emptyset$. Let $U \in \mathcal{U}$ with $U^{-1}(S) \in \mathcal{S}$. Since $(A_\lambda)_{\lambda\in\Lambda}$ is left K-Cauchy, there exists λ_0 such that $A_{\lambda_2} \in U_S(A_{\lambda_1})$ for each $\lambda_2 \geqslant \lambda_1 \geqslant \lambda_0$. Let $\lambda \geqslant \lambda_0$ and let $\lambda_1 \geqslant \lambda$ with $A_{\lambda_1} \cap S \neq \emptyset$, then $A_{\lambda_1} \cap S \subseteq U(A_\lambda)$ and hence $A_\lambda \cap U^{-1}(S) \neq \emptyset$ and this proves the claim with $S' = U^{-1}(S)$.

Now, let $S \in \mathcal{S}_1$, then $(A_\lambda \cap S)_{\lambda\in\Lambda}$ is a net in the compact quasi-uniform space $(\mathcal{P}_0(S), \mathcal{U}_H)$ (note that it is compact by Corollary 3.19). Let $C_S \in \mathcal{P}_0(S)$ be its cluster point. Let $C = \bigcup_{S\in\mathcal{S}_1} C_S$, and let us prove that C is a cluster point of $(A_\lambda)_{\lambda\in\Lambda}$.

Let $U \in \mathcal{U}$, $S \in \mathcal{S}$ and $\lambda_0 \in \Lambda$. We consider two cases:

(a) $S \notin \mathcal{S}_1$.
  (i) $C \cap S = \emptyset$. Since $S \notin \mathcal{S}_1$, then $A_\lambda \cap S = \emptyset$ cofinally so $A_\lambda \in U_S(C)$ cofinally.
  (ii) $C \cap S \neq \emptyset$. Let $x \in C \cap S$, and let $S_1 \in \mathcal{S}_1$ with $x \in C_{S_1} \cap S$. Let $V \in \mathcal{U}$ be such that $V(S) \in \mathcal{S}$. Given λ_1, since C_{S_1} is a $\tau(\mathcal{U}_H)$-cluster point of $(A_\lambda \cap S_1)_{\lambda\in\Lambda}$, there exists $\lambda \geqslant \lambda_1$ such that $x \in C_{S_1} \subseteq V^{-1}(A_\lambda \cap S_1)$. Since $x \in S$, $A_\lambda \cap V(S) \neq \emptyset$. By the claim, there exists $S' \in \mathcal{S}_1$ with $S \subseteq S'$.

(b) $S \in \mathcal{S}_1$ (if $S \notin \mathcal{S}_1$ and $C \cap S \neq \emptyset$, we work with any $S' \in \mathcal{S}_1$ containing S instead of S).
  Let $x \in C \cap S$ and $V \in \mathcal{U}$ with $V^3 \subseteq U$. Since $C \cap S \in \mathcal{S}$, it is precompact with respect to $\mathcal{U}^{-1}$, so there exist $c_1, \dots, c_n \in C \cap S$ such that $C \cap S \subseteq V^{-1}(\{c_1, \dots, c_n\})$. Let $S_i \in \mathcal{S}_1$ with $c_i \in C_{S_i}$ and let $S' = \bigcup_{i=1}^n S_i$. Since $(A_\lambda)_{\lambda\in\Lambda}$ is left K-Cauchy, let $\beta_0 \geqslant \lambda_0$ be such that $A_{\beta_2} \in V_{S'}(A_{\beta_1})$ for each $\beta_2 \geqslant \beta_1 \geqslant \beta_0$.
  Since C_{S_i} is a cluster point of $(A_\lambda \cap S_i)_{\lambda\in\Lambda}$, for each $i \in \{1, \dots, n\}$ there exists $\lambda_i \geqslant \beta_0$ with $C_{S_i} \subseteq V^{-1}(A_{\lambda_i} \cap S_i)$. Let $\beta_1 \geqslant \lambda_i$ for $i \in \{1, \dots, n\}$. If $\beta \geqslant \beta_1$, $C \cap S \subseteq V^{-1}(\{c_1, \dots, c_n\}) \subseteq \bigcup_{i=1}^n V^{-1}(C_{S_i}) \subseteq \bigcup_{i=1}^n V^{-2}(A_{\lambda_i} \cap S_i) \subseteq V^{-3}(A_\beta) \subseteq U^{-1}(A_\beta)$.
  On the other hand, since C_S is a cluster point of $(A_\lambda \cap S)_{\lambda\in\Lambda}$, there exists $\lambda \geqslant \beta_1$ with $A_\lambda \cap S \subseteq V(C_S) \subseteq V(C)$. It follows that $A_\lambda \in U_S(C)$.
  We conclude that C is a cluster point of $(A_\lambda)_{\lambda\in\Lambda}$ and hence a limit point (a cluster point of a left K-Cauchy net is a limit point). Therefore $\mathcal{P}_0(X)$ is left K-complete, and since it is precompact, it is compact. □

