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A note on convexity and semicontinuity of fuzzy mappings[☆,☆☆]

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Abstract

By using parameterized representation of fuzzy numbers, criteria for a lower semicontinuous fuzzy mapping defined on a non-empty convex subset of R^n to be a convex fuzzy mapping are obtained.

Keywords: Convexity; Fuzzy mapping; Fuzzy numbers; Convex fuzzy mapping; Semicontinuity

1. Introduction

Convexity and semicontinuity of fuzzy mappings play central roles in fuzzy mathematics and fuzzy optimization. The concept of convex fuzzy mappings defined through the "fuzz-max" order on fuzzy numbers were studied by several authors, including Furukawa [1], Nanda [2], Syau [3,4], and Wang and Wu [5], aiming at applications to fuzzy nonlinear programming. The concept of upper and lower semicontinuity of fuzzy mappings based on the Hausdorff separation was introduced by Diamond and Kloeden [6]. Recently, Bao and Wu [7] introduced a new concept of upper and lower semicontinuity of fuzzy mappings through the "fuzz-max" order on fuzzy numbers, and obtained the criteria for convex fuzzy mappings under upper and lower semicontinuity conditions, respectively. In an earlier paper [8], we redefined the upper and lower semicontinuity of fuzzy mappings of Bao and Wu [7] by using the concept of parameterized triples of fuzzy numbers.

Bao and Wu [7] established the criteria for a lower semicontinuous fuzzy mapping defined on a non-empty closed convex subset, say C, of R^n to be a convex fuzzy mapping. In this paper, by using parameterized representation of fuzzy numbers, we give the criteria for a lower semicontinuous fuzzy mapping defined on a non-empty convex subset of R^n to be a convex fuzzy mapping. In other words, we deleted the requirement of the closed condition on C. That is, the set C only needs to be a non-empty convex subset of R^n.

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2. Preliminaries

In this section, for convenience, several definitions and results without proof from [3,7–12] are summarized below.

Let R^n denote the n-dimensional Euclidean space. In what follows, let S be a non-empty subset of R^n. For any $x \in R^n$ and $\delta > 0$, let

$$B_\delta(x) = \{y \in R^n : \|y - x\| < \delta\},$$

where $\|\cdot\|$ being the Euclidean norm on R^n.

First, we recall the definitions of upper and lower semicontinuous real-valued functions.

Definition 2.1. A real-valued function $f : S \to R^1$ is said to be

(1) upper semicontinuous at $x_0 \in S$ if $\forall \varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$f(x) < f(x_0) + \varepsilon \quad \text{whenever } x \in S \cap B_\delta(x_0).$$

f is upper semicontinuous on S if it is upper semicontinuous at each point of S.

(2) lower semicontinuous at $x_0 \in S$ if $\forall\, \varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$f(x) > f(x_0) - \varepsilon \quad \text{whenever } x \in S \cap B_\delta(x_0).$$

f is lower semicontinuous on S if it is lower semicontinuous at each point of S.

The support, $\text{supp}(\mu)$, of a fuzzy set $\mu : R^n \to I = [0, 1]$ is defined as

$$\text{supp}(\mu) = \{x \in R^n : \mu(x) > 0\}.$$

A fuzzy set $\mu : R^n \to I$ is normal if $[\mu]_1 \neq \emptyset$. A fuzzy number we treat in this study is a fuzzy set $\mu : R^1 \to I$ which is normal, has bounded support, and is upper semicontinuous and quasiconcave as a function on its support.

Let $\alpha \in I$. The α-level set of a fuzzy set $\mu : R^n \to I$, denoted by $[\mu]_\alpha$, is defined as

$$[\mu]_\alpha = \begin{cases} \{x \in R^n : \mu(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1; \\ \text{cl}(\text{supp}(\mu)), & \text{if } \alpha = 0, \end{cases}$$

where $\text{cl}(\text{supp}(\mu))$ denotes the closure of $\text{supp}(\mu)$.

