On relative oscillation theory for symplectic eigenvalue problems

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Abstract

We develop an analog of classical oscillation theory for discrete symplectic eigenvalue problems with Dirichlet boundary conditions which, rather than measuring the spectrum of one single problem, measures the difference between the spectra of two different problems. This is done by replacing focal points of conjoined bases of one problem by matrix analogs of weighted zeros of Wronskians of conjoined bases of two different problems.

1. Introduction

We consider the discrete symplectic eigenvalue problems [1]

\[ x_{i+1}(\lambda) = A_i x_i(\lambda) + B_i u_i(\lambda), \quad u_{i+1}(\lambda) = C_i x_i(\lambda) + D_i u_i(\lambda) - \lambda W_i x_{i+1}(\lambda), \quad i = 0, \ldots, N, \quad x_0(\lambda) = x_{N+1}(\lambda) = 0, \quad \lambda \in \mathbb{R}, \]

and

\[ \hat{x}_{i+1}(\lambda) = A_i \hat{x}_i(\lambda) + B_i \hat{u}_i(\lambda), \quad \hat{u}_{i+1}(\lambda) = \hat{C}_i \hat{x}_i(\lambda) + \hat{D}_i \hat{u}_i(\lambda) - \lambda \hat{W}_i \hat{x}_{i+1}(\lambda), \quad i = 0, \ldots, N, \quad \hat{x}_0(\lambda) = \hat{x}_{N+1}(\lambda) = 0, \quad \lambda \in \mathbb{R}, \]

where \( \lambda \in \mathbb{R} \), \( x_i(\lambda), u_i(\lambda), \hat{x}_i(\lambda), \hat{u}_i(\lambda) \in \mathbb{R}^n \), and the real \( n \times n \) matrices \( W_i, \hat{W}_i, A_i, B_i, C_i, D_i, \hat{C}_i, \hat{D}_i \) satisfy the conditions

\[ W_i = W_i^T, \quad \hat{W}_i = \hat{W}_i^T, \quad W_i \geq 0, \quad \hat{W}_i \geq 0, \]

\[ B_i^T D_i = D_i^T B_i, \quad B_i^T \hat{D}_i = \hat{D}_i^T B_i, \quad A_i^T C_i = C_i^T A_i, \quad A_i^T \hat{C}_i = \hat{C}_i^T A_i, \quad A_i^T D_i - C_i^T B_i = I, \quad A_i^T \hat{D}_i - \hat{C}_i^T B_i = I. \]

Conditions (1.3), (1.4) imply [see [1]] that matrices of the above difference systems rewritten in the form

\[ W_i(\lambda) y_i(\lambda), \hat{W}_i(\lambda) \hat{y}_i(\lambda), y_i(\lambda) = [x_i(\lambda) u_i(\lambda)]^T, \hat{y}_i(\lambda) = [\hat{x}_i(\lambda) \hat{u}_i(\lambda)]^T \]

are symplectic, i.e.

\[ W_i(\lambda)^T J W_i(\lambda) = J, \quad \hat{W}_i(\lambda)^T J \hat{W}_i(\lambda) = J \quad \text{for} \quad i = 0, \ldots, N, \quad \lambda \in \mathbb{R}. \]

Results of this work rely on the concept of a finite eigenvalue of (1.1), (1.2) which was introduced in the fundamental paper [2]. The so-called Global Oscillation Theorem established in [1,2] relates the number of finite eigenvalues of (1.1) less than or equal to a given number \( \lambda_1 \) to the number of focal points (counting multiplicity) of the principal solution of the...
difference system in (1.1) with $\lambda = \lambda_1$. Our aim is to add a new aspect to this classical result by showing that matrix analogs of weighted zeros [3–5] of the Wronskian for two suitable matrix solutions of the difference systems in (1.1), (1.2) can be used to measure the difference between the number of finite eigenvalues of problems (1.1), (1.2).

Recall now some results of relative oscillation theory developed in [6,3] which we are going to extend to (1.1), (1.2). Consider the Sturm–Liouville eigenvalue problems

\[
\begin{align*}
-\Delta r_i^{(1)}(\Delta x_i) + r_i^{(0)} x_{i+1} &= \lambda x_i, \quad x_0 = x_{N+1} = 0, \\
-\Delta r_i^{(1)}(\Delta \hat{x}_i) + r_i^{(0)} \hat{x}_{i+1} &= \lambda \hat{x}_i, \quad \hat{x}_0 = \hat{x}_{N+1} = 0, \quad r_i^{(1)} \neq 0.
\end{align*}
\]

