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Discrete Applied Mathematics 59 (1995) 103–114

DISCRETE APPLIED MATHEMATICS

Large (d, D, D', s)-bipartite digraphs

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Received 18 February 1992; revised 26 May 1993

Abstract

A (d, D, D', s)-digraph is a directed graph with diameter D and maximum out-degree d such that after the deletion of any s of its vertices the resulting digraph has diameter at most D'. Our concern is to find large, i.e. with order as large as possible, (d, D, D', s)-bipartite digraphs. To this end, it is proved that some members of a known family of large bipartite digraphs satisfy a Menger-type condition. Namely, between any pair of non-adjacent vertices they have $s + 1$ internally disjoint paths of length at most D'. Then, a new family of (d, D, D', s)-bipartite digraphs with order very close to the upper bound is obtained.

1. Introduction

Interconnection networks are usually modeled by graphs, directed or not, in which the vertices represent the switching elements or processors. Communication links are represented by edges if they are bidirectional and by arcs if they are unidirectional. We are concerned here with directed graphs only, called digraphs for short. A *digraph* $G = (V, A)$ consists of a set V of *vertices* and a set A of ordered pairs of vertices called *arcs*. The cardinality of V is called the *order* of the digraph. The set of vertices which are adjacent from (to) a given vertex v is denoted by $\Gamma^+(v)$ $(\Gamma^-(v))$ and its cardinality is the *out-degree* $d^+(v) = |\Gamma^+(v))|$ (*in-degree* $d^-(v) = |\Gamma^-(v)|$). The length of a shortest path from u to v is the *distance from u to v* and is denoted by $d(u, v)$. Its maximum value over all pairs of vertices is the *diameter* of the digraph. The reader is referred to Chartrand and Lesniak [6] for additional graph concepts.

In the design of large interconnection networks several factors have to be taken into account: each processor can be connected just to a few others, communication delays between processors must be short. These requirements lead to the following optimization problems: find digraphs of given maximum out-degree d and diameter D which have large order (the (d, D)-digraph problem) and find digraphs with given order and maximum out-degree which have small diameter. These problems have been widely

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SSDI 0166-218X(93)E0165-U

studied, for graphs (see [2]) as well as for digraphs (see [8]). The case of bipartite graphs (see [4]) and digraphs (see [7]) have been also considered.

An interconnection network must be fault-tolerant. If some processors or communication links cease to function, it is important that the remaining processors can still intercommunicate with reasonable efficiency. One can demand, for example, that the message delay does not increase too much. This means that the (di)graph obtained after deletion of some vertices or edges (arcs) still has a small diameter.

The problem we study in this paper is the (d, D, D', s)-digraph problem, that is, to find large digraphs with maximum out-degree d and diameter D such that the resulting digraph after the deletion of s vertices has diameter at most D'. This problem has been studied in [10] in the case $D' = D = 2$ and in [11] for $D' = 3$. The analogous problem for graphs has been considered in [5, 9, 12]. This paper concentrates upon the case of bipartite digraphs. In Section 2 we present an upper bound for the order of a (d, D, D', s)-bipartite digraph. We recall some facts of the line digraph method in Section 3. A theorem which relates the line digraph method and the (d, D, D', s)-digraph problem is presented. Most of the large (d, D, D', s)-bipartite digraphs we have found are members of a family of bipartite digraphs, called $BD(d, n)$, constructed in [7]. In Section 4 we recall some properties of these digraphs. Finally, in Sections 5–7 large (d, D, D', s)-bipartite digraphs with order very close to the bound (optimal in some cases) and $3 \leqslant D' \leqslant 6$ are given.

Computer explorations have been made in order to find more digraphs $BD(d, n)$ which are large (d, D, D', s)-bipartite digraphs. If $D' \geqslant 7$, it seems that there are no good (d, D, D', s)-bipartite digraphs in this family.

2. A bound for the order of (d, D, D', s)-bipartite digraphs

A digraph with maximum out-degree d and diameter D is called a *(d, D)-digraph*. A *(d, D, D', s)-digraph* is a (d, D)-digraph such that the subdigraphs obtained by deleting any set of s vertices have diameter less or equal than D'.

If a (d, D)-digraph verifies that between any pair of non-adjacent vertices there are $s + 1$ internally disjoint paths of length at most D' $(D' \geqslant D \geqslant 2)$, then it is a (d, D, D', s)-digraph. The existence of $s + 1$ disjoint paths is not necessary, but, of course, there must be at least $s + 1$ paths of length at most D' (disjoint or not) between any pair of non-adjacent vertices of a (d, D)-digraph to be a (d, D, D', s)-digraph.

