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The Koszul dual of a weakly Koszul module

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Abstract

We study the so-called weakly Koszul modules and characterise their Koszul duals. We show that the (adjusted) associated graded module of a weakly Koszul module exactly determines the homology modules of the Koszul dual. We give an example of a quasi-Koszul module which is not weakly Koszul.

Keywords: Koszul algebras; Weakly Koszul modules; A_∞-modules

Let $\Lambda = \bigoplus_{i \geqslant 0} \Lambda_i$ be a Koszul algebra. Such algebras have nice homological properties and appear in various combinatorial and geometrical contexts. There is a corresponding notion of Koszul modules (see Section 1 below for definitions). However, judging from the results in [MZ1], especially Theorems 4.5 and 5.6 of that paper, for some important theoretical purposes the class of Koszul modules is too small, and often the larger class of so-called weakly Koszul modules is needed to complete the picture. Let us briefly recall the definition, a more thorough discussion will follow in Section 1. Let M be a finitely generated graded Λ-module. We say that M is weakly Koszul if $\operatorname{Ext}^*_\Lambda(M, \Lambda_0) = \bigoplus_{i \geqslant 0} \operatorname{Ext}^i_\Lambda(M, \Lambda_0)$ is a Koszul module over the Ext algebra $\operatorname{Ext}^*_\Lambda(\Lambda_0, \Lambda_0) = \bigoplus_{i \geqslant 0} \operatorname{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$. This is a property satisfied by the Koszul modules themselves, so Koszul modules are weakly Koszul.

In the present paper we investigate the behaviour of weakly Koszul modules under the Koszul duality functor of [BGS], which is a functor on the level of derived categories. We show that Koszul duals of weakly Koszul modules can be characterised in terms of their homology (Theorem 3.1). We also show that the Koszul duals of two weakly Koszul modules have isomorphic

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homology modules if and only if the two weakly Koszul modules have isomorphic (adjusted) associated graded modules with respect to the radical filtration (Corollary 3.7). When investigating weakly Koszul modules it is therefore relevant to study the objects in the derived category with given homology modules. The language of A_∞-modules [K2] is particularly well suited for this purpose, and we exploit this in the last section.

Our description of Koszul duals of weakly Koszul modules can be used as a basis for studying more general classes of modules. We say that M is quasi-Koszul if $\mathrm{Ext}^*_\Lambda(M, \Lambda_0)$ is generated in degree 0. Weakly Koszul modules are quasi-Koszul, and quasi-Koszul modules generated in a single degree are weakly Koszul. Based on this evidence, one could be led to believe that the two notions are equivalent. Such speculations can now be laid to rest, as we present a counterexample (Example 4.2). Generalising in another direction we consider modules M such that $\mathrm{Ext}^*_\Lambda(M, \Lambda_0)$ is weakly Koszul. These modules have a surprising property, compare Theorem 3.9 and Example 4.3.

The contents of the different sections are as follows. In Section 1 we give the basic definitions of Koszul algebras and modules, weakly Koszul modules and other related classes of modules. We also recall the fundamental dualities and equivalences present in this setting. Section 2 explains the concept of the "adjusted" associated graded module of M. In Section 3 we give several results and formulas concerning the homology of the Koszul dual of a weakly Koszul module. In the two last sections we discuss how to find the object itself, not just its homology. Section 4 gives a method for straightforward computations, and we use this to produce some important (counter)examples. In Section 5 we take a more systematic approach. We show how to use A_∞-module structures to classify all objects with given homology modules. Via Koszul duality, this classifies all weakly Koszul modules with a given adjusted associated graded module.

1. Dualities and equivalences

Let k be a field and $\Lambda = \bigoplus_{i\geqslant 0} \Lambda_i$ be a graded k-algebra. We assume $\dim_k \Lambda_i < \infty$ for all $i \geqslant 0$ and $\Lambda_0 \simeq k \times \cdots \times k$ as rings. We denote by J the graded Jacobson radical $J = \bigoplus_{i\geqslant 1} \Lambda_i$. We denote by $\mathrm{Gr}\,\Lambda$ the category of graded Λ-modules $M = \bigoplus_{i\in\mathbb{Z}} M_i$ with degree 0 morphisms. By $\mathrm{l.f.}\,\Lambda$ we denote the full subcategory of locally finite modules, that is modules with $\dim_k M_i < \infty$ for all $i \in \mathbb{Z}$. Important (full) subcategories of $\mathrm{l.f.}\,\Lambda$ are the category of finitely generated modules $\mathrm{gr}\,\Lambda$ and the category of finitely cogenerated modules $\mathrm{fcog}\,\Lambda$.

Let M be a graded Λ-module. Its *graded dual* DM is defined to be the graded Λ^{op}-module with graded parts $(DM)_{-i} = \mathrm{Hom}_k(M_i, k)$ and (graded parts of) module structure maps k-dual to those of M. Also using k-dual maps on morphisms we can make D into a contravariant functor $D : \mathrm{Gr}\,\Lambda \to \mathrm{Gr}\,\Lambda^{\mathrm{op}}$. When restricted to locally finite Λ-modules, the functor D is a duality $D : \mathrm{l.f.}\,\Lambda \to \mathrm{l.f.}\,\Lambda^{\mathrm{op}}$.

The *ith graded shift* of M, denoted $M\langle i\rangle$, is the module with graded parts $(M\langle i\rangle)_n = M_{n-i}$ and module structure inherited from M. If M is a module generated in a single degree i, we define $\overline{M} = M\langle -i\rangle$. So $\overline{M}$ is generated in degree 0.

The following lemma, which can for instance be found in [NV, 2.4.7], gives a useful connection between graded and ungraded Ext groups.

Lemma 1.1. *Suppose M is a finitely generated graded Λ-module which has a projective resolution such that all syzygies are finitely generated. Let N be any graded Λ-module.*

Then for every $j \geqslant 0$ we have an isomorphism

$$\operatorname{Ext}^j_\Lambda(M, N) \simeq \bigoplus_{i \in \mathbb{Z}} \operatorname{Ext}^j_{\operatorname{Gr}\Lambda}\big(M, N\langle i \rangle\big)$$

functorial in M and N.

Typical examples of modules M which satisfy the hypothesis of the lemma are finitely generated graded Λ-modules over a Noetherian algebra Λ.

We say that a graded Λ-module M is a *Koszul module* if M is finitely generated and $\operatorname{Ext}^j_{\operatorname{Gr}\Lambda}(M, \Lambda_0\langle i \rangle) \neq 0$ implies $i = j$. In particular we require that $\operatorname{Hom}_{\operatorname{Gr}\Lambda}(M, \Lambda_0\langle i \rangle) \neq 0$ implies $i = 0$, so M must be generated in degree 0. If M is generated in a single degree and $\overline{M}$ is Koszul, we say that M has a *linear resolution*.