Introduce the Wronskian for two solutions $x_i, \hat{x}_i$ of the difference equations in (1.5), (1.6): $w_i(x, \hat{x}) = -r_i^{(1)}(x_i \hat{x}_{i+1} - x_{i+1} \hat{x}_i) = -r_i^{(1)}(x_i \Delta \hat{x}_i - \Delta x_i \hat{x}_i)$, where $\Delta x_i = x_{i+1} - x_i$. According to [6], the Wronskian $w_i(x, \hat{x})$ has a node at $i$ if either $w_i w_{i+1} < 0$ or $w_i = 0$, $w_{i+1} \neq 0$. Then, by [6, Theorem 4.3], for problems (1.5), (1.6) with $r_i^{(0)} = r_i^{(1)}$ and $\lambda_1 < \lambda_2$ we have

\[
\#(x^{(0)}(\lambda_1), x^{(N+1)}(\lambda_2)) = \#(\lambda \in \sigma_1 | \lambda_1 < \lambda < \lambda_2),
\]

where $\#(x, \hat{x})$ denotes the total number of nodes of $w_i(x, \hat{x})$ in $(0, N + 1)$, $\#(\lambda \in \sigma_1 | \lambda_1 < \lambda < \lambda_2)$ denotes the number of eigenvalues of (1.5) (or (1.6)) between $\lambda_1$ and $\lambda_2$ and the solutions $x_i^{(M)}(\lambda), x_i^{(M)}(\lambda), M = 0, N + 1$ of (1.5),(1.6) obey the conditions

\[
\begin{align*}
x_i^{(M)}(\lambda) &= x_i^{(M)}(\lambda) = 0 \text{ at } i = M, M = 0, N + 1.
\end{align*}
\]

The main result in [3] extends (1.7) to the case $r_i^{(0)} \neq r_i^{(1)}$. According to [3, Theorem 1.2], the number of weighted nodes of the Wronskian in $(0, N + 1)$ equals the number of eigenvalues of (1.6) below $\lambda_2$ minus the number of eigenvalues of (1.5) below or equal to $\lambda_1$:

\[
\#(x^{(0)}(\lambda_1), x^{(N+1)}(\lambda_2)) = \#(x^{(N+1)}(\lambda_1), x^{(0)}(\lambda_2)) = \#(\lambda \in \sigma_2 | \lambda < \lambda_2) - \#(\lambda \in \sigma_1 | \lambda \leq \lambda_1).
\]

2. Notation and auxiliary results

We will use the following notation. If $A$ is a symmetric matrix, i.e. $A^T = A$, the inequality $A > 0$ ($\geq, <, \leq$) means that $A$ is positive (nonnegative, negative, nonpositive) definite, ind $A$ denotes the index or the number of negative eigenvalues of a symmetric matrix $A$. By $A^T$ we denote the Moore–Penrose generalized inverse of a matrix $A$.

Introduce the symplectic difference systems

\[
\begin{align*}
y_{i+1} &= W_i y_i, \quad W_i^T J W_i = J, \quad i = 0, \ldots , N, \\
\hat{y}_{i+1} &= \hat{W}_i \hat{y}_i, \quad \hat{W}_i^T J \hat{W}_i = J, \quad i = 0, \ldots , N,
\end{align*}
\]

where $y_i, \hat{y}_i \in \mathbb{R}^{2n}$. Assume that the following condition:

\[
W_i \hat{W}_i^{-1} = \begin{bmatrix} I & 0 \\ R_i & I \end{bmatrix}, \quad R_i = R_i^T, \quad i = 0, \ldots , N
\]

holds for the matrices $W_i, \hat{W}_i$. Then, by (1.3), (1.4), the symplectic difference systems in (1.1), (1.2) can be presented in the form of (2.1), (2.2), where the matrices

\[
W_i = W_i(\lambda_1) = \begin{bmatrix} I & 0 \\ -\lambda_1 W_i & I \end{bmatrix} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \quad \hat{W}_i = \hat{W}_i(\lambda_2) = \begin{bmatrix} I & 0 \\ -\lambda_2 \hat{W}_i & I \end{bmatrix} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}
\]

obey (2.3) for any $\lambda_1, \lambda_2 \in \mathbb{R}$, and

\[
R_i = R_i(\lambda_1, \lambda_2) = R_i(\lambda_1, \lambda_2)^T = \lambda_2 \hat{W}_i - \lambda_1 W_i + C_i D_i^T - D_i C_i^T.
\]

Recall (see [10]) that $2n \times n$ matrix solutions $Y_i, \hat{Y}_i$ of (2.1), (2.2) are said to be conjoined bases if

\[
\text{rank } Y_i = n, \quad \text{rank } \hat{Y}_i = n, \quad Y_i^T J Y_i = 0, \quad \hat{Y}_i^T J \hat{Y}_i = 0.
\]

and the conjoined bases $Y_i^{(M)}$, $\hat{Y}_i^{(M)}$ of (2.1), (2.2) with the initial conditions $Y_i^{(M)} = [0 I]^T$, $\hat{Y}_i^{(M)} = [0 I]^T$ at $i = M$ are said to be the principal solutions at $M$. Note that for conjoined bases $Y_i, \hat{Y}_i$ of (2.1), (2.2) there exist symplectic fundamental matrices $Z_i, \hat{Z}_i$ such that

\[
Y_i = Z_i [0 I]^T, \quad \hat{Y}_i = \hat{Z}_i [0 I]^T
\]
(see [10, Remark 1(ii)]). The Wronskian
\[ w_i(Y, \hat{Y}) = w_i = Y_{i}^{T} \hat{Y}_{i} \]
(2.7)
for conjoined bases of (2.1), (2.2) with condition (2.3) obeys the equation
\[ \Delta w_i(Y, \hat{Y}) = Y_{i+1}^{T} \hat{Y}_{i+1} - Y_{i}^{T} \hat{Y}_{i} = Y_{i+1}^{T} (I - W_{i} W_{i}^{-1}) \hat{Y}_{i+1} = -X_{i+1}^{T} R_{i} \hat{X}_{i+1}. \]
(2.8)