A Moore-like bound for the number of vertices of a (d, D, D', s)-digraph is given in [10]:

$$M(d, D, D', s) = \min\left\{\frac{d^{D+1} - 1}{d - 1}, 1 + d + \left\lfloor \frac{d^2 + d^3 + \cdots + d^{D'}}{s + 1} \right\rfloor\right\}.$$

We can find in [7] a Moore-like bound on the order of (d, D)-bipartite digraphs: $M_b(d, D) = 2(d^{D+1} - d)/(d^2 - 1)$ if D is even and $M_b(d, D) = 2(d^{D+1} - 1)/(d^2 - 1)$ if D is odd.

Proposition 1. *The order of a (d, D, D', s)-bipartite digraph, $D' \geqslant 3$, is upper bounded by the Moore-like bound $M_b(d, D, D', s)$, where*

$$M_b(d, D, D', s) = \min\left\{ M_b(d, D), 2\left(d + \left\lfloor \frac{d^3 + d^5 + \cdots + d^{D'-1}}{s+1} \right\rfloor\right)\right\}$$

if D' is even and

$$M_b(d, D, D', s) = \min\left\{ M_b(d, D), 2\left(1 + \left\lfloor \frac{d^2 + d^4 + \cdots + d^{D'-1}}{s+1} \right\rfloor\right)\right\}$$

if D' is odd.

Proof. Let $G = (V, A)$ be a (d, D, D', s)-bipartite digraph with $V = V_0 \cup V_1$. Let n be the order of this digraph. Since G is a (d, D)-bipartite digraph, $n \leqslant M_b(d, D)$.

Besides, if D' is even, for any $x_0 \in V_0$ and $x_1 \in V_1$ such that (x_0, x_1) is not an arc, there are $s + 1$ paths of length at most $D' - 1$ from x_0 to x_1. There are $d + d^3 + d^5 + \cdots + d^{D'-1}$ paths of length less or equal than $D' - 1$ from v_0 to the vertices of V_1. Therefore, we have

$$|V_1| \leqslant d + \left\lfloor \frac{d^3 + d^5 + \cdots + d^{D'-1}}{s+1} \right\rfloor.$$

Similarly, we have the same inequality for $|V_0|$, so

$$n = |V_0| + |V_1| \leqslant 2\left(d + \left\lfloor \frac{d^3 + d^5 + \cdots + d^{D'-1}}{s+1} \right\rfloor\right).$$

If D' is odd, there are $s + 1$ paths of length at most $D' - 1$ between any pair of different vertices of V_0 (or V_1). Analogously to the case when D' is even, we have

$$|V_i| \leqslant 1 + \left\lfloor \frac{d^2 + d^4 + \cdots + d^{D'-1}}{s+1} \right\rfloor$$

for $i = 0, 1$, and then

$$n \leqslant 2\left(1 + \left\lfloor \frac{d^2 + d^4 + \cdots + d^{D'-1}}{s+1} \right\rfloor\right). \qquad \square$$

3. Line digraph method and (d, D, D', s)-digraphs

The *line digraph method* is a well-known technique which has been used in the study of the (d, D)-digraph problem (i.e., to find large digraphs with maximum out-degree d and diameter D). See, for example [1, 8].

We recall here that in the *line digraph* LG of a digraph G each vertex represents an arc of G, that is, $V(LG) = \{uv \mid (u, v) \in A(G)\}$. A vertex uv is adjacent to a vertex vw if $v = w$, that is, whenever the arc (u, v) of G is adjacent to the arc (w, z). If G has maximum out-degree d, then LG has also maximum out-degree d. Moreover, if G is d-regular with order n, then LG is d-regular and has order dn. If G is a strongly connected digraph different from a directed cycle, then the diameter of LG is the diameter of G plus one (see [8]). We finally recall that if G is bipartite with partite sets V_0 and V_1, so is LG with partite sets which represent the arcs from V_0 to V_1 and the arcs from V_1 to V_0.

The next theorem allows us to apply the line digraph method to the (d, D, D', s)-digraph problem. The next two properties will be necessary.

Property 1. For any pair of different vertices $u, v \in V$ there exist $s + 1$ arc-disjoint paths of length lesser than or equal to D' from u to v.

Property 2. For any vertex x and for any pair of arcs in the form ux, xv, there are s arc-disjoint cycles in x of length at most D' such that neither ux nor xv belong to any of them (see Fig. 2).