Denote by $\mathcal{F}$ the set of all fuzzy numbers. In this paper, we consider mappings F from a non-empty subset of R^n into $\mathcal{F}$. We call such a mapping a fuzzy mapping. It is clear that each $r \in R^1$ can be considered as a fuzzy number $\tilde{r}$ defined by

$$\tilde{r}(t) = \begin{cases} 1, & \text{if } t = r; \\ 0, & \text{if } t \neq r, \end{cases}$$

hence, each real-valued function can be considered as a fuzzy mapping.

It can be easily verified [9] that a fuzzy set $\mu : R^1 \to I$ is a fuzzy number if and only if (i) $[\mu]_\alpha$ is a closed and bounded interval for each $\alpha \in I$, and (ii) $[\mu]_1 \neq \emptyset$. Thus we can identify a fuzzy number μ with the parameterized triples

$$\{(a(\alpha), b(\alpha), \alpha) : \alpha \in I\},$$

where $a(\alpha)$ and $b(\alpha)$ denote the left- and right-hand endpoints of $[\mu]_\alpha$, respectively, for each $\alpha \in I$.

Definition 2.2. Let μ and ν be two fuzzy numbers represented parametrically by

$$\{(a(\alpha), b(\alpha), \alpha) : \alpha \in I\} \quad \text{and} \quad \{(c(\alpha), d(\alpha), \alpha) : \alpha \in I\},$$

respectively. We say that $\mu \preceq \nu$ if

$$a(\alpha) \leq c(\alpha) \quad \text{and} \quad b(\alpha) \leq d(\alpha) \text{ for each } \alpha \in I.$$

We call $\preceq$ the fuzz-max order on $\mathcal{F}$.

We see that $\mu = \nu$, if $\mu \preceq \nu$ and $\nu \preceq \mu$. We say that $\mu \prec \nu$, if $\mu \preceq \nu$ and there exists $\alpha_0 \in [0, 1]$ such that $a(\alpha_0) < c(\alpha_0)$ or $b(\alpha_0) < d(\alpha_0)$.

For fuzzy numbers μ and ν parameterized by

$$\{(a(\alpha), b(\alpha), \alpha) : \alpha \in I\} \quad \text{and} \quad \{(c(\alpha), d(\alpha), \alpha) : \alpha \in I\},$$

respectively, and each nonnegative real number k, we define the addition $\mu + \nu$ and nonnegative scalar multiplication $k\mu$ as follows:

$$\mu + \nu = \{(a(\alpha) + c(\alpha), b(\alpha) + d(\alpha), \alpha) : \alpha \in I\} \tag{2.1}$$

$$k\mu = \{(ka(\alpha), kb(\alpha), \alpha) : \alpha \in I\}. \tag{2.2}$$

It is known that the addition and nonnegative scalar multiplication on $\mathcal{F}$ defined by (2.1) and (2.2) are equivalent to those derived from the usual extension principle, and that $\mathcal{F}$ is closed under the addition and nonnegative scalar multiplication.

Remark 2.1. It is obvious from (2.1) that for each fuzzy number μ parameterized by $\{(a(\alpha), b(\alpha), \alpha) : \alpha \in I\}$ and each real number r,

$$\mu + r = \mu + \tilde{r} = \{(a(\alpha) + r, b(\alpha) + r, \alpha) : \alpha \in I\}. \tag{2.3}$$

Now, we recall the concept of convexity and weak convexity of fuzzy mappings defined through the fuzz-max order on $\mathcal{F}$.

Definition 2.3 ([2,10]). Let C be a non-empty convex subset of R^n. A fuzzy mapping $F : C \to \mathcal{F}$ is said to be

(1) convex if for every $\lambda \in [0, 1]$ and $x, y \in C$,

$$F(\lambda x + (1 - \lambda)y) \preceq \lambda F(x) + (1 - \lambda)F(y).$$

(2) weakly convex, if for all $x, y \in C$, there exists a $\lambda \in (0, 1)$ (λ depends on x, y) such that

$$F(\lambda x + (1 - \lambda)y) \preceq \lambda F(x) + (1 - \lambda)F(y).$$

We recall Bao and Wu's definition of upper and lower semicontinuous fuzzy mappings.

