On the estimation of the boundary temperature on a sphere from measurements at its center

ABSTRACT

An improperly-posed inverse heat conduction problem with radial symmetry is stabilized by reconstructing a mollified image of the unknowns. Strict error bounds are derived and after introducing certain inverse kernels, the numerical problem is solved with an absolute minimum of computation.

1. INTRODUCTION

In this paper we consider a transient heat conduction problem on a sphere in which, after measuring the transient temperature history $F(t)$ at its center, we would like to recover the boundary temperature $f(t)$ and heat flux $q(t)$.

The temperature history $F(t)$ is approximately measurable for all $t$ in $(-\infty, \infty)$.

This inverse problem is an improperly-posed problem in the sense that there are no decent norms for the data and solutions such that the solution depends continuously upon the data.

The inverse problem can be stabilized by restricting attention to those solutions satisfying certain prescribed global bounds, see Miller [2] or Miller and Viano [3] for example. However, it is also possible to restore certain types of continuous dependence on data if, instead of attempting to find point values of $f$ (or $q$), we content ourselves with attempting to reconstruct a slightly "blurred" image of $f$. One natural functional is $J_\delta f$ the $\delta$-mollification" of $f$, that is, the convolution of $f$ with the Gaussian kernel $\rho_\delta$ of "blurring radius" $\delta$. Such an approach was taken by Manselli and Miller [1] on a related problem. See also Murio [4] for a favorable comparison of this method against others commonly in use when applied to the numerical solution of the heat equation for the semi-infinite one-dimensional body problem. In this paper, the unknowns will be assumed to be $\beta$-periodic $N$th order trigonometric sums over the interval of interest instead of the approximation mentioned in [4].

Other approaches, using finite difference methods, are studied by Frank [5] and Davies [6]. Burggraf [7] uses a truncation of the classical power series method for attempting to solve the Cauchy initial value problem; however, for this to be convergent $F(t)$ would have to be analytic and the method would very much amplify errors in the data. An integral equation method which, when discretized allows a step by step recursive calculation of the solution, but becomes unstable if the time intervals are made small, was introduced by Stolz [8]. Integral equation methods are also used by Sparrow et al. [9] and by Beck [10], [11] who, however, adds the distinct improvement of allowing the least squares use of several future data points to compute the solution at the present time.

In the papers mentioned above, the assumptions on the solutions and the choice of parameters which help restore stability are not usually clearly stated and the consequent continuity with respect to the data is not adequately studied.

More recently, Hills and Hullholland [12] presented a method where the ill-posed problem is analyzed and solved after introducing a pseudo-inverse operator and an averaging kernel. This is a global method and if the solution is needed over a long period of time the dimension of the matrices or the number of computations involved might increase very rapidly.

In contrast, for the linear inverse problem treated in this paper, the inverse kernels of section 4 lead very naturally to a step by step method and allow us to find the numerical solution with a minimum of computation. In a completely different setting, we would like to mention that Weber [13] has developed an interesting special procedure which replaces the heat conduction equation with an approximating hyperbolic equation, obtaining a well-posed problem for which numerical procedures are already available.

In section 2, we present the nondiscrete version of the problem with data specified on a continuum of time $t$ and data error measured in the $L^2$ norm, and derive stability bounds for the inverse problem.

Section 3 is devoted to the discretized version of the problem of section 2, involving data at only a discrete sampling of times.

In section 4 we introduce certain direct and inverse convolution kernels with which we shall compute our numerical solution and analyze the solution error. We also present some numerical results of interest.

2. DESCRIPTION OF THE PROBLEM

We consider a sphere in which the transient temperature...
and heat flux are radial functions and after measuring the temperature history \( F(t) \) at the center \((r=0)\), we would like to recover the boundary temperature \( f(t) \) and heat flux \( q(t) \).

We assume linear heat conduction with constant coefficients. After appropriate changes in the radial and time scales we may consider without loss of generality the normalized problem, with constant conductivity 1 and heat capacity 1 on the three dimensional unit sphere centered at the origin.