Theorem 1. *Let G be a (d, D)-digraph. If G verifies Property* 1, *then its line digraph LG is $(d, D+1, D'+1, s)$. If G verifies Properties* 1 *and* 2, *then LG and $L^2G = LLG$ are, respectively $(d, D+1, D'+1, s)$ and $(d, D+2, D'+2, s)$.*

Proof. We will prove that if G verifies Property 1, then between any pair of non-adjacent vertices of LG there are $s + 1$ disjoint paths of length at most $D' + 1$. Let ux and yv be non-adjacent vertices of LG (that is, x and y are different vertices of G). Then, in G there are $s + 1$ arc-disjoint paths of length at most D' from x to y. Each path $x, u_1, \ldots, u_{l-1}, y$ of length $l \leqslant D'$ in G gives in LG a path $ux, xu_1, \ldots, u_{l-1}y, yv$ of length $l + 1$ between the vertices ux and yv. So, the $s + 1$ arc-disjoint paths in G from x to y of length at most D' correspond to $s + 1$ vertex-disjoint paths in LG of length at most $D' + 1$ from ux to yv (see Fig. 1).

We will prove next that if G verifies Properties 1 and 2, then LG is a $(d, D+1)$-digraph which verifies Property 1 and, as a consequence, L^2G is a $(d, D+2, D'+2, s)$-digraph. Since G verifies Property 1, there are $s + 1$ disjoint paths of length at most $D' + 1$ between any pair of non-adjacent vertices of LG. Let ux and xv be two adjacent

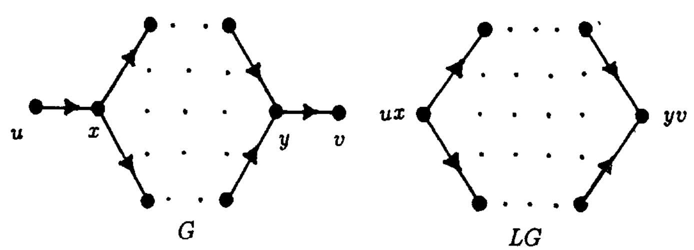

Fig. 1. Disjoint paths in G correspond to disjoint paths in LG.

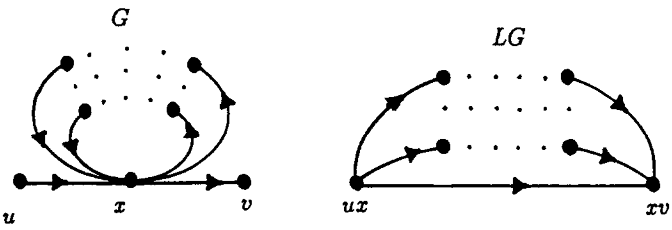

Fig. 2. Disjoint cycles in G correspond to disjoint paths in LG.

vertices in LG. Each cycle $x, u_1, \ldots, u_{l-1}, x$ of length $l \leqslant D'$ corresponds in LG to a path $ux, xu_1, \ldots, u_{l-1}x, xv$ of length $l+1$ from ux to xv. There are, in G, s arc-disjoint cycles in x of length at most D' such that do not go through ux nor xv. These cycles correspond in LG to s disjoint paths of length at most $D'+1$ from ux to xv such that the arc $(ux, xv) \equiv uxv$ does not belong to any of them (see Fig. 2). Adding the arc uxv to those s disjoint paths, we have $s+1$ disjoint paths of length at most $D'+1$ from ux to xv. Then we have, in LG, $s+1$ disjoint paths of length at most $D'+1$ between any pair of different vertices. □

4. Bipartite digraphs $BD(d, n)$

Fiol and Yebra [7] constructed a family of bipartite digraphs, called $BD(d, n)$. The largest known (d, D)-bipartite digraphs (optimal if $D \leqslant 4$) belong to this family.

In the following sections we prove that, for some values of n, the digraphs $BD(d, n)$ are large (d, D, D', s)-bipartite digraphs. We recall here the definition and some properties of these digraphs. See [7] for proofs.

For any positive integers d, n with $d \leqslant n$, the bipartite digraph $BD(d, n)$ has set of vertices $V = \mathbb{Z}_2 \times \mathbb{Z}_n = \{(\alpha, i); \alpha \in \mathbb{Z}_2, i \in \mathbb{Z}_n\}$ and each vertex (α, i) is adjacent to $(\bar{\alpha}, (-1)^{\alpha} d(i + \alpha) + t)$ for any $t \in \{0, 1, \ldots, d-1\}$, where, as usual, $\bar{0} = 1$ and $\bar{1} = 0$.