Theorem 3.24. *Let $\mathcal{S}$ be an E-ideal in a T_1 quasi-uniform space $(X, \mathcal{U})$. Then:*

- *If $\mathcal{S}$ is not a bornology: $(\mathcal{P}_0(X), \mathcal{U}_\mathcal{S})$ is compact if and only if $\overline{S}$ is compact and $(S, \mathcal{U}^{-1})$ is precompact for each $S \in \mathcal{S}$.*
- *If $\mathcal{S}$ is a bornology: $(\mathcal{P}_0(X), \mathcal{U}_\mathcal{S})$ is compact if and only if $(X, \mathcal{U})$ is compact, $\mathcal{U}^{-1}$ is hereditarily precompact and $\mathcal{S} = \mathcal{P}_0(X)$ (and hence $\mathcal{U}_\mathcal{S}$ is the Hausdorff quasi-uniformity $\mathcal{U}_H$).*

Proof. It follows from the previous results. □

Corollary 3.25. *Let $\mathcal{S}$ be an E-ideal in a Hausdorff uniform space $(X, \mathcal{U})$. Then $(\mathcal{P}_0(X), \mathcal{U}_\mathcal{S})$ is compact if and only if $\overline{S}$ is compact for each $S \in \mathcal{S}$.*

Note that if $\mathcal{S}$ is a bornology the latter condition is equivalent to X being compact (and $\mathcal{U}_\mathcal{S}$ being the Hausdorff uniformity $\mathcal{U}_H$).

Now, we look for a characterization of the compactness of $(\mathcal{K}_0(X), \mathcal{U}_\mathcal{S})$.

Corollary 3.26. *Let $\mathcal{S}$ be an E-ideal in a T_1 quasi-uniform space $(X, \mathcal{U})$. If $(\mathcal{K}_0(X), \mathcal{U}_\mathcal{S})$ is compact then $(\mathcal{P}_0(X), \mathcal{U}_\mathcal{S})$ is compact.*

Proof. It follows from Lemmas 3.20 and 3.21 and Theorem 3.24. □

Lemma 3.27. *Let $(X, \mathcal{U})$ be a quasi-uniform space, and let K be a compact subspace of $(X, \mathcal{U})$. Then $\mathrm{cl}_{\tau(\mathcal{U}^{-1})}(K)$ is compact in $(X, \mathcal{U})$.*

Proof. Let $\{O_i\colon\ i \in I\}$ be an open covering of $\mathrm{cl}_{\tau(\mathcal{U}^{-1})}(K)$. Since K is compact, there exist $\{O_{i_1}, \dots, O_{i_n}\}$ with $K \subseteq \bigcup_{k=1}^n O_{i_k}$. Since K is compact, there exists $U \in \mathcal{U}$ with $K \subseteq U(K) \subseteq \bigcup_{k=1}^n O_{i_k}$, and hence, since $\mathrm{cl}_{\tau(\mathcal{U}^{-1})}(K) \subseteq U(K)$, $\{O_{i_1}, \dots, O_{i_n}\}$ is a finite subcovering of $\mathrm{cl}_{\tau(\mathcal{U}^{-1})}(K)$. □

Proposition 3.28. *Let $\mathcal{S}$ be an E-ideal and $(X,\mathcal{U})$ a quasi-uniform space. If $(\mathcal{K}_0(X),\mathcal{U}_\mathcal{S})$ is compact then there exists a compact subspace K of X such that $\bigcup\mathcal{S}\subseteq K$.*