Remark 1.2. If L is a graded Λ-module that is locally finite and bounded below (for instance if L is finitely generated), then there is a projective cover $P \xrightarrow{f} L \to 0$, where P is a projective module which is locally finite and bounded below. It follows that $\operatorname{Ker} f$ is also locally finite and bounded below. Such a module L will therefore have a minimal graded projective resolution. The Ext condition in the definition of a Koszul module is a condition on the degrees where the projective modules in the resolution are generated. More precisely a finitely generated module M is Koszul if and only if M has a graded projective resolution

$$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

with P_i finitely generated in degree i for all $i \geqslant 0$.

A locally finite Λ-module M is called a *co-Koszul module* if DM is a Koszul Λ^{op}-module. We denote the full subcategory of Koszul Λ-modules by $\mathcal{K}(\Lambda)$ and the full subcategory of co-Koszul Λ-module by $c\mathcal{K}(\Lambda)$. The functor D restricts to dualities on subcategories in the way shown by the following diagram.

$$\begin{array}{ccc}
\text{l.f. } \Lambda & \xrightarrow{D} & \text{l.f. } \Lambda^{\mathrm{op}} \\
\uparrow & & \uparrow \\
\operatorname{gr} \Lambda & \xrightarrow{D} & \operatorname{fcog} \Lambda^{\mathrm{op}} \\
\uparrow & & \uparrow \\
\mathcal{K}(\Lambda) & \xrightarrow{D} & c\mathcal{K}(\Lambda^{\mathrm{op}})
\end{array}$$

The algebra Λ is called a *Koszul algebra* if Λ_0 is a Koszul Λ-module. One can prove that Λ is a Koszul algebra if and only if Λ^{op} is a Koszul algebra. Suppose now and for the rest of the paper that Λ is a Koszul algebra. Let $\Gamma = \bigoplus_{i \geqslant 0} \operatorname{Ext}^i_\Lambda(\Lambda_0, \Lambda_0) \simeq \bigoplus_{i \geqslant 0} \operatorname{Ext}^i_{\operatorname{Gr}\Lambda}(\Lambda_0, \Lambda_0\langle i \rangle)$. A fundamental theorem [BGS, 1.2.5] states that Γ is also a Koszul algebra. The algebra Γ is called the *Koszul dual* of Λ. The Koszul dual of Γ is isomorphic to Λ as graded algebras.

Another fundamental theorem states that there is an equivalence between certain triangulated subcategories of the corresponding (unbounded) derived categories. The category $\mathcal{D}^{\downarrow}(\Lambda^{\mathrm{op}})$ can be viewed as the full subcategory of $\mathcal{D}\,\mathrm{Gr}\,\Lambda^{\mathrm{op}}$ formed by all objects M with the property that $(H^i M)_j = 0$ when $i \ll 0$ or $i + j \gg 0$. Similarly, the category $\mathcal{D}^{\uparrow}(\Gamma)$ is the full subcategory of $\mathcal{D}\,\mathrm{Gr}\,\Gamma$ formed by the objects N with the property that $(H^i N)_j = 0$ when $i \gg 0$ or $i + j \ll 0$. The theorem [BGS, 2.12.1] states that there is an equivalence of triangulated categories $G : \mathcal{D}^{\downarrow}(\Lambda^{\mathrm{op}}) \to \mathcal{D}^{\uparrow}(\Gamma)$. A concrete description of the functor G is given in [BGS], and we shall use this description for computations in Section 4. (To avoid confusion we point out that the functor is called K in [BGS].)

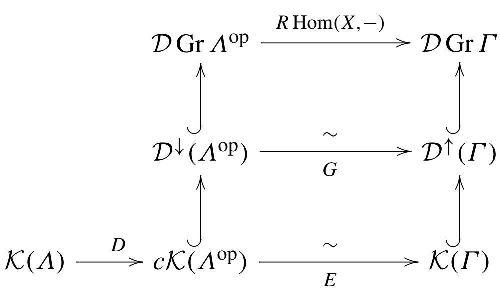
Another description of the functor G follows from the theory of "lifts" in [K1, 7.3,10]. There is a bigraded Λ^{op}-Γ-bimodule complex X, in degree $(*, i)$ quasi-isomorphic to $\Lambda_0^{\mathrm{op}}\langle -i\rangle[-i]$ as a complex of left Λ^{op}-modules, such that the functor $R\,\mathrm{Hom}(X, -) : \mathcal{D}\,\mathrm{Gr}\,\Lambda^{\mathrm{op}} \to \mathcal{D}\,\mathrm{Gr}\,\Gamma$ when restricted to $D^{\downarrow}(\Lambda^{\mathrm{op}})$ is isomorphic to G. In this paper we will not attempt to describe the bimodule X further. For a discussion of to which extent the category equivalence determines X, we refer to [K1, 7].

In the usual way we view modules as stalk complexes concentrated in degree 0. The category $\mathcal{D}^{\downarrow}(\Lambda^{\mathrm{op}})$ contains all finitely cogenerated modules. From the isomorphism $G \simeq R\,\mathrm{Hom}(X, -)$ we get the following result.

Proposition 1.3. *Let M be a finitely cogenerated Λ^{op}-module. Then*

(a) $(H^j G(M))_i \simeq \mathrm{Ext}^{i+j}_{\mathrm{Gr}\,\Lambda^{\mathrm{op}}}(\Lambda_0^{\mathrm{op}}, M\langle i\rangle)$.
(b) $G(M\langle i\rangle) \simeq (G(M))\langle -i\rangle[-i]$.

It follows from part (a) that if M is a co-Koszul Λ^{op}-module, then $G(M) = \bigoplus_{i\geqslant 0} \mathrm{Ext}^i_{\mathrm{Gr}\,\Lambda^{\mathrm{op}}}(\Lambda_0, M\langle i\rangle)$ is a Γ-module. It is possible to show [GM2, 5.1] that in this case $G(M)$ is a Koszul Γ-module. From Lemma 1.1 we get an isomorphism of Γ-modules $\bigoplus_{i\geqslant 0} \mathrm{Ext}^i_{\mathrm{Gr}\,\Lambda^{\mathrm{op}}}(\Lambda_0, M\langle i\rangle) \simeq \bigoplus_{i\geqslant 0} \mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\Lambda_0, M)$ (functorial in M). So when G is restricted to $c\mathcal{K}(\Lambda^{\mathrm{op}})$ it is isomorphic to the functor $E = \bigoplus_{i\geqslant 0} \mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\Lambda_0, -)$. The relation between the various categories and functors is summed up in the following diagram.



Let $\check{E} = ED$. Then $\check{E} : \mathcal{K}(\Lambda) \to \mathcal{K}(\Gamma)$ is also a duality and can be described as $\check{E} = \bigoplus_{i\geqslant 0} \mathrm{Ext}^i_{\Lambda}(-, \Lambda_0)$. This functor can be applied to any graded module, so we view $\check{E}$ with this description as a functor $\check{E} : \mathrm{Gr}\,\Lambda \to \mathrm{Gr}\,\Gamma$. Note that this functor forgets the Λ-grading, so $\check{E}(M) = \check{E}(M\langle i\rangle)$ for any graded Λ-module M and $i \in \mathbb{Z}$. We have the following lemma concerning local finiteness of $\check{E}(M)$.