We can label the arcs of $BD(d, n)$ with the elements of $\{0, 1, \ldots, d-1\}$. We can determine a path in $BD(d, n)$ giving its first vertex and the labels of its arcs.

The digraph $BD(d, n)$ is d-regular and bipartite with partite sets $V_0 = \{0\} \times \mathbb{Z}_n$ and $V_1 = \{1\} \times \mathbb{Z}_n$.

There is an automorphism ϕ of $BD(d, n)$ such that $\phi(V_0) = V_1$ and $\phi(V_1) = V_0$. One important fact about this family is that the line digraph $LBD(d, n)$ is isomorphic to $BD(d, dn)$.

If $n = d^{D-1} + d^{D-4k-3}$, with $0 \leqslant k \leqslant \lfloor (D-3)/4 \rfloor$, $BD(d, n)$ has diameter D. If $n = d^{D-1} + d^{D-3}$, $BD(d, n)$ is a (d, D)-bipartite digraph with order very close to the bound $M_b(d, D)$ and it is optimal if $D = 3$ or $D = 4$. In general, we have that the diameter D of the bipartite digraph $BD(d, n)$ is such that $\lceil \log_d n \rceil \leqslant D \leqslant \lceil \log_d n \rceil + 1$.

For any set A of vertices of a digraph G, let $\Gamma_l^+(A)$ be defined recursively by $\Gamma_l^+(A) = \Gamma^+(\Gamma_{l-1}^+(A))$ beginning with

$$\Gamma_1^+(A) = \Gamma^+(A) = \bigcup_{v\in A} \Gamma^+(v).$$

When $A = \{v\}$ we just write $\Gamma_l^+(v)$.

In $BD(d,n)$ we have:

- $\Gamma_1^+(0,i) = \{(1, di + t_2) \mid t_2 = 0,1,\ldots,d-1\}$,
- $\Gamma_2^+(0,i) = \{(0, -d^2 i - 1 - (t_1 + t_2 d)) \mid t_1, t_2 = 0,1,\ldots,d-1\}$,
- $\Gamma_3^+(0,i) = \{(1, -d^3 i - 1 - (t_0 + t_1 d + t_2 d)) \mid t_0, t_1, t_2 = 0,1,\ldots,d-1\}$.

The values of t_0, t_1, t_2 determine a path from $(0,i)$ to any vertex in $\Gamma_3^+(v)$: the path going through the arcs labeled with $t_2, d - t_1 - 1$ and $d - t_0 - 1$.

The next two lemmas are properties of these digraphs which will be used further.

Lemma 1. *If $n \geqslant d^2$ and x and y are vertices of $BD(d,n)$ such that $d(x,y) = 2$, then there is a unique path of length 2 from x to y.*

Proof. Using the automorphism ϕ, we can suppose $x = (0,i) \in V_0$. Then

$$\Gamma_2^+(x) = \{(0, -d^2 i - 1 - (t_0 + t_1 d)) \mid t_0, t_1 = 0,1,\ldots,d-1\}.$$

Since $n \geqslant d^2$, $\Gamma_2^+(x)$ has d^2 different elements. This would be impossible if there were more than one path between x and a vertex at distance 2 from it. $\square$

Lemma 2. *If $d^2 + 1$ divides n, then there are no cycles of length 2 in $BD(d,n)$.*

Proof. We suppose that there are in $BD(d,n)$ cycles of length 2, that is, we can find a vertex $(0,i) \in V_0$ such that $(0,i) \in \Gamma_2^+(0,i)$. Then, there must be $t_0, t_1 \in \{0,1,\ldots,d-1\}$ such that

$$(d^2+1)i + 1 + t_0 + t_1 d \equiv 0 \pmod{n}.$$

Since $d^2 + 1$ divides n, $1 + t_0 + t_1 d$ must be a multiple of $d^2 + 1$. But that is impossible because $1 \leqslant 1 + t_0 + t_1 d \leqslant d^2$. $\square$

5. Large $(d,3,3,s)$-bipartite digraphs

Some $(d,3,3,s)$-bipartite digraphs with order close to the bound $M_b(d,3,3,s)$ are shown in this section.

First of all, we are going to construct large $(d,3,3,s)$-bipartite digraphs with $s < d-1$ and $s+1$ a divisor of d. Optimal $(d,2,2,s)$-digraphs are constructed in [10] in a similar way. Consider the complete symmetric bipartite digraph $K_{m,m}$, where $m = d/(s+1)$. Let $K_{m,m}^{s+1}$ be the d-regular bipartite digraph obtained from $K_{m,m}$ changing each arc by $s+1$ parallel arcs. $K_{m,m}^{s+1}$ has diameter 2 and between any pair of different vertices of $K_{m,m}^{s+1}$ there are $s+1$ arc-disjoint paths of length at most 2. Then, from Theorem 1, its line digraph $LK^{s+1}_{m,m}$ is a $(d, 3, 3, s)$-bipartite digraph of order $n = 2d^2/(s + 1)$.