Proof. By Lemma 3.20, $\overline{S}$ is compact for each $S\in\mathcal{S}$, and hence $(\overline{S})_{S\in\mathcal{S}}$ is a net in $(\mathcal{K}_0(X),\mathcal{U}_\mathcal{S})$. Since $(\mathcal{K}_0(X),\mathcal{U}_\mathcal{S})$ is compact, the net has a cluster point $K\in\mathcal{K}_0(X)$. For each $U\in\mathcal{U}$ and $S_0\in\mathcal{S}$ there exists $S\in\mathcal{S}$ with $S\supseteq S_0$ and such that $\overline{S}\in U_{S_0}(K)$. It follows that $S_0=\overline{S}\cap S_0\subseteq U(K)$ and hence $\bigcup\mathcal{S}\subseteq U(K)$ for each $U\in\mathcal{U}$, so $\bigcup\mathcal{S}\subseteq \mathrm{cl}_{\tau(\mathcal{U}^{-1})}(K)$. This completes the proof, since $\mathrm{cl}_{\tau(\mathcal{U}^{-1})}(K)$ is compact by Lemma 3.27. □

Proposition 3.29. *Let $\mathcal{S}$ be an E-ideal in a quasi-uniform space $(X,\mathcal{U})$. If $(\mathcal{P}_0(X),\mathcal{U}_\mathcal{S})$ is compact and there exists $K\in\mathcal{K}_0(X)$ with $\bigcup\mathcal{S}\subseteq K$ then $(\mathcal{K}_0(X),\mathcal{U}_\mathcal{S})$ is compact.*

Proof. Let $(K_\lambda)_{\lambda\in\Lambda}$ be a net in $(\mathcal{K}_0(X),\mathcal{U}_\mathcal{S})$. Since $(\mathcal{P}_0(X),\mathcal{U}_\mathcal{S})$ is compact, let $A\in\mathcal{P}_0(X)$ be a cluster point of the net. It easily follows that $\overline{A}$ is also a cluster point of the net, so we can assume that A is closed.

Let us prove that $A\cap K$ is a cluster point of $(K_\lambda)_{\lambda\in\Lambda}$ in $(\mathcal{K}_0(X),\mathcal{U}_\mathcal{S})$.

First, note that $A\cap K$ is a closed subset of K, so it is compact. Let $U_0\in\mathcal{U}$, $S\in\mathcal{S}$, $\lambda_0\in\Lambda$ and $U\in\mathcal{U}$ with $U\subseteq U_0$ and $U^{-1}(S)\in\mathcal{S}$. Then there exists $\lambda\geqslant\lambda_0$ such that $K_\lambda\in U_S(A)$, that is, $K_\lambda\cap S\subseteq U(A)$ and $A\cap S\subseteq U^{-1}(K_\lambda)$. It is clear that $A\cap K\cap S\subseteq U^{-1}(K_\lambda)$.

In order to prove that $K_\lambda\cap S\subseteq U(A\cap K)$, let $x\in K_\lambda\cap S$. There exists $a\in A$ with $x\in U(a)$. Then $a\in U^{-1}(x)\subseteq U^{-1}(S)\subseteq \bigcup\mathcal{S}\subseteq K$, so $x\in U(A\cap K)$. Therefore $K_\lambda\cap S\subseteq U(A\cap K)$ and hence $A\cap K$ is a cluster point of $(K_\lambda)_{\lambda\in\Lambda}$.

Finally, note that if $A\cap K=\emptyset$, by the previous reasoning it follows that $K_\lambda\cap S=\emptyset$. Then we can take $a\in A$ and $\{a\}$ is a cluster point of $(K_\lambda)_{\lambda\in\Lambda}$. □

Theorem 3.30. *Let $\mathcal{S}$ be an E-ideal in a T_1 quasi-uniform space $(X,\mathcal{U})$. Then $(\mathcal{K}_0(X),\mathcal{U}_\mathcal{S})$ is compact if and only if $(\mathcal{P}_0(X),\mathcal{U}_\mathcal{S})$ is compact and there exists $K\in\mathcal{K}_0(X)$ with $\bigcup\mathcal{S}\subseteq K$.*

Corollary 3.31. *Let $\mathcal{S}$ be an E-ideal in a Hausdorff quasi-uniform space $(X,\mathcal{U})$. The following statements are equivalent*:

1. *$(\mathcal{K}_0(X),\mathcal{U}_\mathcal{S})$ is compact*;
2. *$(\mathcal{P}_0(X),\mathcal{U}_\mathcal{S})$ is compact and $\overline{\bigcup\mathcal{S}}$ is compact*;
3. *$\overline{\bigcup\mathcal{S}}$ is compact and $(S,\mathcal{U}^{-1})$ is precompact for each $S\in\mathcal{S}$.*

Corollary 3.32. *Let $\mathcal{S}$ be an E-ideal in a Hausdorff uniform space $(X,\mathcal{U})$. Then $(\mathcal{K}_0(X),\mathcal{U}_\mathcal{S})$ is compact if and only if $\overline{\bigcup\mathcal{S}}$ is compact.*

The proof of the following result is straightforward (note that $A\in U_S(\overline{A})$ for each $U\in\mathcal{U}$ and $S\in\mathcal{S}$).

Proposition 3.33. *Let $\mathcal{S}$ be an E-ideal in a quasi-uniform space $(X,\mathcal{U})$. Then $(\mathcal{C}_0(X),\mathcal{U}_\mathcal{S})$ is compact if and only if so is $(\mathcal{P}_0(X),\mathcal{U}_\mathcal{S})$.*

4. Right K-completeness of bornological convergences

In this section, we study a certain notion of completeness for the quasi-uniformity compatible with a bornological convergence. For quasi-uniform spaces, there exist many notions for completeness [25]. It has been proved [28,27] that the notion which has a good behavior for hyperspaces is right K-completeness.

Definition 4.1. ([40,44]) A net $(x_\lambda)_{\lambda\in\Lambda}$ is said to be *right K-Cauchy* if for each $U\in\mathcal{U}$ there exists $\lambda_0\in\Lambda$ such that $x_{\lambda_1}\in U(x_{\lambda_2})$ whenever $\lambda_2\geqslant\lambda_1\geqslant\lambda$.

The quasi-uniformity $\mathcal{U}$ is called *right K-complete* provided that each right K-Cauchy net converges.

This concept allows to obtain an elegant extension of the characterization due to Burdick [15] of those uniform spaces which have a complete Hausdorff uniformity to the quasi-uniform setting [28] (see also [7] for a characterization of cofinal completeness of the Hausdorff metric). Here, we obtain a similar characterization of the quasi-uniformity $\mathcal{U}_\mathcal{S}$ associated with an E-ideal $\mathcal{S}$.

We also recall some other concepts that will be useful.

Definition 4.2. Let $(X,\mathcal{U})$ be a quasi-uniform space.

- A net $(x_\lambda)_{\lambda\in\Lambda}$ on $(X,\mathcal{U})$ is said to be $\mathcal{U}^*$-Cauchy if for each $U\in\mathcal{U}$ there exists $\lambda_0\in\Lambda$ such that $x_{\lambda_1}\in U^*(x_{\lambda_2})$ for all $\lambda_2,\lambda_1\geqslant\lambda_0$;
- $(X,\mathcal{U})$ is said to be *half complete* if every $\mathcal{U}^*$-Cauchy net converges in $(X,\mathcal{U})$.

Lemma 4.3. *Let $\mathcal{S}$ be an E-bornology with $X \notin \mathcal{S}$ in a quasi-uniform space $(X,\mathcal{U})$. Then there exists a $\mathcal{U}_{\mathcal{S}}^*$-Cauchy net in $\mathcal{F}_0(X)$ without a cluster point in $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$.*

Proof. For each $S \in \mathcal{S}$, let $x_S \in X \setminus S$. If $S \subseteq S_1$ then $x_{S_1} \in U_S^*(x_S)$. It follows that $(\{x_S\})_{S\in\mathcal{S}}$ is a $\mathcal{U}_{\mathcal{S}}^*$-Cauchy net in $\mathcal{F}_0(X)$.