In spherical polar coordinates the problem can be described mathematically as follows:

The unknown temperature \( v(r,t) \) satisfies
\[
2V_r + V_{rr}; \quad 0 < r < 1; \quad -\infty < t < \infty. \tag{2.1}
\]

\[ v(0,t) = F(t), \tag{2.2} \]
with corresponding approximate data function \( F(t) \).

\[ v(0,t) = Q(t) = 0; \quad -\infty < t < \infty. \tag{2.3} \]

\[ v(1,t) = f(t), \tag{2.4} \]
the desired but unknown temperature function.

\[ v_r(1,t) = q(t), \tag{2.5} \]
the desired but unknown heat flux function.

Under the transformation
\[ u(r,t) = rv(r,t), \tag{2.6} \]
\[ u_r(r,t) = v(r,t) + rV_r(r,t). \tag{2.7} \]

The system (2.1)-(2.5) becomes
\[ u_t = u_{rr}; \quad 0 < r < 1; \quad -\infty < t < \infty. \tag{2.8} \]

\[ u(0,t) = 0; \quad -\infty < t < \infty. \tag{2.9} \]

\[ u_r(0,t) = F(t), \tag{2.10} \]
with corresponding approximate data function \( F(t) \).

\[ u(1,t) = f(t), \tag{2.11} \]
the desired but unknown temperature function.

\[ u_r(1,t) = f(t) + q(t) = g(t), \tag{2.12} \]
the desired but unknown heat flux function.

\[ F(t) = [\cosh(\mu + i\omega t)]^{-1} g(t). \tag{2.13} \]

From (2.12), (2.13), and (2.14) it follows that

\[ F(t) = Bq(t) = [\cosh(\mu + i\omega t) - \sinh(\mu + i\omega t)/(\mu + i\omega t)]^{-1} q(t). \tag{2.15} \]

This shows that the inverse problem attempting to go from \( F \) to \( q \) is also ill-posed in the high frequency components.

We shall concentrate in the analysis of the computation of \( f(t) \), satisfying (2.13), for which special methods must be used. Of course, the same techniques can be applied for \( q(t) \) in (2.15).

2.1. The stabilized inverse problem

For the moment, in order to use Fourier integral analysis, we are going to assume that all functions involved are \( L^2 \) functions on the whole line \((-\infty, \infty)\) and we will use the corresponding \( L^2 \) norm to measure errors. This is rather unnatural since in many applications one might expect the temperature to never tend to 0 as \( t \to \pm \infty \), but to oscillate about in bounded fashion forever. Nevertheless, this assumption will be later loosened by switching to \( L^2 \) norms on bounded intervals of interest.

In what follows, \( f(w) \) denotes the Fourier transform of \( f(t) \).

\[ f(w) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{-iwt} dt \tag{2.16} \]

We assume only a known \( L^2 \) global error bound on \( f \).

\[ \|f\| \leq E \text{ or } [2\pi \int_{-\infty}^{+\infty} |f(w)|^2 dw]^{1/2} \leq E, \tag{2.17} \]

and since there is nothing that adequately forces down the high frequency part of \( f(w) \), we seek to reconstruct some useful functional of \( f \) which strongly damps the high frequency part of \( f(w) \). One such function is \( \rho \delta \) the "\( \delta \)-mollification" of \( f \) at time \( t \), defined as

\[ \rho \delta f(t) = (\rho \delta * f)(t) = \int_{-\infty}^{+\infty} \rho \delta (t-s) f(s) ds, \tag{2.18} \]

where

\[ \rho \delta (t) = (\delta \sqrt{\pi})^{-1} e^{-t^2/\delta^2} \tag{2.19} \]

is the Gaussian kernel of "blurring radius \( \delta \)".