Lemma 1.4. *Let M be a finitely generated graded Λ-module.*

(a) *Suppose $GD(M)$ has bounded homology and that $\dim_k(H^i GD(M))_j < \infty$ for all $i, j \in \mathbb{Z}$. Then M has a projective resolution such that all syzygies are finitely generated.*
(b) *The Γ-module $\check{E}(M)$ is locally finite if and only if M has a projective resolution such that all syzygies are finitely generated. Moreover, in this case $\check{E}(M) \simeq \bigoplus_{i\in\mathbb{Z}}(H^i GD(M))\langle i\rangle$ as graded Γ-modules.*

Proof. Suppose M has a minimal graded projective resolution

$$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

(a): From minimality it follows that $\mathrm{Hom}_{\mathrm{Gr}\,\Lambda}(P_j, \Lambda_0\langle l\rangle) \simeq \mathrm{Ext}^j_{\mathrm{Gr}\,\Lambda}(M, \Lambda_0\langle l\rangle) \simeq \mathrm{Ext}^j_{\mathrm{Gr}\,\Lambda^{\mathrm{op}}}(\Lambda_0^{\mathrm{op}}, DM\langle l\rangle) \simeq (H^{l-j}GD(M))_l$ for all $l \in \mathbb{Z}$, $j \geqslant 0$. For any given $j \geqslant 0$, since $GD(M)$ has bounded homology, this is non-zero only for a finite number of values of l (and finite-dimensional in those cases). Therefore P_j is finitely generated.

(b): From minimality of the above sequence it follows that $\check{E}(M)_j = \mathrm{Ext}^j_\Lambda(M, \Lambda_0) \simeq \mathrm{Hom}_\Lambda(P_j, \Lambda_0)$ for all $j \in \mathbb{Z}$.

Suppose P_j is not finitely generated for some $j \geqslant 0$. Since P_j as a graded module is locally finite and bounded below, we must have $\dim_k(\mathrm{Hom}_{\mathrm{Gr}\,\Lambda}(P_j, \Lambda_0)) = \infty$. Since $\mathrm{Hom}_{\mathrm{Gr}\,\Lambda}(P_j, \Lambda_0) \subseteq \mathrm{Hom}_\Lambda(P_j, \Lambda_0)$, the module $\check{E}(M)$ is not locally finite.

Suppose P_j is finitely generated for all $j \geqslant 0$. Then $\check{E}(M)_j \simeq \mathrm{Hom}_\Lambda(P_j, \Lambda_0)$ is finite-dimensional for all $j \geqslant 0$ (and zero for $j < 0$), so $\check{E}(M)$ is locally finite. By Lemma 1.1 we have $\check{E}(M)_j \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^j_{\mathrm{Gr}\,\Lambda}(M, \Lambda_0\langle i\rangle)$ for all $j \in \mathbb{Z}$. From Proposition 1.3 we get $(\bigoplus_{i\in\mathbb{Z}}(H^i GD(M))\langle i\rangle)_j = \bigoplus_{i\in\mathbb{Z}}((H^i GD(M))\langle j - i\rangle) \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^j_{\mathrm{Gr}\,\Lambda^{\mathrm{op}}}(\Lambda_0^{\mathrm{op}}, DM\langle i\rangle) \simeq \check{E}(M)_j$ for all $j \in \mathbb{Z}$. Also the module structure is preserved, so $\check{E}(M) \simeq \bigoplus_{i\in\mathbb{Z}}(H^i GD(M))\langle i\rangle$ as graded Γ-modules. □

We are now ready to define weakly Koszul modules.

Definition 1.5. A finitely generated graded Λ-module M is called a *weakly Koszul module* if $\check{E}(M)$ is a Koszul Γ-module.

A slightly weaker condition is that $\check{E}(M)$ is generated in degree 0. A finitely generated graded Λ-module M satisfying this condition is called a *quasi-Koszul module*.

Remark 1.6. There are other equivalent ways of defining weakly Koszul modules. For instance, a finitely generated graded Λ-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ is weakly Koszul if and only if for all $j \in \mathbb{Z}$, the submodule of M generated by M_j has a linear resolution. Quasi-Koszul and weakly Koszul (originally under the name *strongly quasi-Koszul*) modules were introduced in [GM1]. In addition to the mentioned paper [MZ1], weakly Koszul modules are also studied in the paper [HI].

Suppose M is generated in a single degree. In this case it can be shown that M is quasi-Koszul if and only if M is weakly Koszul, which is again equivalent to M having a linear resolution. At the end of Section 4 we give an example of a module M generated in multiple degrees which is quasi-Koszul but not weakly Koszul.

For which finitely generated modules M is $\check{E}(M)$ locally finite and indecomposable? The following proposition gives the answer.

Proposition 1.7. *Let M be a finitely generated graded Λ-module.*

The Γ-module $\check{E}(M)$ is locally finite and indecomposable if and only if M is a graded shift of an indecomposable Koszul Λ-module.

Proof. If M is a graded shift of an indecomposable Koszul Λ-module, then $\check{E}(M) = \check{E}(\overline{M})$ is indecomposable and Koszul, so in particular locally finite.

For the converse assume first that $H^0GD(M) \simeq \mathrm{Hom}_{\mathrm{Gr}\,\Lambda}(M, \Lambda_0) \neq 0$. If $\check{E}(M)$ is locally finite and indecomposable, then it follows from Lemma 1.4 that $\check{E}(M) \simeq H^0GD(M)$ as graded modules and that $H^iGD(M) = 0$ for $i \neq 0$. Since $\mathrm{Ext}^j_{\mathrm{Gr}\,\Lambda}(M, \Lambda_0\langle i\rangle) \simeq \mathrm{Ext}^j_{\mathrm{Gr}\,\Lambda^{\mathrm{op}}}(\Lambda_0^{\mathrm{op}}, DM\langle i\rangle) \simeq (H^{j-i}GD(M))_i \neq 0$ implies $i = j$, the module M is Koszul. If M is Koszul, then M is indecomposable if and only if $\check{E}(M)$ is indecomposable.

If M is an arbitrary finitely generated Λ-module, then $\mathrm{Hom}_{\mathrm{Gr}\,\Lambda}(M\langle i\rangle, \Lambda_0) \neq 0$ for some $i \in \mathbb{Z}$. If $\check{E}(M) = \check{E}(M\langle i\rangle)$ is locally finite and indecomposable, then it follows from the arguments above that M is a graded shift of an indecomposable Koszul Λ-module. □

We denote the category of weakly Koszul Λ-modules by $w\mathcal{K}(\Lambda)$. It is closed under direct summands and finite direct sums. We call a module dual under D to a weakly Koszul module a *weakly co-Koszul* module. The corresponding category of weakly co-Koszul Λ^{op}-modules we denote by $wc\mathcal{K}(\Lambda^{\mathrm{op}})$.