Definition 2.4 ([7]). A fuzzy mapping $F : S \to \mathcal{F}$ is said to be

(1) upper semicontinuous at $x_0 \in S$ if $\forall\, \varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$F(x) \preceq F(x_0) + \tilde{\varepsilon} \quad \text{whenever } x \in S \cap B_\delta(x_0).$$

$F : S \to \mathcal{F}$ is upper semicontinuous if it is upper semicontinuous at each point of S.

(2) lower semicontinuous at $x_0 \in S$ if $\forall \varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$F(x_0) \preceq F(x) + \tilde{\varepsilon} \quad \text{whenever } x \in S \cap B_\delta(x_0).$$

$F : S \to \mathcal{F}$ is lower semicontinuous if it is lower semicontinuous at each point of S.

Theorems 2.1 and 2.2 for upper and lower, respectively, semicontinuous fuzzy mappings can be easily derived from (2.3).

Theorem 2.1 ([8]). *Let $F : S \to \mathcal{F}$ be a fuzzy mapping parameterized by*

$$F(x) = \{(a(\alpha, x), b(\alpha, x), \alpha) : \alpha \in I\}, \quad \forall x \in S.$$

The following conditions are equivalent:

(1) *F is upper semicontinuous at $x_0 \in S$.*

(2) *$a(\alpha, x)$ and $b(\alpha, x)$ are upper semicontinuous at x_0 uniformly in $\alpha \in I$.*

Theorem 2.2 (*[8]*). *Let $F : S \to \mathcal{F}$ be a fuzzy mapping parameterized by*

$$F(x) = \{(a(\alpha, x), b(\alpha, x), \alpha) : \alpha \in I\}, \quad \forall\, x \in S.$$

The following conditions are equivalent:
(1) *F is lower semicontinuous at $x_0 \in S$.*
(2) *$a(\alpha, x)$ and $b(\alpha, x)$ are lower semicontinuous at x_0 uniformly in $\alpha \in I$.*

Finally, we recall two important results concerning convex functions.

Theorem 2.3 (*[11]*). *Let C be a non-empty convex subset of R^n, and let $f : C \to R^1$ be a lower semicontinuous function. If for all $x, y \in C$, there exists a $\lambda \in (0, 1)$ (λ depends on x, y) such that*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y),$$

then f is a convex function on C.

Theorem 2.4 (*[12]*). *Let C be a non-empty convex subset of R^n, and let $f : C \to R^1$ be an upper semicontinuous function. If, there exists a $\lambda \in (0, 1)$ such that*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \quad \forall x, y \in C,$$

then f is a convex function on C.

3. Main results

In what follows, let C be a non-empty convex subset of R^n. Motivated by Theorem 2.4 and the concept of weakly convex fuzzy mappings proposed in [10], we propose the concept of intermediate-point convex fuzzy mappings in Definition 3.1.

Definition 3.1. Let C be a non-empty convex subset of R^n. A fuzzy mapping $F : C \to \mathcal{F}$ is said to be intermediate-point convex if, there exists a $\lambda \in (0, 1)$ such that

$$F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda) F(y), \quad \forall x, y \in C.$$

We establish a characterization of convex fuzzy mappings in parametric representation in the following result.

Theorem 3.1. *Let C be a non-empty convex subset of R^n, and let $F : C \to \mathcal{F}$ be a fuzzy mapping parameterized by*

$$F(x) = \{(a(\alpha, x), b(\alpha, x), \alpha) : \alpha \in I\}, \quad \forall x \in C. \tag{3.1}$$

Then F is convex on C if and only if for each $\alpha \in [0, 1]$,

$$a(\alpha, x) \quad \text{and} \quad b(\alpha, x) \text{ are convex with respect to } x \text{ on } C. \tag{3.2}$$