The fact that among all possible mollification kernels of "spread" \( \delta \), the Gaussian kernel has the smallest spread of its Fourier transform makes it a natural choice to work with.
The Fourier transform of $J_\delta f$ is given by
\[ \hat{J}_\delta f(w) = 2\pi \rho_\delta \hat{f}(w). \] (2.20)
This mollification damps those Fourier components of $f$ with wavelength $2\pi/w$ much shorter than $2\pi\delta$; the longer wavelengths are damped hardly at all. Moreover, $J_\delta f(t) = J_\delta f(0)$ at $t = 0$ gives
\[ J_\delta f(0) = \int_{-\infty}^{\infty} \rho_\delta (-s) \hat{f}(s) ds = \int_{-\infty}^{\infty} e^{-w^2\delta^2/4} \hat{f}(w) dw. \] (2.21)
We thus have the following stabilized problem:
Attempt to find the linear function $J_\delta f(t)$ at some time of interest and for some assigned blurring radius $\delta$, given that $f$ is a particular function satisfying
\[ \|Af - \hat{f}\| < \epsilon \] (2.22)
\[ \|f\| \leq E. \] (2.23)
It will turn out in the stability analysis that the bound (2.23) is not necessary to stabilize this problem; the data error bound (2.22) by itself is sufficient to assure Lipschitz continuous dependence on the data as $\epsilon$ tends to zero, provided we keep $\delta$ fixed. However, the prescribed bound (2.23) can actually aid the stability in the case of $\epsilon$ which are not small, or in the case of data at only discrete sampling points in a limited data interval, as will occur in the physically practical numerical methods of section 4.

2.2. Stability analysis
The problem is now in a form that can be solved by the method of least squares, see Miller [21 and [3]. This method is a "nearly-best-possible-method" in the sense that for any seminorm $<,>$ which might be used to measure the error, it gives an approximation $\hat{f}$ to $f$ which satisfies the error bound
\[ <\hat{f} - f> \leq 2M(\epsilon, E), \] (2.24)
where $M(\epsilon, E)$ is the "best-possible-stability-bound"
\[ M(\epsilon, E) = \sup \{<f>: \|Af\| \leq \epsilon; \|f\| \leq E\}. \] (2.25)
If $\hat{f}$ satisfies (2.21) and (2.22) it also satisfies
\[ \|\hat{F} - Af\|^2 + (\frac{\epsilon}{E})^2 \|\hat{F}\|^2 \leq 2\epsilon^2 \] (2.26)
and we have lost at most a factor of $\sqrt{2}$ going from the two constraints to the one. Let our approximation $\hat{f}$ be chosen such as to minimize
\[ \|\hat{F} - Af\|^2 + (\frac{\epsilon}{E})^2 \|\hat{F}\|^2. \] (2.27)
The canonical equation for this minimization is given by
\[ (A^*A + (\frac{\epsilon}{E})^2 I) \hat{f} = A^* \hat{F}, \] (2.28)
where $A^*$ indicates the transpose conjugate of the matrix $A$. We can now derive an estimate for $M(\epsilon, E)$ for the linear functional $J_\delta f(t)$. We may of course assume the time of interest to be $\xi = 0$.
We want the supremum of
\[ \|J_\delta f(0)\| = \|\int_{-\infty}^{\infty} e^{-w^2\delta^2/4} \hat{f}(w) dw\| \] (2.29)
with respect to the two constraints
\[ \|Af\|^2 = 2\pi \int_{-\infty}^{\infty} \left| \frac{\mu + i\nu}{\sinh (\mu + i\nu)} \right|^2 |\hat{f}(w)|^2 dw \leq \epsilon^2 \] (2.30)
and
\[ \|f\|^2 = 2\pi \int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw \leq E^2 \] (2.31)
However, it is sufficient to bound (2.29) by the single constraint (2.30) alone. Using the Cauchy inequality, we obtain
\[ \|J_\delta f(0)\| \leq \epsilon \left( \int_{-\infty}^{\infty} \left| e^{-w^2\delta^2/4} \frac{\sinh (\mu + i\nu)}{(\mu + i\nu)} \right|^2 dw \right)^{1/2} \]
\[ \leq \epsilon \pi^{1/2} \int_{0}^{\infty} e^{-w^2\delta^2/2} e^{2\sqrt{\nu} dw} \] (2.32)
Since $e^{-w^2\delta^2/2}$ is about .6 for $w \leq w_1 \equiv 1/\delta$ and falls rapidly to zero for $w > w_1$, while on the other hand $e^{2\sqrt{\nu}}$ grows only slowly with $w$, it follows that
\[ \|J_\delta f(0)\| \leq \epsilon \pi^{1/2} \int_{0}^{w_1} e^{2\sqrt{\nu} dw} \] (2.33)
which as $\epsilon \to 0$ becomes the best possible bound but for a factor of two, for fixed $\delta$.
This shows that the error can be guaranteed to go down in Lipschitz fashion as $\epsilon -\to 0$ for fixed $\delta$.