Theorem 2. *Let s, d be integers such that $s < d$ and $s + 1$ divides $d^2 + 1$ and let $n = (d^2 + 1)/(s + 1)$. The bipartite digraph $BD(d, n)$ is $(d, 3, 3, s)$.*

Proof. We shall prove that between any pair of non-adjacent vertices x, y of $BD(d, n)$ there exist $s + 1$ disjoint paths of length at most 3. Since there is an automorphism ϕ in $BD(d, n)$ which transforms V_0 in V_1, we can assume that $x \in V_0$.

First we prove that there are $s + 1$ disjoint paths of length 2 from x to any different vertex of V_0. Let us take $x = (0, i)$ and $y = (0, j), i \neq j$. For each $k = 0, 1, \ldots, s$, exist $t_0(k), t_1(k) \in \{0, 1, \ldots, d - 1\}$ such that

$$j + k(d^2 + 1)/(s + 1) \equiv i - 1 - (t_0(k) + t_1(k)d) \pmod{d^2 + 1}.$$

Since $j \equiv j + k(d^2 + 1)/(s + 1) \pmod{n}$ for all k, each value of k gives us a path of length 2 from $(0, i)$ to $(0, j)$, the path which goes through the arcs labeled with $t_1(k)$ and $d - t_0(k) \; - 1$. It is easy to prove that $t_1(k) \neq t_1(k')$ if $k \neq k'$. Therefore, these $s + 1$ paths from $(0, i)$ to $(0, j)$ are disjoint.

Finally, we prove that there are $s + 1$ disjoint paths of length 3 from x to any non-adjacent vertex $y \in V_1$. We take $s + 1$ different vertices $v_1, v_2, \ldots, v_{s+1}$ in $\Gamma^+(x)$. Since $v_i \in V_1$ and $v_i \neq y$, there are $s + 1$ disjoint paths of length 2 from v_i to y. Let $v_1 w_1 y$ be one of the paths from v_1 to y. At least one of the $s + 1$ disjoint paths from v_2 to y does not go through w_1. Then, there is a path $v_2 w_2 y$ such that $w_2 \neq w_1$. In the same way, for each $i \leqslant s + 1$, we can find a path $v_i w_i y$ with $w_i \neq w_j$ for any $j < i$. Then, there are $s + 1$ disjoint paths from x to y (see Fig. 3).

We have found $s + 1$ disjoint paths of length at most 3 between any pair of non-adjacent vertices of $BD(d, n)$. Besides, $BD(d, n)$ has diameter 3. Therefore, $BD(d, n)$ is a $(d, 3, 3, s)$ -bipartite digraph. □

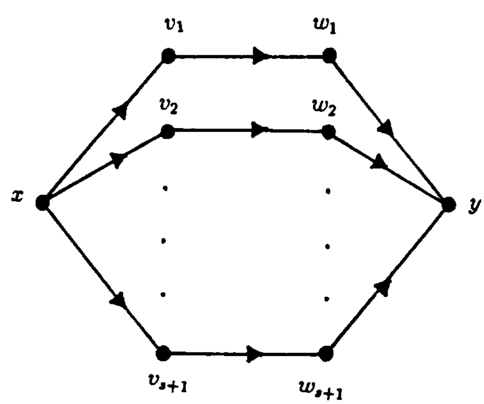

Fig. 3. $s + 1$ disjoint paths from $x \in V_0$ to a non-adjacent vertex $y \in V_1$.

6. Large $(d, D, 4, s)$-bipartite digraphs

In this section we prove that the bipartite digraphs $BD(d,n)$, with $n = (d^3 + d)/(s + 1)$, are $(d, D, 4, s)$-bipartite digraphs.