The essential image of $wc\mathcal{K}(\Lambda^{\mathrm{op}})$ under $G \simeq R\,\mathrm{Hom}(X, -)$ we denote by $\mathcal{X}$. In other words $\mathcal{X}$ is the full subcategory of $\mathcal{D}\,\mathrm{Gr}\,\Gamma$ consisting of objects isomorphic to an object of the form $G(L)$, where L is an object of $wc\mathcal{K}(\Lambda^{\mathrm{op}})$. Our aim is to describe this category further. It is a full subcategory of $D^{\uparrow}(\Gamma)$ as we see from the following diagram.

$$
\begin{array}{ccccc}
 & & \mathcal{D}\,\mathrm{Gr}\,\Lambda^{\mathrm{op}} & \xrightarrow{R\,\mathrm{Hom}(X,-)} & \mathcal{D}\,\mathrm{Gr}\,\Gamma \\
 & & \uparrow & & \uparrow \\
 & & \mathcal{D}^{\downarrow}(\Lambda^{\mathrm{op}}) & \xrightarrow{\sim} & \mathcal{D}^{\uparrow}(\Gamma) \\
 & & \uparrow & & \uparrow \\
w\mathcal{K}(\Lambda) & \xrightarrow{D} & wc\mathcal{K}(\Lambda^{\mathrm{op}}) & \xrightarrow{\sim} & \mathcal{X} \\
 & & \uparrow & & \uparrow \\
\mathcal{K}(\Lambda) & \xrightarrow{D} & c\mathcal{K}(\Lambda^{\mathrm{op}}) & \xrightarrow[E]{\sim} & \mathcal{K}(\Gamma)
\end{array}
$$

2. Associated graded module

In this section we explain some technicalities concerning filtrations of finitely generated modules and the associated graded modules.

If M is a finitely generated Λ-module, then its associated graded module (with respect to the radical filtration) is

$$\mathfrak{gr}(M) = \bigoplus_{i \geqslant 0} J^iM/J^{i+1}M.$$

This is also a finitely generated graded Λ-module. If M is generated in a single degree, then $\mathfrak{gr}(M) \simeq \overline{M}$.

Suppose M is finitely generated in degrees $j_0 < \cdots < j_p$. We always assume that the set of generators is minimal, in other words M is finitely generated and $\mathrm{Hom}_{\mathrm{Gr}\,\Lambda}(M, \Lambda_0\langle i\rangle) \neq 0$ if and only if $i \in \{j_0, \ldots, j_p\}$. In [MZ1] we find the following result.

Proposition 2.1. *(See [MZ1, 2.5].) Suppose M is a finitely generated module generated in degrees $j_0 < \cdots < j_p$. Let $K^{(0)}$ be the submodule of M generated by M_{j_0}. Then there is a split-exact sequence of Λ-modules*

$$0 \to \mathfrak{gr}\big(K^{(0)}\big) \to \mathfrak{gr}(M) \to \mathfrak{gr}\big(M/K^{(0)}\big) \to 0.$$

We define $M^{(0)} = M$, and in general for all $0 < i \leqslant p$ we define $M^{(i)} = M^{(i-1)}/K^{(i-1)}$ and let $K^{(i)}$ denote the submodule of $M^{(i)}$ generated in degree j_i (the "highest degree"). For each i the module $M^{(i)}$ is generated in degrees $j_i < \cdots < j_p$. In particular we have $M^{(p)} = K^{(p)}$.

With this notation we get the following corollary.

Corollary 2.2. *If M is a finitely generated module generated in degrees $j_0 < \cdots < j_p$, then*

$$\mathfrak{gr}(M) \simeq \mathfrak{gr}\left(\bigoplus_{i=0}^{p} K^{(i)}\right) \simeq \bigoplus_{i=0}^{p} \overline{K^{(i)}}.$$

Mention should now be made of the following theorem in [MZ1].

Theorem 2.3. *(See [MZ1, 2.5].) A finitely generated module M is a weakly Koszul module if and only if $\mathfrak{gr}(M)$ is Koszul.*

Motivated by Corollary 2.2 we define the following "adjusted" version of the associated graded module. If M is a finitely generated module, then $\tilde{\mathfrak{gr}}(M)$ is defined to be the module

$$\tilde{\mathfrak{gr}}(M) \simeq \bigoplus_{i=0}^{p} K^{(i)}.$$

With this definition M and $\tilde{\mathfrak{gr}}(M)$ are generated in the same degrees, but each indecomposable summands of $\tilde{\mathfrak{gr}}(M)$ is generated in a single degree. We also have

$$\tilde{\mathfrak{gr}}\big(M^{(s)}\big) \simeq \bigoplus_{i=s}^{p} K^{(i)}$$

whenever $0 \leqslant s \leqslant p$.

3. Homology of the Koszul dual

We now return to the question of describing the objects in $\mathcal{X}$. The following theorem shows that such objects can be characterised by their homology.

Theorem 3.1. *Let N be an object in $\mathcal{D}\operatorname{Gr}\Gamma$.*
Then $N \in \mathcal{X}$ if and only if

(i) *N has bounded homology and*
(ii) *for all $i \in \mathbb{Z}$, $H^i N$ is generated in degree $-i$ and has a linear resolution.*

Proof. If $N \in \mathcal{X}$, then $N \simeq GD(M)$ for a weakly Koszul Λ-module M. Also $\check{E}(M) \simeq \bigoplus_{i \geqslant 0}(H^i N)\langle i\rangle$ is a Koszul Γ-module and in particular it is generated in degree 0. Therefore for each $i \in \mathbb{Z}$, we have that $(H^i N)\langle i\rangle$ is Koszul and $H^i N$ is generated in degree $-i$. We know $\check{E}(M)_0$ is finite-dimensional over k, and therefore $((H^i N)\langle i\rangle)_0 = (H^i N)_{-i}$ is non-zero only for a finite number of values for i. This means that N has bounded homology.

Let N be an object in $\mathcal{D}\operatorname{Gr}\Gamma$ with bounded homology and suppose $H^i N$ is generated in degree $-i$ for all $i \in \mathbb{Z}$. Choose a representing complex for N, with differential d, such that $N^i = 0$ for $i \leqslant a$ and $i > b$, for suitable integers a, b with $a < b$. For each integer p, the *soft truncation* $\tau^{\leqslant p} N$ of N at p is defined by

$$\left(\tau^{\leqslant p} N\right)^i = \begin{cases} N^i & \text{if } i < p, \\ \operatorname{Ker} d^i & \text{if } i = p, \\ 0 & \text{if } i > p. \end{cases}$$

Its homology is given by

$$H^i\left(\tau^{\leqslant p} N\right) = \begin{cases} H^i N & \text{if } i \leqslant p, \\ 0 & \text{if } i > p. \end{cases}$$

We have a filtration of complexes $0 = \tau^{\leqslant a} N \subseteq \cdots \subseteq \tau^{\leqslant b} N = N$. All these objects are in $D^{\uparrow}(\Gamma)$. Consider the triangles in $D^{\uparrow}(\Gamma)$

$$\tau^{\leqslant i-1} N \to \tau^{\leqslant i} N \to Y_i \to \left(\tau^{\leqslant i-1} N\right)[1]$$

for all i with $a < i \leqslant b$. Here $Y_{(i)}$ has non-zero homology only possibly in degree i and $H^i(Y_{(i)}) \simeq H^i N$. By assumption, $H^i N \simeq L_{(i)}\langle -i\rangle$ for some Koszul Γ-module $L_{(i)}$. So there is a Koszul Λ-module $K_{(i)}$ such that $GD(K_{(i)}\langle -i\rangle) \simeq G((DK_{(i)})\langle i\rangle) \simeq L_{(i)}\langle -i\rangle[-i] \simeq Y_{(i)}$.