3. DISCRETIZED PROBLEM
In this section we assume that $f$ is locally $L^2$ bounded, uniformly on every sufficiently long interval, and that the data for $Af$ is measured at a discrete set of equally spaced data points in some finite interval of length $\beta$; we then seek to reconstruct $J_\delta f(\xi)$ at some point $\xi$ approximately opposite the middle of the data set. If we choose our point of interest to be $\xi = 0$, the data set consists of $K$ points $d_1, d_2, ..., d_K$ in the $[-\beta/2, \beta/2]$ interval, with equal spacing $A \delta = \beta/(K-1)$. The data function $\hat{F}$ is a discrete function measured at these sampling points. The interval $[-\beta/2, \beta/2]$ should contain all the data for $Af$ which might reasonably be expected to enter into the reconstruction of $f$ at time $\xi = 0$, provided we make $\beta$ sufficiently large. If that is the case, since the operator $A$ makes $Af$ so smooth, we have reason to believe that a discrete sampling of $Af$ in $[-\beta/2, \beta/2]$ contains just as much information as a continuous sampling, provided that the sampling interval $\Delta t$ is made sufficiently small. $\delta f$ is given by the integral operator
\[ J_\delta f(t) = \int_{-1}^{1} P(t-s)f(s) ds \] (3.1)
where the kernel function $P(t)$ has the Fourier series
\[ P(t) = \begin{cases} 2\pi^2 \sum_{n=1}^{\infty} (-1)^{n+1} n^2 \exp(-n^2 \frac{t}{\delta}) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \] (3.2)

Journal of Computational and Applied Mathematics, volume 8, no 2, 1982 113
The kernel function shows that $A_f(t)$ depends strongly on almost the entire "past history of $f$". This means that $\beta/2$ should be taken so large that the number of points in the discrete data sampling interval will become enormous when the sampling interval $\Delta t$ is made sufficiently small.

In order to avoid this situation, we subtract from the actual data $A_f(t)$ the influence of the past history of $f$. Of course, this implies the knowledge of $f$ for all previous times. Because of this difficulty, we will lower our goal for the moment and assume as an alternative,

$$A_f(t) \equiv 0 \quad \text{for } t \leq 0. \quad (3.3)$$

The choice of $\beta$ sufficiently large, certainly allows us to rigorously approximate our problem by a completely discretized one with $f$ replaced by a $\beta$-periodic trigonometric sum of the form

$$f(t) = \sum_{N} x_j \exp(\text{i}w_j \cdot t), \quad \text{with } w_j = (2 \pi / \beta) j \quad (3.4)$$

Since the operators $A$ and $J_\beta$ are very smoothing, it is easy to pick $N$ sufficiently large such that $A_f(t) - J_\beta f$ is negligibly small at the reconstruction point of interest. Thus, any unknown function $f$ satisfying (2.21), (2.22), and (3.3) can be well approximated by some finite sum of the form (3.4), in the precise sense that

(i) $J_\beta f - J_\beta f$ is negligibly small at the time $\xi$ of interest
(ii) $\|F - A_f\|_{[-\beta/2, \beta/2]} < 1.1 e$
(iii) $\|F - A_f\|_{[-\beta/2, \beta/2]} \leq E$
(iv) $F = 0$ on $[-\beta/2, 0]$

In the following we are going to wipe out all distinctions in the notation between $\epsilon$ and $1.1e$, $f$ and $f^\prime$ and we will concentrate on solving the discretized problem.

3.1. Numerical method

Given a function $f$ on $[-\beta/2, \beta/2]$ of the form (3.4) satisfying

$$\|A_f - F\|_{[-\beta/2, \beta/2]} \leq \epsilon, \quad (3.6)$$

with $F = 0$ on $[-\beta/2, 0]$, and

$$\|f\|_{[-\beta/2, \beta/2]} \leq E, \quad (3.7)$$

We wish to approximately determine the linear function $J_\beta f(\xi)$ at the point of interest $\xi = 0$.

Our least squares approximation is choose to be that element $f^0$ of the form (3.4) (with coefficient vector $x^0$), which minimizes

$$\phi(f) = \|A_f - F\|_{[-\beta/2, \beta/2]}^2 + \epsilon x^2 \|f\|_{[-\beta/2, \beta/2]}^2 \quad (3.8)$$

Next, we write the quadratic functional $\phi(f)$ in terms of the coefficient vector $x$ of $f$.