According to [8,9], we define the comparative index for $2n \times n$ matrices $Y, \hat{Y}$ with conditions (2.5) using the notation
\[
\begin{align*}
\mathcal{M} &= (I - XX^{T}) \hat{X}, \quad X = [I \ 0] Y, \quad \hat{X} = [I \ 0] \hat{Y}, \\
\mathcal{T} &= I - M^{T} M, \\
\mathcal{D} &= \mathcal{T} w(Y, \hat{Y})^{T} X^{T} \mathcal{T}.
\end{align*}
\]
(2.9)
The comparative index is defined by $\mu(Y, \hat{Y}) = \mu_1(Y, \hat{Y}) + \mu_2(Y, \hat{Y})$, where $\mu_1(Y, \hat{Y}) = \text{rank } \mathcal{M}$ and $\mu_2(Y, \hat{Y}) = \text{ind } \mathcal{D}$.

Introduce the dual index $\mu^{*}(Y, \hat{Y}) = \mu(Y, \hat{Y}) + \mu^{*}_2(Y, \hat{Y})$, where $\mu^{*}_2(Y, \hat{Y}) = \text{ind}(-\mathcal{D})$.

If $Z_i$ is a symplectic fundamental matrix for (2.1) and (2.6) holds, then, according to [8, Lemmas 3.1 and 3.2] we have
\[
\begin{align*}
m_{i}(Y) &= \mu(Y_{i+1}, W_{i}[0 \ 0]^{T}) = \mu^{*}(Z_{i+1}^{-1}[0 \ 0]^{T}, Z_{i}^{-1}[0 \ 0]^{T}), \\
m^{*}_{i}(Y) &= \mu^{*}(Y_{i}, W_{i}^{-1}[0 \ 0]^{T}) = \mu(Z_{i}^{-1}[0 \ 0]^{T}, Z_{i+1}^{-1}[0 \ 0]^{T}).
\end{align*}
\]
(2.10) (2.11)
where $m_{i}(Y) = \text{rank } M_{i} + \text{ind } P_{i}$, $m^{*}_{i}(Y) = \text{rank } \tilde{M}_{i} + \text{ind } \tilde{P}_{i}$ denote the numbers of focal points of $Y_{i} = [X_{i}^{T} \ U_{i}^{T}]^{T}$ in $(i, i + 1)$, $(i, i + 1)$ respectively (see [11, Definition 1], [8, Definition 3.2]), and
\[
\begin{align*}
M_{i} &= (I - X_{i+1} X_{i+1}^{T}) B_{i}, \\
T_{i} &= I - M_{i}^{T} M_{i}, \\
P_{i} &= T_{i} X_{i+1} B_{i} T_{i},
\end{align*}
\]
\[
\begin{align*}
\tilde{M}_{i} &= (I - X_{i} X_{i}^{T}) B_{i}^{T}, \\
\tilde{T}_{i} &= I - M_{i}^{T} M_{i}, \\
\tilde{P}_{i} &= \tilde{T}_{i} X_{i+1} X_{i+1}^{T} B_{i} \tilde{T}_{i}.
\end{align*}
\]
By formula (3.5) in [8, p. 451] we have
\[ m^{*}_{i}(Y) - m_{i}(Y) = \Delta \text{rank } X_{i}, \quad X_{i} = [I \ 0] Y_{i}. \]
(2.12)
Introduce the notation
\[ i(Y) = \sum_{i=0}^{N} m_{i}(Y), \quad i^{*}(Y) = \sum_{i=0}^{N} m_{i}^{*}(Y) \]
(2.13)
for the numbers of focal points of $Y_{i}$ in $(0, N + 1)$ and $[0, N + 1)$; then, by (2.12),
\[ i^{*}(Y) - i(Y) = \Delta \text{rank } X_{N+1} - \text{rank } X_{0}. \]
(2.14)
Consider an arbitrary symplectic transformation $P_{i}$ for a conjoined basis $Y_{i}$ of (2.1). Then, according to [9, Lemma 3.1] we have the relation
\[ m_{i}(Y_{i}) + \Delta \mu(\hat{Y}_{i}, P_{i}[0 \ 0]^{T}) + \mu(P_{i+1}[0 \ 0]^{T}, \tilde{W}_{i} [0 \ 0]^{T}) = m_{i}(\hat{Y}_{i}) + \mu^{*}(P_{i}^{-1}[0 \ 0]^{T}, W_{i}^{-1}[0 \ 0]^{T}) \]
(2.15)
for the numbers of focal points $m_{i}(Y_{i})$ and $m_{i}(\hat{Y}_{i})$. Here $\hat{Y}_{i} = P_{i} Y_{i}$ obeys the transformed symplectic difference system
\[ \hat{Y}_{i+1} = \hat{W}_{i} \hat{Y}_{i}, \quad \hat{W}_{i} = P_{i+1} W_{i} P_{i}^{-1}. \]
(2.16)
If $\hat{Z}_{i}$ is a symplectic fundamental matrix for (2.2), we can put $P_{i} := \hat{Z}_{i}^{-1}$ in (2.15) and then, by assumption (2.3), we have that the conjoined basis $\hat{Y}_{i} = \hat{Z}_{i}^{-1} Y_{i}$ obeys the transformed symplectic system (2.16) with the matrix
\[ \hat{W}_{i} = \hat{Z}_{i+1}^{-1} W_{i} \hat{Z}_{i+1}^{-1} \hat{W}_{i+1} \hat{Z}_{i+1}^{-1} = \left[ \begin{array}{cc} \mathfrak{A}_{i} & \mathfrak{B}_{i} \\ \mathfrak{C}_{i} & \mathfrak{D}_{i} \end{array} \right], \quad \mathfrak{B}_{i} = - \hat{X}_{i+1}^{T} R_{i} \hat{X}_{i+1}. \]
(2.17)
Moreover, we have
\[ \mu^{*}(P_{i}^{-1}[0 \ 0]^{T}, W_{i}^{-1}[0 \ 0]^{T}) = \mu^{*}(\hat{Z}_{i}[0 \ 0]^{T}, \hat{W}_{i}^{-1} \hat{W}_{i}^{-1} [0 \ 0]^{T}) = \mu^{*}(\hat{Z}_{i}[0 \ 0]^{T}, \hat{W}_{i}^{-1} [0 \ 0]^{T}) = m^{*}_{i}(\hat{Y}_{i}) = m_{i}(\hat{Y}_{i}) + \Delta \text{rank } \hat{X}_{i}, \]
where we use (2.11) and (2.12) for a conjoined basis $\hat{Y}_{i}$ of (2.2). Next, according to properties 3 and 9 of the comparative index (see [8, Section 2]) we derive
\[ \mu(P_{i+1}[0 \ 0]^{T}, \tilde{W}_{i} [0 \ 0]^{T}) = \mu(\hat{Z}_{i+1}^{-1} [0 \ 0]^{T}, \hat{Z}_{i+1}^{-1} \hat{W}_{i} \hat{Z}_{i+1}^{-1} [0 \ 0]^{T}) = \mu^{*}(\hat{Z}_{i+1}^{-1} [0 \ 0]^{T}, \hat{W}_{i}^{-1} \hat{Z}_{i+1}^{-1} [0 \ 0]^{T}) = \text{ind} (\mathfrak{B}_{i}). \]
\[ \Delta \mu(\hat{Y}_{i}, P_{i}[0 \ 0]^{T}) = \Delta \mu(\hat{Z}_{i+1}^{-1} Y_{i}, \hat{Z}_{i+1}^{-1} [0 \ 0]^{T}) = - \Delta \mu(Y_{i}, \hat{Y}_{i}) + \Delta \mu([0 \ 0]^{T}, \hat{Y}_{i}) = - \Delta \mu(Y_{i}, \hat{Y}_{i}) + \Delta \text{rank } \hat{X}_{i}, \]
where $\mathfrak{B}_{i}$ is given by (2.17).
Finally, relation (2.15) for $P_i = \hat{Z}_i^{-1}$ takes the form
\[
m_i(\hat{Y}) - m_i(Y) + \Delta \mu(Y_i, \hat{Y}_i) = \#_i(Y, \hat{Y}) - \mu(\hat{Y}) = \text{ind}(\mathcal{B}_i) - m_i(\hat{Z}_i^{-1}Y).
\] (2.18)
Here $m_i(Y), m_i(\hat{Y}), m_i(\hat{Z}_i^{-1}Y)$ are the numbers of focal points in $(i, i + 1)$ for conjoined bases of (2.1), (2.2) and (2.16) with matrix (2.17).