So for all i we have that $Y_{(i)}$ is isomorphic to G of a finitely cogenerated module (viewed as a stalk complex). Let $F : D^{\uparrow}(\Gamma) \to D^{\downarrow}(\Lambda^{\mathrm{op}})$ denote a quasi-inverse of G. By induction (starting with $\tau^{\leqslant a} N = 0$), using the triangles above, we get for all $a < i \leqslant b$ that $F(\tau^{\leqslant i} N)$ is a module and there are exact sequences

$$0 \to F\left(\tau^{\leqslant i-1} N\right) \to F\left(\tau^{\leqslant i} N\right) \to F(Y_{(i)}) \to 0.$$

The modules $F(Y_{(i)})$ are finitely cogenerated and again by induction every $F(\tau^{\leqslant i} N)$ is finitely cogenerated. So in particular $N = \tau^{\leqslant b} N$ is isomorphic to $GD(M)$ for the finitely generated Λ-module $M = DF(N)$. From Lemma 1.4 we get $\check{E}(M) \simeq \bigoplus_{i \in \mathbb{Z}}(H^i GD(M))\langle i\rangle$ and by assumption this is a Koszul Γ-module. But then M is a weakly Koszul module by definition. □

Remark 3.2. In Example 4.2 we give an example of a graded Λ-module M with the property that $N = GD(M)$ has bounded homology and $H^i N$ is generated in degree $-i$ for all $i \in \mathbb{Z}$, but there is an $i \in \mathbb{Z}$ such that $H^i N$ does not have a linear resolution. This means that the module M is quasi-Koszul but not weakly Koszul.

Let M be a weakly Koszul module. We next try to find formulas for the homology of $GD(M)$. We start with the simple case when the module is a graded shift of a Koszul module.

Proposition 3.3. *Let K be a Koszul module. Then*

$$H^{-n}GD\big(K\langle i\rangle\big) \simeq \begin{cases} \check{E}(K)\langle i\rangle & \text{if } n = i, \\ 0 & \text{if } n \neq i. \end{cases}$$

Proof. We have

$$\begin{aligned} H^{-n}GD\big(K\langle i\rangle\big) &\simeq H^{-n}G\big((DK)\langle -i\rangle\big) \simeq H^{-n}(GDK)[i]\langle i\rangle \simeq H^{-n+i}(GDK)\langle i\rangle \\ &\simeq H^{-n+i}\big(\check{E}(K)\big)\langle i\rangle. \qquad \square \end{aligned}$$

The following proposition from [MZ1] will help us resolve the general case.

Proposition 3.4. *(See [MZ1, 2.4].) Let M be a weakly Koszul module generated in degrees $j_0 < \cdots < j_p$. Let $K^{(0)}$ be the submodule of M generated by M_{j_0}.*
Then $K^{(0)}$ has a linear resolution and $M^{(1)} = M/K^{(0)}$ is weakly Koszul.

Keeping the notation from the previous section we have the following obvious corollary.

Corollary 3.5. *Let M be a weakly Koszul module generated in degrees $j_0 < \cdots < j_p$.*
Then for each $0 \leqslant i \leqslant p$, the module $K^{(i)}$ has a linear resolution and $M^{(i)}$ is weakly Koszul.

We know that the homology of $GD(M)$ is bounded and in each degree it is of the same form as in Proposition 3.3. Therefore there must exist a module $\tilde{M}$, being a finite direct sum of modules with linear resolutions, such that $H^n GDM \simeq H^n GD(\tilde{M})$ for all n. But what is this module $\tilde{M}$? The next proposition shows that the answer is the (adjusted) associated graded module of M.

Proposition 3.6. *Let M be a weakly Koszul module. Then*

$$H^n GDM \simeq H^n GD\big(\tilde{\mathfrak{gr}}(M)\big)$$

for all $n \in \mathbb{Z}$.

Proof. We prove $H^n GDM^{(s)} \simeq H^n GD(\tilde{\mathfrak{gr}}(M^{(s)}))$ for all $0 \leqslant s \leqslant p$ by downward induction on s going from $M^{(p)} = K^{(p)}$ to $M^{(0)} = M$.

Since $M^{(p)}$ is generated in a single degree, we have $M^{(p)} \simeq \tilde{\mathfrak{gr}}(M^{(p)})$, so this case is clear.

From each exact sequence

$$0 \to K^{(s)} \to M^{(s)} \to M^{(s+1)} \to 0$$

with $0 \leqslant s < p$ we get a triangle

$$DM^{s+1} \to DM^s \to DK^s \to DM^{s+1}[1]$$

in $\mathcal{D}\operatorname{Gr}\Lambda^{\mathrm{op}}$. Applying G to this triangle we get a triangle

$$GDM^{s+1} \to GDM^s \to GDK^s \to GDM^{s+1}[1]$$

in $\mathcal{D}\operatorname{Gr}\Gamma$. We have a long-exact sequence in homology

$$\begin{aligned} \cdots \to H^{n-1}GDK^s \to H^n GDM^{s+1} \to H^n GDM^s \\ \to H^n GDK^s \to H^{n+1}GDM^{s+1} \to \cdots \end{aligned}$$

We assume that $H^n GDM^{(s+1)} \simeq H^n GD(\tilde{\mathfrak{gr}}(M^{(s+1)}))$ for a given s. The module $M^{(s+1)}$ is generated in degrees $j_{s+1} < \cdots < j_p$ and the same is true for $\tilde{\mathfrak{gr}}(M^{(s+1)}) \simeq \bigoplus_{i=s+1}^{p} K^{(i)}$. Since the $K^{(i)}$ have linear resolutions, Proposition 3.3 says that $H^n GDM^{(s+1)}$ is non-zero only for $n \in \{-j_p, \dots, -j_{s+1}\}$. In particular $H^n GDM^{(s+1)} = 0$ when $n \geqslant -j_s$. Also by Proposition 3.3 $H^n GDK^{(s)} \neq 0$ if and only if $n = -j_s$. Using these facts and the isomorphism

$$\tilde{\mathfrak{gr}}\big(M^{(s)}\big) \simeq K^{(s)} \oplus \tilde{\mathfrak{gr}}\big(M^{(s+1)}\big)$$

we get

$$H^n GDM^{(s)} \simeq H^n GDK^{(s)} \simeq H^n GD\big(\tilde{\mathfrak{gr}}\big(M^{(s)}\big)\big)$$

when $n \geqslant -j_s$ and

$$H^n GDM^{(s)} \simeq H^n GDM^{(s+1)} \simeq H^n GD\big(\tilde{\mathfrak{gr}}\big(M^{(s+1)}\big)\big) \simeq H^n GD\big(\tilde{\mathfrak{gr}}\big(M^{(s)}\big)\big)$$

when $n < -j_s$. This finishes the induction step. □

As a corollary we have the following.