Suppose that $A \in \mathcal{P}_0(X)$ is a cluster point of $(\{x_S\})_{S\in\mathcal{S}}$, and let $a \in A$. Since $\mathcal{S}$ is a bornology, there exists $S \in \mathcal{S}$ with $a \in S$. Let $U \in \mathcal{U}$ with $U(S) \in \mathcal{S}$, then there exists $S_0 \supseteq U(S)$ with $x_{S_0} \in U_S(A)$. Therefore $a \in A \cap S \subseteq U^{-1}(x_{S_0})$ and hence $x_{S_0} \in U(a) \subseteq U(S) \subseteq S_0$, a contradiction. □

Proposition 4.4. *Let $\mathcal{S}$ be an E-bornology in a quasi-uniform space $(X,\mathcal{U})$ and $\mathcal{F}_0(X) \subseteq \mathcal{M} \subseteq \mathcal{P}_0(X)$. If $(\mathcal{M},\mathcal{U}_{\mathcal{S}})$ is half complete then $\mathcal{S} = \mathcal{P}_0(X)$.*

Corollary 4.5. *Let $\mathcal{S}$ be an E-bornology in a quasi-uniform space $(X,\mathcal{U})$ and $\mathcal{F}_0(X) \subseteq \mathcal{M} \subseteq \mathcal{P}_0(X)$. If $(\mathcal{M},\mathcal{U}_{\mathcal{S}})$ is complete then $\mathcal{S} = \mathcal{P}_0(X)$.*

Corollary 4.6. *Let $\mathcal{S}$ be an E-bornology in a quasi-uniform space $(X,\mathcal{U})$ and $\mathcal{F}_0(X) \subseteq \mathcal{M} \subseteq \mathcal{P}_0(X)$. If $(\mathcal{M},\mathcal{U}_{\mathcal{S}})$ is compact then $\mathcal{S} = \mathcal{P}_0(X)$.*

Definition 4.7. We say that a filter $\mathcal{F}$ is *stable in an ideal* $\mathcal{S}$ of a quasi-uniform space $(X,\mathcal{U})$ if:

- there exists $S' \in \mathcal{S}$ with $S' \cap F \neq \emptyset$ for each $F \in \mathcal{F}$, and
- for each $U \in \mathcal{U}$ and $S \in \mathcal{S}$ there exists $F_0 \in \mathcal{F}$ such that $F_0 \cap S \subseteq U(F)$ for each $F \in \mathcal{F}$.

The following two results and their proofs are based on [28, Lemma 6 and Proposition 6].

Lemma 4.8. *Suppose that $(X,\mathcal{U})$ is a quasi-uniform space in which each stable filter in $\mathcal{S}$ has a cluster point. Let $\mathcal{F}$ be a stable filter in $\mathcal{S}$ and C its set of cluster points. Then for each $U \in \mathcal{U}$ and $S \in \mathcal{S}$ there exists $F \in \mathcal{F}$ with $F \cap S \subseteq U(C)$.*

Proof. Suppose that there exist $U_0 \in \mathcal{U}$ and $S \in \mathcal{S}$ such that $E \cap S \setminus U_0^2(C) \neq \emptyset$ for each $E \in \mathcal{F}$. In particular, note that $E \cap S \neq \emptyset$ for each $E \in \mathcal{F}$.

Let $H_{UE} = \{a \in X\colon$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$, $V^{-2}(a) \cap U_0(C) = \emptyset$ and $a \in E \cap \bigcap_{F\in\mathcal{F}} V(F)\}$ for each $E \in \mathcal{F}$ and $U \in \mathcal{U}$.

First note that $H_{UE} \neq \emptyset$. To check this, let $V \in \mathcal{U}$ with $V^2 \subseteq U \cap U_0$. Since $\mathcal{F}$ is stable in $\mathcal{S}$, there exists $F_0 \in \mathcal{F}$ with $F_0 \cap S \subseteq \bigcap_{F\in\mathcal{F}} V(F)$. Then $F_V = F_0 \cap E \in \mathcal{F}$, so there exists $a \in S \cap F_V \setminus U_0^2(C)$. It follows that $a \in H_{UE} \cap S$.

On the other hand, it is clear that $H_{U_1E_1} \subseteq H_{U_2E_2}$ whenever $U_1, U_2 \in \mathcal{U}$ with $U_1 \subseteq U_2$ and $E_1, E_2 \in \mathcal{F}$ with $E_1 \subseteq E_2$.

Thus $\{H_{UE}\colon U \in \mathcal{U},\ E \in \mathcal{F}\}$ is a base for a filter $\mathcal{H}$ on X. Let us prove that $\mathcal{H}$ is stable in $\mathcal{S}$. First, note that we have already proved that $H_{UE} \cap S \neq \emptyset$ for each $U \in \mathcal{U}$ and $E \in \mathcal{F}$. Let $U, V \in \mathcal{U}$, $E \in \mathcal{F}$ and $S' \in \mathcal{S}$.