Proof. Assume that for each $\alpha \in [0, 1]$,

$$a(\alpha, x) \quad \text{and} \quad b(\alpha, x) \text{ are convex with respect to } x \text{ on } C.$$

Let $\alpha \in [0, 1]$ be given. From (3.2), we have

$$a(\alpha, \lambda x + (1-\lambda)y) \le \lambda a(\alpha, x) + (1-\lambda) a(\alpha, y)$$

and

$$b(\alpha, \lambda x + (1-\lambda)y) \le \lambda b(\alpha, x) + (1-\lambda) b(\alpha, y)$$

for all $x, y \in C$ and $\lambda \in [0, 1]$. Then, by (3.1), (2.1) and (2.2), we obtain

$$\begin{aligned} F(\lambda x + (1-\lambda)y) &= \{(a(\alpha, \lambda x + (1-\lambda)y), b(\alpha, \lambda x + (1-\lambda)y), \alpha) : \alpha \in I\} \\ &\preceq \{(\lambda a(\alpha, x), \lambda b(\alpha, x), \alpha) : \alpha \in I\} + \{((1-\lambda)a(\alpha, y), (1-\lambda)b(\alpha, y), \alpha) : \alpha \in I\} \\ &= \lambda F(x) + (1-\lambda) F(y), \end{aligned}$$

for all $x, y \in C$ and $\lambda \in [0, 1]$. Hence F is convex on C.

Conversely, let F be convex on C. Then for every $\lambda \in [0, 1]$ and $x, y \in C$, we have

$$F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda)F(y).$$

From (3.1), we have

$$F(\lambda x + (1-\lambda)y) = \{(a(\alpha, \lambda x + (1-\lambda)y), b(\alpha, \lambda x + (1-\lambda)y), \alpha) : \alpha \in I\} \tag{3.3}$$

for every $\lambda \in [0, 1]$ and $x, y \in C$. From (3.1), (2.1) and (2.2), we obtain

$$\begin{aligned}\lambda F(x) + (1-\lambda)F(y) &= \{(a(\alpha, \lambda x), b(\alpha, \lambda x), \alpha) : \alpha \in I\} + \{(a(\alpha, (1-\lambda)y), b(\alpha, (1-\lambda)y), \alpha) : \alpha \in I\} \\ &= \{(\lambda a(\alpha, x) + (1-\lambda)a(\alpha, y), \lambda b(\alpha, x) + (1-\lambda)b(\alpha, y), \alpha) : \alpha \in I\},\end{aligned}$$

for all $x, y \in C$ and $\lambda \in [0, 1]$. Then, by (3.3) and the convexity of F, we have for every $\lambda \in [0, 1]$ and $x, y \in C$,

$$a(\alpha, \lambda x + (1-\lambda)y) \le \lambda a(\alpha, x) + (1-\lambda)a(\alpha, y)$$

and

$$b(\alpha, \lambda x + (1-\lambda)y) \le \lambda b(\alpha, x) + (1-\lambda)b(\alpha, y)$$

for each $\alpha \in [0, 1]$. Hence, we conclude that

$a(\alpha, x)$ and $b(\alpha, x)$ are convex with respect to x on C.

This completes the proof. □

Theorem 3.2. *Let C be a non-empty convex subset of R^n, and let $F : C \to \mathcal{F}$ be a lower semicontinuous fuzzy mapping. If for all $x, y \in C$, there exists a $\lambda \in (0, 1)$ (λ depends on x, y) such that*

$$F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda)F(y), \tag{3.4}$$

then F is a convex fuzzy mapping on C.