Corollary 3.7. *Let M and M' be two weakly Koszul modules. Then*

$$\tilde{\mathfrak{gr}}(M) \simeq \tilde{\mathfrak{gr}}(M')$$

if and only if

$$H^n GDM \simeq H^n GDM'$$

for all $n \in \mathbb{Z}$.

Proof. The modules $\tilde{\mathfrak{gr}}(M)$ and $\tilde{\mathfrak{gr}}(M')$ are both direct sums of modules with linear resolutions. From Proposition 3.3 it follows that $\tilde{\mathfrak{gr}}(M) \simeq \tilde{\mathfrak{gr}}(M')$ if and only if $H^n GD(\tilde{\mathfrak{gr}}(M)) \simeq H^n GD(\tilde{\mathfrak{gr}}(M'))$ for all $n \in \mathbb{Z}$. But by Proposition 3.6 this is true if and only if $H^n GDM \simeq H^n GDM'$ for all $n \in \mathbb{Z}$. □

Combining Proposition 3.6 with Proposition 3.3 we get the following formula. Here $j^{-1}n$ is the number i such that $j_i = n$.

Corollary 3.8. *Let M be a weakly Koszul module generated in degrees $J = \{j_0, \dots, j_p\}$. Then*

$$H^{-n}GD(M) \simeq \begin{cases} \check{E}(K^{(j^{-1}n)})\langle n\rangle & \text{if } n \in J, \\ 0 & \text{if } n \notin J. \end{cases}$$

In [MZ2], the authors ask which finitely generated graded Λ-modules M have the property that $\check{E}(M)$ is weakly Koszul. We present the following proposition as a first step towards understanding such modules. The case when $\check{E}(M)$ is indecomposable is also covered by Proposition 1.7.

Theorem 3.9. *Suppose M is a finitely generated graded Λ-module such that $\check{E}(M)$ is weakly Koszul. Then $\check{E}(M)$ has a non-zero direct summand which is Koszul.*

Proof. Without loss of generality we may assume that M is generated in degrees $0 = j_0 < j_1 < \cdots < j_p$. In this case $H^n GD(M) = 0$ for $n > 0$ and there is a triangle

$$\tau^{\leqslant -1}GD(M) \to GD(M) \to H^0GD(M) \to \tau^{\leqslant -1}GD(M)[1]$$

in $D^{\uparrow}(\Gamma)$. Since $H^0GD(M)$ is a direct summand of $\check{E}(M)$ by Lemma 1.4, it is weakly Koszul by assumption. Since $(H^0GD(M))_i \simeq \mathrm{Ext}^i_{\mathrm{Gr}\,\Lambda^{\mathrm{op}}}(\Lambda_0^{\mathrm{op}}, DM\langle i\rangle) \simeq \mathrm{Ext}^i_{\mathrm{Gr}\,\Lambda}(M, \Lambda_0\langle i\rangle)$, we know that $(H^0GD(M))$ has support only in non-negative degrees. Also $(H^0GD(M))_0 \simeq \mathrm{Hom}_{\mathrm{Gr}\,\Lambda}(M, \Lambda_0) \neq 0$.

Let $F : D^{\uparrow}(\Gamma) \to D^{\downarrow}(\Lambda^{\mathrm{op}})$ denote a quasi-inverse of G. If K is a Koszul Γ-module, then $F(K)$ is a co-Koszul Λ^{op}-module. Let N be a weakly Koszul Γ-module generated in degrees $J = \{h_0, \dots, h_p\}$. Since $F(K\langle i\rangle) \simeq (F(K)\langle -i\rangle)[-i]$ for all $i \in \mathbb{Z}$, by induction using Proposition 3.4 we get that $H^i(F(N))$ is cogenerated in degree $-i$ for $i \in \mathbb{Z}$. Also $H^i(F(N)) \neq 0$ if and only if $i \in J$.

Let $N = H^0GD(M)$. Applying F to the triangle above, we get the triangle

$$F\big(\tau^{\leqslant -1}GD(M)\big) \to DM \to F(N) \to F\big(\tau^{\leqslant -1}GD(M)\big)[1]$$

in $D^{\downarrow}(\Lambda^{\mathrm{op}})$. Since N is generated in degrees $0 = h_0 < h_1 < \cdots < h_p$, we have $H^iF(N) = 0$ for $i < 0$. Since DM is concentrated in homological degree 0, there is an isomorphism $H^iF(N) \simeq H^{i+1}(F(\tau^{\leqslant -1}GD(M)))$ when $i > 0$. Now $\tau^{\leqslant -1}GD(M)$ is by assumption obtained by a finite number of extensions of objects of the form $N'[s]$ with N' weakly Koszul and $s > 0$. The module $H^{i+1}(F(N')[s])$ is cogenerated in degree $-i - s - 1$, so $(H^{i+1}(F(N')[s]))_{-i} = 0$ for all $i \in \mathbb{Z}$ when $s > 0$. By induction $(H^{i+1}(F(\tau^{\leqslant -1}GD(M))))_{-i} = 0$ for all $i \in \mathbb{Z}$. But $H^iF(N)$ is cogenerated in degree $-i$, so $H^iF(N) \simeq H^{i+1}(F(\tau^{\leqslant -1}GD(M))) = 0$ when $i > 0$.

So $H^iF(N) \neq 0$ only when $i = 0$. This means that the weakly Koszul module N is generated in degree 0 and is therefore Koszul. So $\check{E}(M)$ has a direct summand $N = H^0GD(M)$ which is Koszul. □

Surprisingly, this is not the beginning of an inductive procedure. An example in the next section (Example 4.3) shows that the other direct summands of $\check{E}(M)$ are not necessarily Koszul or shifts of Koszul modules. In other words $\check{E}(M)$ can have indecomposable direct summands which are generated in multiple degrees.

4. Computation of the object $GD(M)$

So far we have found a formula for the homology of the object $GD(M)$ when M is a weakly Koszul module, but we have not described the object $GD(M)$ itself. Based on the description of the functor G in [BGS], we give a method for computing $GD(M)$ for any finitely generated Λ-module when Λ is given as a path algebra (that is an algebra given as a quiver with relations). We refer to [MS] for more details on a construction that is essentially the same as ours, but there it is done in an abelian setting.