Interchanging the roles of $Y_i, \hat{Y}_i$ in (2.18) we have
\[
m_i(Y) - m_i(\hat{Y}) + \Delta \mu(Y_i, \hat{Y}_i) = \#_i(Y, \hat{Y}) - \mu(\hat{Y}) = \text{ind}(\mathcal{B}_i) - m_i(Z_i^{-1}\hat{Y}),
\] (2.19)
where $Z_i^{-1}\hat{Y}$ obeys transformed system (2.16) with $P_i = Z_i^{-1}$, and
\[
\tilde{W}_i = Z_i^{-1}\tilde{W}_iZ_i = Z_i^{-1}\tilde{W}_iW_i^{-1}Z_i = \begin{bmatrix} \tilde{a}_i & \tilde{b}_i \\ \tilde{c}_i & \tilde{d}_i \end{bmatrix}, \quad \mathcal{B}_i = X_i^T R X_i + 1.
\] (2.20)
Summing (2.18) and (2.19) we get
\[
\Delta \text{rank } w_i(Y, \hat{Y}) = \#_i(Y, \hat{Y}) + \#_i(\hat{Y}, Y),
\] (2.21)
where we use $\mu(\hat{Y}_i, Y_i) + \mu(Y_i, \hat{Y}_i) = \text{rank } w_i(Y, \hat{Y})$ for the Wronskian $w_i(Y, \hat{Y})$ defined by (2.7) (see [9, Theorem 2.4(iv)]). From (2.21) we have the second representation
\[
\#_i(Y, \hat{Y}) = m_i^*(Z_i^{-1}\hat{Y}) - \text{ind}(\mathcal{B}_i)
\] (2.22)
for the number $\#_i(Y, \hat{Y})$ in (2.18). Here $m_i^*(Z_i^{-1}\hat{Y})$ is the number of focal points of $Z_i^{-1}\hat{Y}$ in the interval $[i, i + 1]$ connected with $m_i(\hat{Z}_i^{-1}Y)$ by formula (2.17) Summing (2.18) from $i = 0$ to $i = N$ we prove the following:

**Lemma 1.** For any conjoined bases $Y_i = [X_i^T U_i^T]^T, \hat{Y}_i = [\hat{X}_i^T \hat{U}_i^T]^T$ of systems (2.1), (2.2) with condition (2.3) we have
\[
l(\hat{Y}) - l(Y) + \mu(Y_{N+1}, \hat{Y}_{N+1}) - \mu(Y_0, \hat{Y}_0) = \sum_{i=0}^N \#_i(Y, \hat{Y}).
\] (2.23)
where the number $\#_i(Y, \hat{Y})$ is defined by (2.18), (2.17) or (2.22), (2.20) and $\hat{Z}_i, Z_i$ are symplectic fundamental matrices of (2.2), (2.1) with condition (2.6).

**Corollary 2.** Let $Y_i^{(0)}, \hat{Y}_i^{(0)}$ and $Y_i^{(N+1)}, \hat{Y}_i^{(N+1)}$ be the principal solutions of (2.1), (2.2) at 0 and $N + 1$; then
\[
l(\hat{Y}^{(0)}) - l(Y^{(0)}) = \sum_{i=0}^N \#_i(Y^{(0)}, \hat{Y}^{(N+1)}),
\] (2.24)
\[
l(\hat{Y}^{(0)}) - l(Y^{(0)}) = \sum_{i=0}^N \#_i(Y^{(N+1)}, \hat{Y}^{(0)}).
\] (2.25)

**Proof.** Put $\hat{Y}_i := \hat{Y}_i^{(N+1)}, Y_i := Y_i^{(0)}$ in (2.23); then $\mu(Y_i^{(0)}, \hat{Y}_i^{(N+1)}) = 0, \mu(Y_0^{(0)}, \hat{Y}_0^{(N+1)}) = \text{rank}(\hat{X}_0^{(N+1)}), \text{rank}(\hat{X}_N^{(N+1)}), l(\hat{Y}^{(N+1)}) = l(\hat{Y}^{(0)})$. Moreover, $l(\hat{Y}^{(N+1)}) = l(\hat{Y}^{(0)})$ by [8, Lemma 3.3]. Then, for the given case, we have in the left hand side of (2.23) $l(\hat{Y}^{(N+1)}) - l(Y^{(0)}) = \text{rank}(\hat{X}_0^{(N+1)}) = l(\hat{Y}^{(0)}) - l(Y^{(0)})$ and (2.24) is proved.

Similarly, putting $\hat{Y}_i := \hat{Y}_i^{(0)}, Y_i := Y_i^{(N+1)}$ in (2.23) we get $\mu(Y_i^{(N+1)}, \hat{Y}_i^{(0)}) = \text{rank}(\hat{X}_N^{(N+1)}), \mu(Y_0^{(N+1)}, \hat{Y}_0^{(0)}) = 0$, and $l(\hat{Y}^{(0)}) = \text{rank}(\hat{X}_N^{(N+1)}) = l(\hat{Y}^{(0)})$ by (2.14). Moreover, $l(\hat{Y}^{(N+1)}) = l(Y^{(0)}) + \text{rank}(\hat{X}_N^{(N+1)}) = l(\hat{Y}^{(0)})$, where we use [9, Corollary 2.9] and (2.14). Finally, we have in the left hand side of (2.23) $l(Y^{(0)}) - l(Y^{(N+1)}) + \text{rank}(\hat{X}_N^{(N+1)}) = l(\hat{Y}^{(0)}) - l(\hat{Y}^{(0)})$ and the proof of (2.25) is completed. $\square$

Note that by (2.10), (2.11)
\[
m_i(\hat{Z}_i^{-1}Y) = \mu(\hat{Z}_i^{-1}Y_{i+1}, [\mathcal{B}_i \mathcal{D}_i^T]^T) = \mu^*(\hat{Z}_i^{-1}Y_{i+1}, Z_i^{-1}Y_i),
\] (2.26)
\[
m_i^*(Z_i^{-1}Y) = \mu^*(Z_i^{-1}Y_i, [-\mathcal{B}_i \mathcal{D}_i]^T) = \mu(\hat{Z}_i^{-1}Y_i, \hat{Z}_i^{-1}Y_{i+1}).
\] (2.27)
where $\mathcal{B}_i, \mathcal{D}_i, \tilde{\mathcal{B}}_i, \tilde{\mathcal{D}}_i$ are the blocks of (2.17), (2.20), and
\[
[I \, 0]Z_i^{-1}\hat{Y}_i = [I \, 0]Z_i^{-1}Y_i = [0 \, -I]Z_i^{-1}Y_i = -w_i(Y, \hat{Y}) = w_i(\hat{Y}, Y)^T = -(I \, 0)Z_i^{-1}Y_i^T.
\] (2.28)
Then, the numbers of focal points $m_i(\hat{Z}^{-1}Y)$, $m_i^*(Z^{-1}\hat{Y})$ in (2.18), (2.22) can be presented in terms of the Wronskian
 $w_i = w_i(Y, \hat{Y})$ given by (2.7) and $B_1$, $B_2$. We have