Proof. Let

$$F(x) = \{(a(\alpha, x), b(\alpha, x), \alpha) : \alpha \in I\}, \quad \forall x \in C, \tag{3.5}$$

be the parametric representation of the fuzzy mapping $F : C \to \mathcal{F}$. Since $F : C \to \mathcal{F}$ is lower semicontinuous, by Theorem 2.2, we have for each $x \in C$, both

$a(\alpha, x)$ and $b(\alpha, x)$ are lower semicontinuous at x uniformly in $\alpha \in I$. (3.6)

In view of (3.4) and (3.5) it can be written as for all $x, y \in C$, there exists a $\lambda \in (0, 1)$ (λ depends on x, y) such that

$$a(\alpha, \lambda x + (1-\lambda)y) \le \lambda a(\alpha, x) + (1-\lambda)a(\alpha, y) \tag{3.7}$$

and

$$b(\alpha, \lambda x + (1-\lambda)y) \le \lambda b(\alpha, x) + (1-\lambda)b(\alpha, y) \tag{3.8}$$

for each $\alpha \in [0, 1]$. Combining (3.6), (3.7), and Theorem 2.3, we have

$a(\alpha, x)$ is convex with respect to x on C. (3.9)

Similarly, combining (3.6), (3.8), and Theorem 2.3, we have

$b(\alpha, x)$ is convex with respect to x on C. (3.10)

From (3.9), (3.10), and Theorem 3.1, it follows that F is a convex fuzzy mapping on C. □

Similarly, by Theorems 2.1 and 2.4, we obtain an analogous result to Theorem 3.2 for the case of upper semicontinuous fuzzy mappings:

Theorem 3.3. *Let C be a non-empty convex subset of R^n, and let $F : C \to \mathcal{F}$ be an upper semicontinuous fuzzy mapping. If, there exists a* $\lambda \in (0, 1)$ *such that*

$$F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda)F(y) \quad \forall x, y \in C,$$

then F is a convex fuzzy mapping on C.

From Part (2) of Definition 2.3, and Theorem 3.2, we obtain the following result:

Theorem 3.4. *Let C be a non-empty convex subset of R^n, and let $F : C \to \mathcal{F}$ be a lower semicontinuous fuzzy mapping. Then, F is a convex fuzzy mapping on C if and only if it is a weakly convex fuzzy mapping on C.*

Similarly, by Definition 3.1 and Theorem 3.3, we obtain:

Theorem 3.5. *Let C be a non-empty convex subset of R^n, and let $F : C \to \mathcal{F}$ be an upper semicontinuous fuzzy mapping. Then, F is a convex fuzzy mapping on C if and only if it is an intermediate-point convex fuzzy mapping on C.*

Recall that, the epigraph of a fuzzy mapping $F : S \to \mathcal{F}$, denoted by $\mathrm{epi}(F)$, is defined as

$$\mathrm{epi}(F) = \{(x, \mu) : x \in S, \mu \in \mathcal{F}, F(x) \preceq \mu\}.$$

Theorem 3.4, combined with Theorem 3.3 in [10], implies the following result.

Theorem 3.6. *Let C be a non-empty convex subset of R^n, and let $F : C \to \mathcal{F}$ be a lower semicontinuous fuzzy mapping. The following conditions are equivalent.*

(1) *$F : C \to \mathcal{F}$ is a convex fuzzy mapping.*

(2) *for all (x, μ), $(y, \nu) \in \mathrm{epi}(F)$, with $x, y \in C$ and $\mu, \nu \in \mathcal{F}$,*

$$\{\gamma(x, \mu) + (1-\gamma)(y, \nu) | 0 < \gamma < 1\} \cap \mathrm{epi}(F) \neq \emptyset.$$

(3) *for all $x, y \in C$, there exists a $\lambda \in (0, 1)$ (λ depends on x, y) such that*

$$F(\lambda x + (1-\lambda)y) \prec \lambda\mu + (1-\lambda)\nu$$

whenever $\mu, \nu \in \mathcal{F}_0$, $F(x) \prec \mu$, $F(y) \prec \nu$.

(4) *for all (x, μ), $(y, \nu) \in G(F)$ with $x, y \in C$ and $\mu, \nu \in \mathcal{F}$,*

$$\{\gamma(x, \mu) + (1-\gamma)(y, \nu) | 0 < \gamma < 1\} \cap G(F) \neq \emptyset,$$

where $G(F) = \{(x, \mu) : x \in C, F(x) \prec \mu\}$.

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