Theorem 3. *Let s and d be integers such that $s < d$ and $s + 1$ is a divisor of $d^3 + d$. Let $n = (d^3 + d)/(s + 1)$. In the bipartite digraph $BD(d,n)$ there are $s + 1$ disjoint paths of length at most 4 between any pair of non-adjacent vertices.*

Proof. Let x and y be two non-adjacent vertices of $BD(d,n)$. *As before, we can suppose that* $x = (0,i) \in V_0$. First we will prove that there are $s+1$ disjoint paths of length 3 from x to any non-adjacent vertex $y = (1,j) \in V_1$. If $(1,j)$ is not adjacent from $(0,i)$, for each $k = 0, 1, \ldots, s$, there exist $t_0(k), t_1(k), t_2(k) \in \{0, 1, \ldots, d-1\}$ such that

$$j + k(d^3 + d)/(s + 1) \equiv di - 1 - (t_0(k) + t_1(k)d + t_2(k)d^2) \pmod{d^3 + d}.$$

Since $j \equiv j + k(d^3 + d)/(s + 1) \pmod{n}$ for all k, each value of k gives a path of length 3 from $(0,i)$ to $(1,j)$, the path xu_kv_ky which goes through the arcs labeled with $t_2(k), d - t_1(k) - 1$ and $d - t_0(k) - 1$. It is easy to prove that $t_2(k) \neq t_2(k')$ if $k \neq k'$ and so, $u_k \neq u_{k'}$ if $k \neq k'$. Since $n > d^2$, from Lemma 1 we see that $v_k \neq v_{k'}$ if $k \neq k'$. Finally, since (x,y) is not an arc, $x \neq v_k$ and $y \neq u_k$ for all $k \in \{0, 1, \ldots, s\}$ (see Fig. 4). Therefore, we have $s + 1$ disjoint paths of length 3 from $(0,i)$ to $(1,j)$.

We have to prove now that there are $s + 1$ disjoint paths of length at most 4 from x to any vertex $y \in V_0, y \neq x$.

If $d(x,y) = 4$, we take $s + 1$ different vertices $v_1, \ldots, v_{s+1} \in \Gamma^+(x)$. There are $s + 1$ disjoint paths of length 3 from each v_i to y ($v_i \in V_1$ and are not adjacent to y). Let $v_1u_1w_1y$ be one of the paths from v_1 to y. At least one of the $s + 1$ paths from v_2 to y does not go through w_1. Then, there is a path $v_2u_2w_2y$ such that $w_2 \neq w_1$. In the same way, for each $i \leqslant s + 1$, we can find a path $v_iu_iw_iy$ with $w_i \neq w_j$ for any $j < i$.

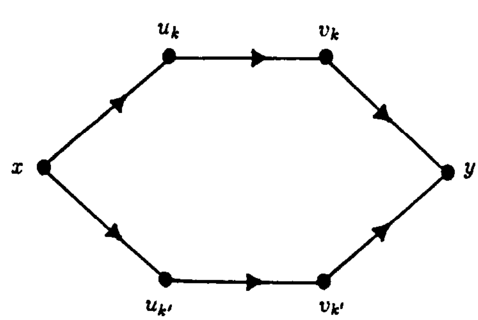

Fig. 4. Disjoint paths from $x \in V_0$ to a non-adjacent vertex $y \in V_1$.

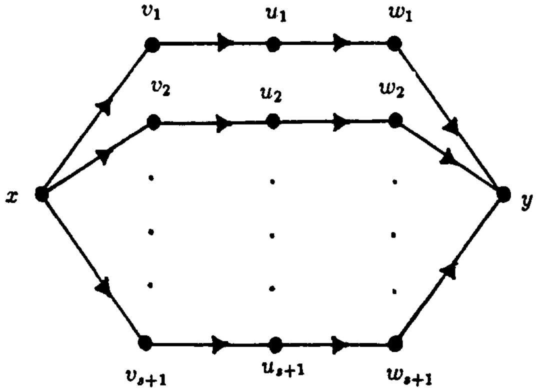

Fig. 5. $s + 1$ disjoint paths between two vertices of V_0 at distance 4.

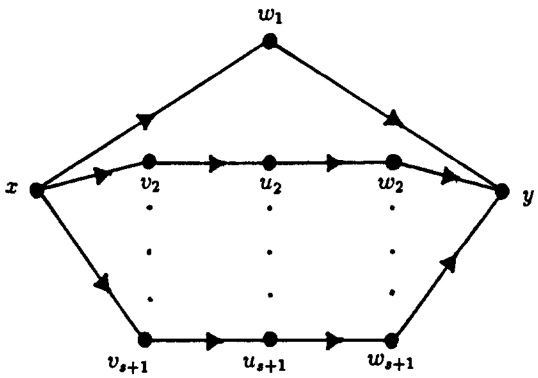

Fig. 6. $s + 1$ disjoint paths between two vertices of V_0 at distance 2.

Since $d(x, y) = 4, v_i \neq w_j$ for all i, j and $u_i \neq x, y$ for all i. From Lemma 1, $u_i \neq u_j$ if $i \neq j$. Then, there are $s + 1$ disjoint paths of length 4 from x to y (see Fig. 5).