Thus, the least squares problem becomes minimize $\phi(x) = \|Hx - h\|_K^2 + \epsilon x^2 \|R^{1/2}x\|_K^2 \quad (3.9)$

where $h$ is the vector with elements

$$h_k = F(d_k) / \sqrt{K} \quad k = 1, \ldots, K. \quad (3.10)$$

$H$ is the $K \times (2N+1)$ matrix with entries

$$H_{kj} = K^{-1/2} e^{-i k \cdot k \cdot j} \sinh (\mu_j + i o_j) j \quad (3.11)$$

$k = 1, \ldots, K; j = -N, \ldots, N$.

$R$ is the $K \times (2N+1)$ matrix with entries

$$R_{kj} = K^{-1/2} e^{-i k \cdot k \cdot j} \quad (3.12)$$

$k = 1, \ldots, K; j = -N, \ldots, N$.

The vector $x^0$ minimizing (3.9) is the solution of the normal equations

$$Zx^0 = (H*H + \epsilon I)^{-1} R*V \quad (3.13)$$

where

$$v_j = \frac{1}{E} \int_{-\infty}^{+\infty} e^{-(\xi - \xi_j)^2/2} e^{i \xi_j \cdot s} ds. \quad (3.15)$$

From (3.13) and (3.14), it follows that

$$J_\beta f(\xi) = (Z^{-1} H^*h, V) = (h, V). \quad (3.16)$$

The vector $V = HZ^{-1} V$ can be computed and stored once and for all for that time $\xi$ of interest. Therefore,

$$J_\beta f(\xi) = \sum_{k=1}^K h_k V_k \quad (3.17)$$

If our data $F$ is measured at a whole long sequence of sample points with equal spacing $\Delta t$, we can just translate our data set along the $t$ axis by the multiples $T_j = j \Delta t$, $j$ integer, and attempt to reconstruct our linear functional $J_\beta f$ at those new points using the previous weights given by (3.16) if and only if we are able to repeat the conditions for the reconstruction at $\xi = 0$, which requires that $F \equiv 0$ for $t < T_j$. This can actually be achieved if we subtract the influence upon the data of the last reconstructed point. In doing so, we subtract the influence upon the data of the last $J_\beta f$ instead of $f$, but this is allowed since $A$ is a smoothing operator and therefore the high frequency components die out very fast in $Af$.

Hence, our approximation at $T_j$ is given by the very cheap and simple discrete convolution against the data sequence, updated as mentioned:

$$J_\beta f(T_j) = \sum_{k=1}^K \tilde{F}(T_j + d_k) (V_k / \sqrt{K}). \quad (3.18)$$

4. NUMERICAL RESULTS. INVERSE KERNELS

In order to investigate the stability of our numerical method, we would like to know the amplification factor associated with errors in the data when using numerical procedures.
We notice that any piecewise constant data function can be expressed by a discrete convolution against the numerical delta function $N(t)$ defined by
\[
N(t) = \begin{cases} 
\frac{1}{\Delta t} & \text{if } (-1/2)\Delta t \leq t \leq (1/2)\Delta t \\
0 & \text{otherwise}
\end{cases} \quad (4.1)
\]
Moreover, if we know the solution for $N(t)$ as data, our Inverse Kernel (IK), it follows by linearity that the total error in the solution can be obtained as the discrete convolution of the data error against the inverse kernel.

Our inverse kernel is the temperature solution for the problem in the slab:
\[
\begin{align*}
&u_t = u_{rr}; \quad 0 < r < 1; \quad -\infty < t < \infty \\
&u(0,t) = 0, \text{ data.} \\
&u_r(0,t) = N(t), \text{ data.} \\
&u(1,t) = \text{IK}(t), \text{ unknown.}
\end{align*}
\]

In figures 2, 3, 4 and 5 we plot the inverse kernels for different values of the parameters.