Let us prove that $H_{UX} \cap S' \subseteq U(H_{VE})$. Let $a \in H_{UX} \cap S'$, then there is $W \in \mathcal{U}$ such that $W^{-1}(S') \in \mathcal{S}$, $W^2 \subseteq U$, $W^{-2}(a) \cap U_0(C) = \emptyset$ and $a \in \bigcap_{F\in\mathcal{F}} W(F)$. Let $Z \in \mathcal{U}$ with $Z^2 \subseteq V \cap W$. Since $\mathcal{F}$ is stable in $\mathcal{S}$, there exists $F_0 \in \mathcal{F}$ with $F_0 \cap W^{-1}(S') \subseteq \bigcap_{F\in\mathcal{F}} Z(F)$. Define $F_Z = F_0 \cap E \in \mathcal{F}$. Since $a \in W(F_Z)$, there exists $y \in F_Z \cap W^{-1}(a)$. It follows that $Z^{-2}(y) \subseteq W^{-2}(a)$ and hence $Z^{-2}(y) \cap U_0(C) = \emptyset$. Finally $y \in F_Z \cap W^{-1}(a) \subseteq F_0 \cap W^{-1}(S') \subseteq \bigcap_{F\in\mathcal{F}} Z(F)$, and hence $a \in W(y) \subseteq U(y)$ and $y \in H_{VE}$, so $a \in U(H_{VE})$.

Therefore $\mathcal{H}$ is stable in $\mathcal{S}$, so, by hypothesis, it has a cluster point $x \in X$. Since $H_{UF} \subseteq F$ for each $U \in \mathcal{U}$ and $F \in \mathcal{F}$, then $\mathcal{F} \subseteq \mathcal{H}$ and $x \in C$. But this is a contradiction, since $H_{UE} \cap U_0(C) = \emptyset$ for each $U \in \mathcal{U}$ and $E \in \mathcal{F}$. □

Theorem 4.9. *Let $\mathcal{S}$ be an E-ideal in a quasi-uniform space $(X,\mathcal{U})$. Then $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ is right K-complete if and only if any stable filter in $\mathcal{S}$ has a cluster point in $(X,\mathcal{U})$ and ($\mathcal{S}$ is not a bornology or $X \in \mathcal{S}$).*

Proof. If $\mathcal{S}$ is a bornology and $X \notin \mathcal{S}$, then $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ is not half complete by Corollary 4.5. If $X \in \mathcal{S}$, then $\mathcal{U}_{\mathcal{S}}$ is the Hausdorff quasi-uniformity $\mathcal{U}_H$ and the result follows from [28, Proposition 6] (note that if $X \in \mathcal{S}$, a filter is stable in $\mathcal{S}$ if and only if it is stable).

So we can assume that $\mathcal{S}$ is not a bornology.

Suppose that $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$ is right K-complete, and let $\mathcal{F}$ be a stable filter in $\mathcal{S}$. It easily follows that $(F)_{F\in\mathcal{F}}$ is a right K-Cauchy net in $(\mathcal{P}_0(X),\mathcal{U}_{\mathcal{S}})$, so it $\mathcal{S}$-converges to some $C \in \mathcal{P}_0(X)$.

Let $S \in \mathcal{S}$ with $S \cap F \neq \emptyset$ for each $F \in \mathcal{F}$, and let $V \in \mathcal{U}$ with $V^{-1}(S) \in \mathcal{S}$. Then there exists $F_0 \in \mathcal{F}$ such that $F \in V_S(C)$ for each $F \subseteq F_0$, so $F \cap S \subseteq V(C)$. Since $F \cap S \neq \emptyset$, it follows that $C \cap V^{-1}(S) \neq \emptyset$. Choose $x \in C \cap V^{-1}(S)$.

Now we will prove that x is a cluster point of $\mathcal{F}$. Let $U \in \mathcal{U}$ and $W = U \cap V$. Then there exists $F_1 \in \mathcal{F}$ such that $F \in W_{V^{-1}(S)}(C)$ for each $F \subseteq F_1$. Hence $x \in C \cap V^{-1}(S) \subseteq W^{-1}(F) \subseteq U^{-1}(F)$ for each $F \subseteq F_1$. Therefore x is a cluster point of $\mathcal{F}$.