$$m_i(\hat{Z}^{-1}Y) = \text{rank } M_i + \text{ind}(\tilde{P}_i), \quad m_i^*(Z^{-1}\hat{Y}) = \text{rank } \tilde{M}_i + \text{ind}(\tilde{\tilde{P}}_i),$$

$$(2.29)$$

$$M_i = (I - w_i^+ w_i+1 )B_i, \quad \tilde{P}_i = T_i B_i w_i^+ w_i T_i, \quad \tilde{T}_i = I - \tilde{M}_i^+ \tilde{M}_i,$$

$$(2.30)$$

$$\tilde{M}_i = (I - w_i w_i^+ )B_i, \quad \tilde{\tilde{P}}_i = \tilde{T}_i w_i w_i^+ \tilde{T}_i, \quad \tilde{T}_i = I - \tilde{M}_i^+ \tilde{M}_i,$$

$$(2.31)$$

where

$$\text{rank } M_i = \text{rank}(I - w_i^+ w_i+1 )w_i, \quad \text{rank } \tilde{M}_i = \text{rank}(I - w_i w_i^+ )w_i^+$$

by (2.26), (2.27) (see also [12, Theorem 1(iii)]). Note that by [11, Lemma 1(iv)] we have the following inequalities:

$m_i(\hat{Z}^{-1}Y) \leq \text{rank}(B_i), \quad m_i^*(Z^{-1}\hat{Y}) \leq \text{rank}(\tilde{B}_i)$

for the numbers of focal points $m_i(\hat{Z}^{-1}Y)$, $m_i^*(Z^{-1}\hat{Y})$ in (2.18), (2.22). Then, for $#_i(Y, \hat{Y})$ in (2.23) we derive

$$\left| #_i(Y, \hat{Y}) \right| \leq \min(\text{rank}(B_i), \text{rank}(\tilde{B}_i)) \leq n,$$

(2.33)

where we use $\text{ind}(B_i) \leq \text{rank}(B_i), \text{ind}(\tilde{B}_i) \leq \text{rank}(\tilde{B}_i)$ for the symmetric matrices $B_i, \tilde{B}_i$ given by (2.17) and (2.20). For the particular case when the symmetric matrix $R_i$ given by (2.3) is nonnegative (nonpositive) definite the number $#_i(Y, \hat{Y})$ is nonnegative (nonpositive) and

$$#_i(Y, \hat{Y}) = m_i^*(Z^{-1}\hat{Y}) \quad \text{for } R_i \geq 0,$$

(2.34)

$$#_i(Y, \hat{Y}) = -m_i(\hat{Z}^{-1}Y) \quad \text{for } R_i \leq 0.$$

Remark 1. Note that by (2.8), (2.33) the condition $\left| #_i(Y, \hat{Y}) \right| = n$ implies $\text{det}(\Delta w_i) = \text{det}(X_i^T R_i \tilde{X}_{i+1}) \neq 0$. For the Sturm–Liouville equations in (1.5), (1.6) rewritten in the form (2.1), (2.2) for $y_i = [x_i \ r_i^{(1)} \Delta x_i]$, $\hat{y}_i = [\hat{x}_i \ r_i^{(1)} \Delta \hat{x}_i]$ we have $r_i(\lambda_1, \lambda_2) = r_i^{(0)} + \lambda_2 - \lambda_1$, and $#(y, \hat{y}) = 1$ iff $m_i^*(Z^{-1}\hat{Y}) = 1, r_i(\lambda_1, \lambda_2) > 0, x_{i+1} \neq 0$ because of (2.22) and (2.33). By (2.29), (2.31), (2.32) and (2.8) the last conditions are equivalent with $r_i(\lambda_1, \lambda_2) > 0$ and either $w_i w_{i+1} < 0$ or $w_i = 0, w_{i+1} \neq 0$. Similarly, by (2.18), (2.29), (2.30), (2.32), we have $#_i(y, \hat{y}) = -1 \Leftrightarrow m_i(\hat{Z}^{-1}Y) = 1, r_i(\lambda_1, \lambda_2) < 0, \hat{x}_{i+1} \neq 0$, and the last conditions coincide with $r_i(\lambda_1, \lambda_2) < 0$ and either $w_i w_{i+1} < 0$ or $w_i = 0, w_{i+1} \neq 0$. Then we can say that the Wronskian $w_i = y_i^T \hat{y}_i$ for solutions $x_i, \hat{x}_i$ of (1.5), (1.6) has a weighted node at $i$ according to the definition in [3] iff $#_i(y, \hat{y}) = \pm 1$.