The most important feature of the inverse kernels is the fact that they nearly have compact support. In table 1, the apparent "support" indicates the interval in which the absolute value of the reconstructed function is greater than $10^{-4}$. This allows us to actually compute the solution of the general problem by means of the discrete convolution.

\[
J_\delta f(t) = (\text{IK} * \hat{F})(t) \quad (4.3)
\]

without needing the history of $\hat{F}$ for very long times either before or after the time $t$ of interest; i.e. the kernels are taken to be zero outside their "support".

The inverse kernels are computed once and for all for fixed $\Delta t$ at the points $s_j = \pm \Delta t, j = 0, 1, \ldots, m$ which in-
include the "support" interval of IK. Then if we want to reconstruct the functional $J_{\delta t}$ at any time $T_i = i\Delta t$, integer, we read the data in the interval $[T_i - s_m, T_i + s_m]$ and merely use (4.3) in the form

$$J_{\delta t} f(T_i) = \Delta t \sum_{j=-m}^{m} \delta (T_i - j\Delta t) IK(j\Delta t)$$

(4.4)

4.1. Direct kernels

In order to test the accuracy of our method, we would like to approximately reconstruct a delta function at time $t = 0$ in $u(1,t)$ by solving the problem

$$u_t = u_{rr}; 0 < r < 1; -\infty < t < \infty.$$  

(4.5)

$u(0,t)$ = 0, data.

$u_r(0,t)$ = data.

$u(1,t)$ = $\delta_0(t)$, unknown.

We generate the exact data as the solution of the well-posed problem.

$$u_t = u_{rr}; 0 < r < 1; -\infty < t < \infty.$$  

(4.6)

$u(0,t)$ = 0.

The solution $u(r,t)$ is given by the kernel

$$u(r,t) = \sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n^2 \pi^2 t} \sin(n\pi r); t > 0.$$  

(4.7)

and we have the exact data

$$u_r(0,t) = 2 \pi^2 \sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n^2 \pi^2 t}.$$  

(4.8)

Figures 6, 7, 8, and 9 show several numerical solutions of the problem (4.5), the direct kernels, for different values of the parameters involved.

Figure 10 shows the reconstructed temperature solution (with $\beta = 1$, $N = 39$, $\Delta t = .0125$ and $\beta = 2\Delta t$) for the problem with $u(0,t) = 0$ and $u(1,t)$ = a step function of height 1 between $t = .1$ and .3. The exact data for this problem is given by

$$F(t) = u_{\gamma}(0,t-.01) - u_{\gamma}(0,t-.3),$$  

(4.9)

where

$$u_{\gamma}(0,t) = 1 + 2 \sum_{n=1}^{\infty} (-1)^{n} e^{-n^2 \pi^2 t}.$$  

(4.10)

Finally, in figure 11, the reconstructed mollified temperature shows over and undershoots of only 5 % when
Fig. 6. Direct Kernel
Reconstructed temperature \( f(t) \) corresponding to
\( \tilde{F}(t) = \) exact data.
\( \beta = 1; \ N = 9; \ \delta = 4\Delta t = .2 \)

1 % random error is added to \( F(t) \); that is
\( \tilde{F}(t) = F(t) + .01 \theta \max |F(t)|, \) (4.11)
where \( \theta \) is a random variable with values in \([-1,1]\) and
\( \max |F(t)| = 1. \)

Fig. 7. Direct Kernel
Reconstructed temperature \( f(t) \) corresponding to
\( \tilde{F}(t) = \) exact data.
\( \beta = 1; \ N = 19; \ \delta = 4\Delta t = .1 \)

**BIBLIOGRAPHY**


Fig. 8. Direct kernel
Reconstructed temperature $f(t)$ corresponding to
$F(t) = $ exact data.
$\beta = 1; \ N = 39; \ \delta = 4\Delta t = .05$

Fig. 9. Direct Kernels
Reconstructed temperature $f(t)$ corresponding to
$\tilde{F}(t) = $ exact data.
Mollification method with $\delta = 4\Delta t = .2; .1$ and .05
Fig. 10. Reconstructed temperature $f(t)$ corresponding to $\tilde{F}(t) = \text{exact data.}$
$\beta = 1; \ N = 39; \ \delta = 2\Delta t = .025$

Fig. 11. Reconstructed temperature $f(t)$ corresponding to data $\tilde{F}(t)$ with $1\%$ random error.
$\beta = 1; \ N = 39; \ \delta = 2\Delta t = .025$