Suppose Λ is a Koszul algebra given as the path algebra of a quiver with relations. Then Λ_0 is the k-linear span of the vertices, while Λ_1 is the k-linear span of the arrows. It can be shown that the relations are *quadratic*, that is they are given by a Λ_0-sub-bimodule R of $\Lambda_1 \otimes_{\Lambda_0} \Lambda_1$. The quiver of Λ^{op} is the opposite quiver to the one of Λ. The relations for Λ^{op} are similarly given by a Λ_0^{op}-sub-bimodule $\check{R}$ of $\Lambda_1^{\mathrm{op}} \otimes_{\Lambda_0^{\mathrm{op}}} \Lambda_1^{\mathrm{op}}$. The Koszul dual $\Gamma = \bigoplus_{i \geqslant 0} \operatorname{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$ is isomorphic to the path algebra with the same quiver as Λ (so $\Gamma_0 \simeq \Lambda_0$, $\Gamma_1 \simeq \Lambda_1$ and $\Gamma_1 \otimes_{\Gamma_0} \Gamma_1 \simeq \Lambda_1 \otimes_{\Lambda_0} \Lambda_1$) and *orthogonal* relations $R^\perp$, that is the kernel of the Λ_0-bimodule morphism $\operatorname{Hom}_{\Lambda_0^{\mathrm{op}}}(\Lambda_1^{\mathrm{op}} \otimes_{\Lambda_0^{\mathrm{op}}} \Lambda_1^{\mathrm{op}}, \Lambda_0^{\mathrm{op}}) \to \operatorname{Hom}_{\Lambda_0^{\mathrm{op}}}(\check{R}, \Lambda_0^{\mathrm{op}})$ (see [BGS]). For path algebras, the orthogonal relations can be found with the help of the bilinear form used in [GM2, 2.2].

Now let Q denote the quiver of Λ, let $\check{Q}$ denote the quiver of Λ^{op} and let Q^* denote the quiver of Γ. Here $Q^* = Q$, but it is still helpful to keep separate notation. If the vertices of Q are indexed by $\{1, \ldots, t\}$, let $\{\check{1}, \ldots, \check{t}\}$ denote the corresponding vertices of $\check{Q}$ and let $\{1^*, \ldots, t^*\}$ denote the corresponding vertices of Q^*. If $a \xrightarrow{\alpha} b$ is an arrow in Q, then let $\check{a}$ denote the corresponding arrow $\check{a} \xleftarrow{\check{\alpha}} \check{b}$ in $\check{Q}$, let α^* denote the corresponding arrow $a^* \xrightarrow{\alpha^*} b^*$ in Q^*. If $\check{a}$ is a vertex of $\check{Q}$ and let $S_{\check{a}}$ denote the corresponding simple Λ^{op}-module (and Λ_0^{op}-module). Let P_{a^*} denote the projective Γ-module corresponding to the vertex a^* of Q^*.

Now let M be a finitely generated Λ-module. For the moment we do not assume that M is weakly Koszul. Then DM is a finitely cogenerated Λ^{op}-module. We now try to find a complex representing $GD(M)$, so we need to know the result of applying G to a finitely cogenerated module.

The terms of $GD(M)$ we find from the graded parts of DM (or just as easily directly from the graded parts of M). If $(DM)_i \simeq (S_{\check{1}})^{n_1} \bigoplus \cdots \bigoplus (S_{\check{t}})^{n_t}$ is a decomposition of $(DM)_i$ into simple Λ_0^{op}-modules, then we put $GD(M)^i = [(P_{1^*})^{n_1} \bigoplus \cdots \bigoplus (P_{t^*})^{n_t}]\langle -i \rangle$.

To describe the differential is slightly more complicated. Suppose $\check{a}$ and $\check{b}$ are two (not necessarily distinct) vertices in $\check{Q}$ with r arrows going from $\check{b}$ to $\check{a}$. Denote the arrows by $\check{\alpha}_1, \ldots, \check{\alpha}_r$.

$$\check{a} \quad \begin{array}{c} \xleftarrow{\;\check{\alpha}_1\;} \\ \xleftarrow{\;\check{\alpha}_2\;} \\ \vdots \\ \xleftarrow[\;\check{\alpha}_r\;]{} \end{array} \quad \check{b}$$

(There might also be arrows in the other direction.) Suppose x is an element in the summand $(S_{\check{b}})^{n_{b,i}}$ of $(DM)_i$ and let $\check{a} \xleftarrow{\check{\alpha}_j} \check{b}$ be an arrow. Then $\check{\alpha}_j \cdot x$ is an element in the summand $(S_{\check{a}})^{n_{a,i+1}}$ of $(DM)_{i+1}$. Choose k-bases for $(S_{\check{b}})^{n_{b,i}}$ and $(S_{\check{a}})^{n_{a,i+1}}$ and let $A_{(\check{\alpha}_j,i)}$ be the $(n_{a,i+1} \times n_{b,i})$-matrix with entries in k which represents the map $(S_{\check{b}})^{n_{b,i}} \to (S_{\check{a}})^{n_{a,i+1}}$ induced by $\check{\alpha}_j$.

For each arrow $a^* \xrightarrow{\alpha_j^*} b^*$ and each $i \in \mathbb{Z}$ we have a map $(\bar{\alpha_j^*})_i : P_{b^*}\langle -i\rangle \to P_{a^*}\langle -i-1\rangle$ which we can view as multiplication with α_j^* from the right. The part of $d^i : GD(M)^i \to GD(M)^{i+1}$ which maps $(P_{b^*}\langle -i\rangle)^{n_{b,i}}$ to $(P_{a^*}\langle -i-1\rangle)^{n_{a,i+1}}$ is given by

$$\sum_{j=1}^{r} \bigl(A_{(\check{\alpha}_j,i)} \times \bigl(\bar{\alpha_j^*}\bigr)_i\bigr),$$

where $\times$ means that each entry in the matrix $A_{(\check{\alpha}_j,i)}$ is to be multiplied by $(\bar{\alpha_j^*})_i$. We illustrate with an example.

Example 4.1. Let Λ be the path algebra of the quiver

$$1 \xrightarrow{\alpha} 2 \circlearrowright \beta \;.$$

This is a Koszul algebra.

Then Λ^{op} is the path algebra of the quiver

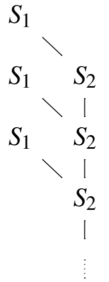
$$\check{1} \xleftarrow{\check{\alpha}} 2 \circlearrowleft \check{\beta} \;,$$

and $\Gamma \simeq \bigoplus_{i \geqslant 0} \operatorname{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$ is isomorphic to the path algebra given by the quiver

$$1^* \xrightarrow{\alpha^*} 2^* \circlearrowright \beta^*$$

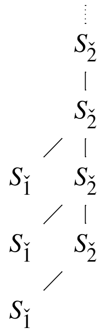
and with relations $\beta^*\alpha^* = 0$ and $(\beta^*)^2 = 0$.

Let M be the following infinite-dimensional Λ-module generated in degrees 0, 1 and 2. As Λ_0-modules, we have $M_i = S_2$ for $i \geqslant 3$.



Here $\tilde{\mathfrak{gr}}(M) \simeq P_1 \oplus S_1\langle 1\rangle \oplus S_1\langle 2\rangle$. Since $\mathfrak{gr}(M) \simeq P_1 \oplus (S_1)^2$ is a Koszul module, the module M is weakly Koszul.