3. Main results

According to [2, Theorem 2] we have

$$l(Y^{(0)}(\lambda_1)) = \# \{ \lambda \in \sigma_1 | \lambda \leq \lambda_1 \} + p, \quad l(\hat{Y}^{(0)}(\lambda_2)) = \# \{ \lambda \in \sigma_2 | \lambda \leq \lambda_2 \} + \hat{p},$$

(3.1)

where $\sigma_1, \sigma_2$ are the finite spectra of (1.1), (1.2), and $\# \{ \lambda \in \sigma | \lambda \leq \lambda_1 \} = \sum_{\lambda \leq \lambda_1} \theta(\lambda), \# \{ \lambda \in \sigma | \lambda \leq \lambda_2 \} = \sum_{\lambda \leq \lambda_2} \hat{\theta}(\lambda)$, and $\theta(\lambda), \hat{\theta}(\lambda)$ are the multiplicities of $\lambda$ defined by

$$\theta(\lambda) = r_{\max} - \text{rank } X_{N+1}^{(0)}(\lambda), \quad \hat{\theta}(\lambda) = \hat{r}_{\max} - \text{rank } \hat{X}_{N+1}^{(0)}(\lambda),$$

$$r_{\max} = \max_{\lambda}(\text{rank } X_{N+1}^{(0)}(\lambda)), \quad \hat{r}_{\max} = \max_{\lambda}(\text{rank } \hat{X}_{N+1}^{(0)}(\lambda)).$$

(3.2)

By [2], there are always only finitely many finite eigenvalues of (1.1), (1.2); hence for any $\lambda_0$ such that

$$\lambda_0 < \min(\lambda_{\min}, \hat{\lambda}_{\min}), \quad \lambda_{\min} = \min \sigma_1, \quad \hat{\lambda}_{\min} = \min \sigma_2$$

(3.3)

we have

$$p = l(Y^{(0)}(\lambda_0)), \quad \hat{p} = l(\hat{Y}^{(0)}(\lambda_0)), \quad r_{\max} = \text{rank } X_{N+1}^{(0)}(\lambda_0), \quad \hat{r}_{\max} = \text{rank } \hat{X}_{N+1}^{(0)}(\lambda_0).$$

(3.4)

Introduce the notation

$$#^0(Y(\lambda_1), \hat{Y}(\lambda_2)) = \sum_{i=0}^{N} #_i(Y(\lambda_1), \hat{Y}(\lambda_2)) - \sum_{i=0}^{N} #_i(Y(\lambda_0), \hat{Y}(\lambda_0)).$$

(3.5)

where the numbers $#_i(Y, \hat{Y})$ are defined by (2.18), (2.22) and $\lambda_0$ obeys (3.3).

**Theorem 3.** Let $Y^{(0)}_{l_1}, Y^{(N+1)}_{l_1}$ and $\hat{Y}^{(0)}_{l_1}, \hat{Y}^{(N+1)}_{l_1}$ be the principal solutions of (1.1) and (1.2); then we have

$$#^0(Y^{(0)}_{l_1}, \hat{Y}^{(N+1)}_{l_1}(\lambda_2)) = \# \{ \lambda \in \sigma_2 | \lambda \leq \lambda_2 \} - \# \{ \lambda \in \sigma_1 | \lambda \leq \lambda_1 \}.$$

(3.6)

$$#^0(Y^{(N+1)}_{l_1}(\lambda_1), \hat{Y}^{(0)}_{l_1}(\lambda_2)) = \# \{ \lambda \in \sigma_2 | \lambda < \lambda_2 \} - \# \{ \lambda \in \sigma_1 | \lambda < \lambda_1 \}.$$

(3.7)
Proof. From (3.1) we have
\[
\#\{\lambda \in \sigma_2|\lambda \leq \lambda_2\} - \#\{\lambda \in \sigma_1|\lambda \leq \lambda_1\} = \{l(\hat{Y}(0)(\lambda_2)) - l(Y(0)(\lambda_1))\} - \{\hat{p} - p\},
\]  
(3.8)
where by Corollary 2 and (3.4),
\[
l(\hat{Y}(0)(\lambda_2)) - l(Y(0)(\lambda_1)) = \sum_{i=0}^{N} \#_i(Y(0)(\lambda_1), \hat{Y}(N+1)(\lambda_2)),
\]
\[
\hat{p} - p = l(\hat{Y}(0)(\lambda_0)) - l(Y(0)(\lambda_0)) = \sum_{i=0}^{N} \#_i(Y(0)(\lambda_0), \hat{Y}(N+1)(\lambda_0)).
\]
Substituting the last relations in (3.8) and using the notation of (3.5) we complete the proof of (3.6).