If $d(x, y) = 2$, we take w_1 such that xw_1y is the unique path (see Lemma 1) of length 2 from x to y and s different vertices $v_2, \ldots, v_{s+1} \in \Gamma^+(x) - \{w_1\}$. Similarly, to the case $d(x, y) = 4$, for all $i = 2, \ldots, s + 1$, we can find a path $xv_iu_iw_iy$ such that $w_i \neq w_j$ if $i \neq j$. From Lemma 1, $v_i \neq w_j$ for all $i, j, u_i \neq u_j$ if $i \neq j$ and $u_i \neq x, y$ for all i. There are, then, $s + 1$ disjoint paths of length at most 4 from x to y (see Fig. 6). □

If $n = (d^3 + d)/(s + 1), 2n > M_b(d, 3)$ if $s < d - 1$. Then, the diameter of $BD(d, n)$ is 4 if $s < d - 1$ and is 3 if $s = d - 1$. Therefore, as a consequence of Theorem 3, we obtain the result pursued in this section.

Theorem 4. *If $s < d$ and $s + 1$ divides $d^3 + d, BD(d,(d^3 + d)/(s + 1))$ is a $(d, D, 4, s)$-bipartite digraph, where $D = 3$ if $s = d - 1$ and $D = 4$ otherwise.*

7. Large (d, D, D', s)-bipartite digraphs with $D' = 5, 6$

In this section we use the results of Section 3 in order to find large (d, D, D', s)-bipartite digraphs with $D' = 5, 6$.

Theorem 5. *Let s, d be integers such that $s + 1$ divides d. Let $n = (d^3 + d)/(s + 1)$. The bipartite digraph $BD(d, n)$ verifies:*

1. *Between any pair of different vertices, there are $s + 1$ disjoint paths of length at most 4.*
2. *For any vertex x of $BD(d,n)$ and for any pair of arcs in the form ux, xv, there are s disjoint cycles in x of length 4 such that none of them goes neither through ux nor xv.*

Proof. From Theorem 3, we have $s + 1$ disjoint paths of length at most 4 between any pair of non-adjacent vertices in $BD(d,n)$. We have to prove the existence of $s + 1$ disjoint paths of length at most 3 between any pair x, y of adjacent vertices. It is enough to prove it for $x = (0, i) \in V_0$ and $y = (1, j) \in \Gamma^+(x)$. Since $(1, j)$ is adjacent from $(0, i)$, $j = di + t$ for some $t \in \{0, 1, \ldots, d - 1\}$. For each $k = 1, \ldots, s$, there exist $t_0(k), t_1(k), t_2(k) \in \{0, 1, \ldots, d - 1\}$ such that

$$j + k(d^3 + d)/(s + 1) \equiv di - 1 - (t_0(k) + t_1(k)d + t_2(k)d^2) \pmod{d^3 + d}.$$

Since for all k, $j \equiv j + k(d^3 + d)/(s + 1) \pmod{n}$, each value of k gives us a path of length 3 from $(0, i)$ to $(1, j)$, the path xu_kv_ky which goes through the arcs labeled with $t_2(k), d - t_1(k) - 1$ and $d - t_0(k) - 1$. It is easy to prove that $t_2(k) \neq t_2(k')$ if $k \neq k'$ and so, $u_k \neq u_{k'}$, if $k \neq k'$. From Lemma 1 we see that $v_k \neq v_{k'}$ if $k \neq k'$. Since $s + 1$ divides

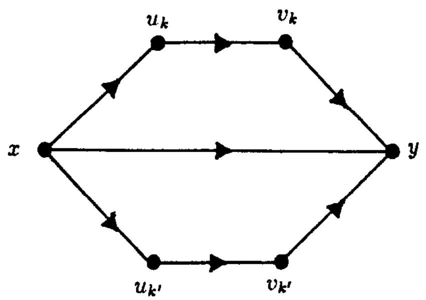

Fig. 7. Disjoint paths from $x \in V_0$ to an adjacent vertex $y \in V_1$.