Conversely, suppose that any stable filter in $\mathcal{S}$ has a cluster point in $(X,\mathcal{U})$, and let $(A_\lambda)_{\lambda\in\Lambda}$ be a right K-Cauchy net in $\mathcal{P}_0(X)$.

For each $\lambda \in \Lambda$, let $F_\lambda = \bigcup_{\beta \geqslant \lambda} A_\beta$ and define $\mathcal{F}$ as the filter generated by the filter base $\{F_\lambda\colon\ \lambda \in \Lambda\}$. Now we consider two cases:

1. For each $S \in \mathcal{S}$ there exists λ_0 such that $A_\lambda \cap S = \emptyset$ for each $\lambda \geqslant \lambda_0$.
   Since $\mathcal{S}$ is not a bornology, we can take $x \notin \bigcup \mathcal{S}$. It easily follows that $(A_\lambda)_{\lambda\in\Lambda}$ converges to $\{x\}$.
2. There exists $S_0 \in \mathcal{S}$ such that for each λ_0 there exists $\lambda \geqslant \lambda_0$ with $A_\lambda \cap S_0 \neq \emptyset$.
   Let us prove that $\mathcal{F}$ is stable in $\mathcal{S}$. It follows that $F_\lambda \cap S_0 \neq \emptyset$ for each $\lambda \in \Lambda$, and hence $F \cap S_0 \neq \emptyset$ for each $F \in \mathcal{F}$.
   Let $U \in \mathcal{U}$ and $S \in \mathcal{S}$, then there exists λ_0 such that $A_{\lambda_1} \in U_S(A_\lambda)$ for each $\lambda \geqslant \lambda_1 \geqslant \lambda_0$. Then $A_{\lambda_1} \cap S \subseteq U(A_\lambda)$ for each $\lambda \geqslant \lambda_1 \geqslant \lambda_0$. It follows that $F_{\lambda_0} \cap S \subseteq U(F_\beta)$ for each $\beta \in \Lambda$, and hence $\mathcal{F}$ is stable in $\mathcal{S}$.
   By hypothesis $\mathcal{F}$ has a cluster point $x \in X$. Let $C \in \mathcal{P}_0(X)$ be the set of cluster points of $\mathcal{F}$ and let us prove that C is a cluster point of the net $(A_\lambda)_{\lambda\in\Lambda}$.
   Let $U, W \in \mathcal{U}$ and $S \in \mathcal{S}$ such that $W^2 \subseteq U$ and $W(S) \in \mathcal{S}$. There exists λ_0 such that $A_{\lambda_1} \in W_{W(S)}(A_{\lambda_2})$ for each $\lambda_2 \geqslant \lambda_1 \geqslant \lambda_0$. We prove that $S \cap C \subseteq U^{-1}(A_\lambda)$ for each $\lambda \geqslant \lambda_0$. Let $x \in S \cap C$ and $\lambda \geqslant \lambda_0$. Then $x \in W^{-1}(F_\lambda)$. Let $a \in F_\lambda$ with $x \in W^{-1}(a)$, then $a \in A_\beta$ for some $\beta \geqslant \lambda$. It follows that $a \in A_\beta \cap W(x) \subseteq A_\beta \cap W(S) \subseteq W^{-1}(A_\lambda)$ and hence $x \in W^{-2}(A_\lambda) \subseteq U^{-1}(A_\lambda)$. Therefore $S \cap C \subseteq U^{-1}(A_\lambda)$ for each $\lambda \geqslant \lambda_0$.
   On the other hand, by Lemma 4.8 there exists λ with $F_\lambda \cap S \subseteq U(C)$, and hence $A_\beta \cap S \subseteq U(C)$ for each $\beta \geqslant \lambda$.
   We conclude that $(A_\lambda)_{\lambda\in\Lambda}$ converges to C. □

Corollary 4.10. *Let $\mathcal{S}$ be an E-ideal in a uniform space $(X, \mathcal{U})$. Then $(\mathcal{P}_0(X), \mathcal{U}_{\mathcal{S}})$ is complete if and only if any stable filter in $\mathcal{S}$ has a cluster point in $(X, \mathcal{U})$ and ($\mathcal{S}$ is not a bornology or $X \in \mathcal{S}$).*

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