Note that (3.1) can be rewritten in the form \(l(Y(0)(\lambda_1)) = \#\{\lambda \in \sigma_1|\lambda < \lambda_1\} + \theta(\lambda_1) + p\), where \(\theta(\lambda_1)\) is the multiplicity of \(\lambda_1\), given by (3.2) and \(r_{\text{max}} = \text{rank} X_{N+1}^{(0)}(\lambda_0)\) due to (3.4). Using (2.14) we have \(l(Y(0)(\lambda_1)) = \text{rank} X_{N+1}^{(0)}(\lambda_1)\) and \(r_{\text{max}} = \text{rank} X_{N+1}^{(0)}(\lambda_0) = \text{rank} X_{N+1}^{(0)}(\lambda_0)\). Then we derive
\[
l''(Y(0)(\lambda_1)) = \#\{\lambda \in \sigma_1|\lambda < \lambda_1\} + p^*.
\]
and a similar relation holds for \(l''(Y(0)(\lambda_2))\). Using (3.9), (2.25) in Corollary 2 for the differences \(l''(Y(0)(\lambda_2)) - l''(Y(0)(\lambda_1))\), and \(p^* - p = l''(Y(0)(\lambda_0)) - l''(Y(0)(\lambda_0))\) we obtain (3.7). The proof is completed. □

From (3.6), (3.7) we also have
\[
\#^\ast(Y(0)(\lambda_1), \hat{Y}(N+1)(\lambda_2)) = \#\{\lambda \in \sigma_2|\lambda < \lambda_2\}.
\]
(3.10)

For the case \(C_i = \hat{C}_i, D_i = \hat{D}_i, W_i = \hat{W}_i, \lambda_2 > \lambda_1\) Theorem 3 presents the number of finite eigenvalues of (1.1) in \([\lambda_1, \lambda_2]\) and \((\lambda_1, \lambda_2)\) [see also [13]].

Theorem 4. Let \(Y(0), Y(N+1)\) be the principal solutions of (1.1) at 0 and \(N+1\) associated with fundamental matrices \(Z_i(0), Z_i(N+1)\) such that (2.6) hold. Then, for \(\lambda_1 < \lambda_2\) we have
\[
l''(Z(0)(\lambda_1)-1Y(N+1)(\lambda_2)) = \#\{\lambda \in \sigma_1|\lambda < \lambda_2\},
\]
\[
l''(Z(N+1)(\lambda_1)-1Y(0)(\lambda_2)) = \#\{\lambda \in \sigma_1|\lambda \leq \lambda_2\},
\]
\[
(3.11)
where \(l''(Z^{-1}(\lambda_1)Y(\lambda_2))\) is the number of focal points in \([0, N+1)\) given by (2.13), (2.29), (2.31), (2.20) for \(R_l(\lambda_1, \lambda_2) = (\lambda_2 - \lambda_1) W_i \geq 0\).

Proof. For the given particular case we have \(p = \hat{p}\); then \(\#^\ast_1(Y(0)(\lambda_1), Y(N+1)(\lambda_2)) = \sum_{i=0}^{N} \#_i(Y(0)(\lambda_1), Y(N+1)(\lambda_2)) = l''(Z(0)(\lambda_1)-1Y(N+1)(\lambda_2)),\) where \(\#_i(Y(0)(\lambda_1), Y(N+1)(\lambda_2)) = m''(Z(0)(\lambda_1)-1Y(N+1)(\lambda_2))\) because of (2.34) for \(R_l = (\lambda_2 - \lambda_1) W_i \geq 0\). Then, the proof of the first relation in (3.11) is completed. The proof of the second relation is similar and follows from (3.7). □

For problem (1.1) and \(\lambda_2 > \lambda_1\) we also obtain from (3.10)
\[
l''(Z(0)(\lambda_1)-1Y(N+1)(\lambda_2)) - \theta(\lambda_2) = l''(Z(N+1)(\lambda_1)-1Y(0)(\lambda_2)) - \theta(\lambda_1) = \#\{\lambda \in \sigma_1|\lambda_1 < \lambda < \lambda_2\}.
\]
(3.12)

Remark 2. For problems (1.5), (1.6) we have \(p = \hat{p} = 0\); then formula (3.10) takes the form
\[
\sum_{i=0}^{N} \#_i(Y(0)(\lambda_1), Y(N+1)(\lambda_2)) - \theta(\lambda_2) = \sum_{i=0}^{N} \#_i(Y(N+1)(\lambda_1), \hat{Y}(N+1)(\lambda_2)) - \theta(\lambda_1)
\]
\[
= \#\{\lambda \in \sigma_2|\lambda < \lambda_2\} - \#\{\lambda \in \sigma_1|\lambda \leq \lambda_1\},
\]
where \(\theta(\lambda_2) = 1\) iff \(\lambda_2\) is an eigenvalue of (1.6). The last condition is equivalent to \(w_0 = w_0(y(0)(\lambda_1), \hat{x}(N+1)(\lambda_2)) = 0\) if \(l''(Y(N+1)(\lambda_2)) = 0\), and, by Remark 1, we see that \(\sum_{i=0}^{N} \#_i(Y(0)(\lambda_1), Y(N+1)(\lambda_2)) - \theta(\lambda_2) = \#(k(0)(\lambda_1), \hat{x}(N+1)(\lambda_2)),\) where \(\#(k(0)(\lambda_1), \hat{x}(N+1)(\lambda_2))\) is the number of the weighted nodes of the Wronskian in the open interval \((0, N+1)\) according to (1.9) in [3]. Similarly, it can be shown that \(\sum_{i=0}^{N} \#_i(Y(N+1)(\lambda_1), \hat{Y}(0)(\lambda_2)) - \theta(\lambda_1) = \#(\lambda(N+1)(\lambda_1), \hat{x}(N+1)(\lambda_2));\) then, for problems (1.5), (1.6) formula (1.8) follows from (3.10). Arguing as above, we also see that (1.7) follows from (3.12).

References