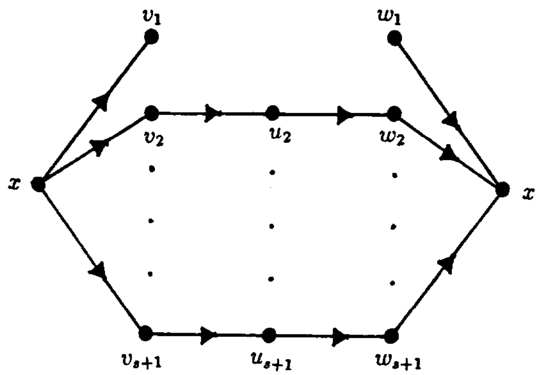

Fig. 8. Disjoint cycles in a vertex $x \in V_0$.

d, $(d^3 + d)/(s + 1)$ is a multiple of $d^2 + 1$. Then, from Lemma 2, $x \neq v_k$ and $y \neq u_k$ for all $k \in \{1, \ldots, s\}$ (see Fig. 7). Therefore, we have $s + 1$ disjoint paths of length at most 3 from $(0, i)$ to $(1, j)$ (one path is the arc xy and the other paths are given by the different values of k).

We are now going to prove that for any vertex x and for any pair of arcs in the form w_1x, xv_1, there are s disjoint cycles in x of length 4 such that any of them goes through neither w_1x nor xv_1. We suppose $x \in V_0$ and take s different vertices $v_2, \ldots, v_{s+1} \in \Gamma^+(x) - \{v_1\}$. Since $x \in V_0$ and $v_i \in V_1$ are not adjacent to x (see Lemma 2), there are $s + 1$ disjoint paths of length 3 from each v_i to x. At least one of the $s + 1$ paths from v_2 to x does not go through w_1. Then, there is a path $v_2u_2w_2x$ such that $w_2 \neq w_1$. In the same way, for each $i \leqslant s + 1$, we can find a path $v_iu_iw_ix$ with $w_i \neq w_j$ for any $j < i$. From Lemma 2, $v_i \neq w_j$ for all i, j, and $u_i \neq x$ for all i. From Lemma 1, $u_i \neq u_j$ if $i \neq j$. Then, there are s disjoint cycles in x of length 4 such that any of them goes through neither w_1x nor xv_1 (see Fig. 8). □

Theorem 6. *Let s, d be integers such that $s + 1$ divides d. Then, the bipartite digraphs $BD(d, (d^4 + d^2)/(s + 1))$ and $BD(d, (d^5 + d^3)/(s + 1))$ are, respectively, $(d, D, 5, s)$ and $(d, D, 6, s)$, where $D = D' - 1$ if $s = d - 1$ and $D = D'$ otherwise.*

Proof. From Theorem 4, $BD(d, (d^3 + d)/(s + 1))$ is a (d, D)-digraph with $D = 3$ if $s = d - 1$ and $D = 4$ otherwise. Besides, from Section 4, we have

$$LBD(d, (d^3 + d)/(s + 1)) \cong BD(d, (d^4 + d^2)/(s + 1)),$$

$$L^2BD(d, (d^3 + d)/(s + 1)) \cong BD(d, (d^5 + d^3)/(s + 1)).$$

Apply Theorems 1 and 5. □

In Table 1 we can see the orders of the (d, D, D', s)-bipartite digraphs found in this paper and compare them with the upper bound $M_b(d, D, D', s)$.

Table 1
Orders of the (d, D, D', s)-bipartite digraphs found in this paper

(d, D, D', s)	$M_b(d, D, D', s)$	Order
$(d, 3, 3, s)$ $s < d - 1, (s + 1) \mid d$	$2\left(1 + \frac{d^2}{s+1}\right)$	$2\frac{d^2}{s+1}$
$(d, 3, 3, s)$ $(s + 1) \mid (d^2 + 1)$	$2\left(1 + \left\lfloor\frac{d^2}{s+1}\right\rfloor\right)$	$2\frac{d^2+1}{s+1}$
$(d, 4, 4, s)$ $s < d - 1, (s + 1) \mid (d^3 + d)$	$2\left(d + \left\lfloor\frac{d^3}{s+1}\right\rfloor\right)$	$2\frac{d^3+d}{s+1}$
$(d, 3, 4, d - 1)$	$2(d^2 + 1)$	$2(d^2 + 1)$
$(d, 5, 5, s)$ $s < d - 1, (s + 1) \mid d$	$2\left(1 + \frac{d^4+d^2}{s+1}\right)$	$2\frac{d^4+d^2}{s+1}$
$(d, 4, 5, d - 1)$	$2(d + d^3)$	$2(d + d^3)$
$(d, 6, 6, s)$ $s < d - 1, (s + 1) \mid d$	$2\left(d + \frac{d^3+d^5}{s+1}\right)$	$2\frac{d^3+d^5}{s+1}$
$(d, 5, 6, d - 1)$	$2(1 + d^2 + d^4)$	$2(d^2 + d^4)$

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