Then DM is the finitely cogenerated Λ^{op}-module.



From this we can read off the object $GD(M)$:

$$\cdots \to P_{2^*}\langle 4\rangle \xrightarrow{\bar{\beta}^*} P_{2^*}\langle 3\rangle \xrightarrow{\binom{\bar{\alpha}^*}{\bar{\beta}^*}} (P_{1^*} \oplus P_{2^*})\langle 2\rangle \xrightarrow{\left(\begin{smallmatrix} 0 & \bar{\alpha}^* \\ 0 & \bar{\beta}^* \end{smallmatrix}\right)} (P_{1^*} \oplus P_{2^*})\langle 1\rangle \xrightarrow{(0\ \bar{\alpha}^*)} P_{1^*} \to 0.$$

Here

$$\begin{aligned} H^0GD(M) &\simeq S_{1^*} \simeq \check{E}(P_1) \simeq H^0GD\big(\tilde{\mathfrak{gr}}(M)\big), \\ H^{-1}GD(M) &\simeq P_{1^*}\langle 1\rangle \simeq \check{E}(S_1)\langle 1\rangle \simeq H^{-1}GD\big(\tilde{\mathfrak{gr}}(M)\big), \\ H^{-2}GD(M) &\simeq P_{1^*}\langle 2\rangle \simeq \check{E}(S_1)\langle 2\rangle \simeq H^{-2}GD\big(\tilde{\mathfrak{gr}}(M)\big) \end{aligned}$$

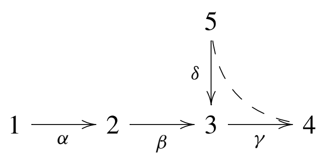
and

$$H^nGD(M) \simeq 0 \simeq H^nGD\big(\tilde{\mathfrak{gr}}(M)\big)$$

for $n \notin \{-2, -1, 0\}$ as expected.

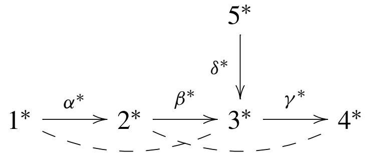
In the next example we compute $GD(M)$ for a module M which is not weakly Koszul but turns out to be quasi-Koszul.

Example 4.2. Let Λ be the path algebra of the quiver



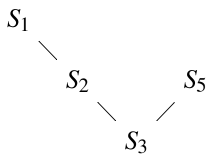
and relation $\gamma\delta = 0$. This algebra is Koszul.

Then $\Gamma \simeq \bigoplus_{i \geqslant 0} \mathrm{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$ is isomorphic to the path algebra of the quiver



with relations $\beta^*\alpha^* = 0$ and $\gamma^*\beta^* = 0$.

Let M be the following module generated in degrees 0 and 1.

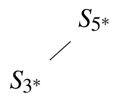


Let L be the submodule generated by M_0. Since $\mathfrak{gr}(M) \simeq L \oplus S_5$ is not Koszul, the module M is not weakly Koszul.

Then $GD(M)$ is represented by the complex

$$0 \to P_{3^*}\langle 2\rangle \xrightarrow{\binom{\bar{\beta}^*}{\bar{\delta}^*}} (P_{2^*} \oplus P_{5^*})\langle 1\rangle \xrightarrow{(\bar{\alpha}^*\ 0)} P_{1^*} \to 0.$$

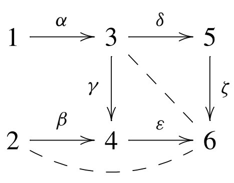
Here $H^n GD(M) = 0$ for $n \notin \{-1, 0\}$, $H^0 GD(M) \simeq S_{1^*}$ and $H^{-1} GD(M)$ is the module



generated in degree 1. In this example, although $H^i GD(M)$ is generated in degree $-i$ for all $i \in \mathbb{Z}$, the module $H^{-1} GD(M)$ does not have a linear resolution. Therefore the conditions in Theorem 3.1 are not satisfied. Since $\check{E}(M) \simeq \bigoplus_{i \in \mathbb{Z}} (H^i GD(M))\langle i\rangle$ is generated in degree 0, we have that M is quasi-Koszul.

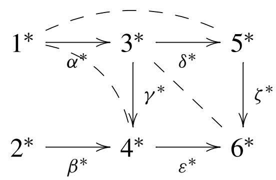
In Theorem 3.9 we have shown that if $\check{E}(M)$ is weakly Koszul, then $\check{E}(M)$ has a Koszul direct summand. The following example shows that $\check{E}(M)$ can have other indecomposable direct summands which are generated in multiple degrees.

Example 4.3. Let Λ be the path algebra of the quiver



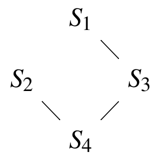
and relations $\varepsilon\beta = 0$ and $\varepsilon\gamma - \zeta\delta = 0$. This algebra is Koszul.

Then $\Gamma \simeq \bigoplus_{i \geqslant 0} \operatorname{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$ is isomorphic to the path algebra of the quiver



with relations $\gamma^*\alpha^* = 0$, $\delta^*\alpha^* = 0$ and $\varepsilon^*\gamma^* + \zeta^*\delta^* = 0$.

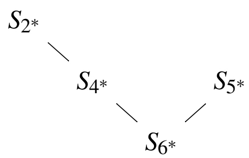
Let M be the following Λ-module generated in degrees 0 and 1.



Then $GD(M)$ is represented by the complex

$$0 \to P_{4^*}\langle 2\rangle \xrightarrow{\binom{\bar\beta^*}{\bar\gamma^*}} (P_{2^*} \oplus P_{3^*})\langle 1\rangle \xrightarrow{(0\ \bar\alpha^*)} P_{1^*} \to 0.$$

Here $H^n GD(M) = 0$ for $n \notin \{-1, 0\}$, $H^0 GD(M) \simeq S_{1^*}$ and $H^{-1} GD(M)$ is the Γ-module



generated in degrees 1 and 2. Call this module L. Since $\mathfrak{gr}(L) \simeq P_{2^*} \oplus S_{5^*}$ is Koszul, the module L is weakly Koszul. From Lemma 1.4 we get $\check{E}(M) \simeq L\langle -1\rangle \oplus S_{1^*}$. So $\check{E}(M)$ is weakly Koszul and has an indecomposable summand which is generated in multiple degrees.

5. A_∞-modules

In this section we discuss an alternative way of viewing objects in $\mathcal{D}\operatorname{Gr}\Gamma$, namely as A_∞-modules. Instead of thinking of objects as complexes, we think of them as homology groups with some additional structure. If we fix homology groups satisfying the conditions of Theorem 3.1, then each possible A_∞-module structure on that homology gives an object in $\mathcal{X}$. From Corollary 3.7 we know that if two objects in $\mathcal{X}$ share the same homology, then the two corresponding weakly Koszul Λ-modules have isomorphic adjusted associated graded modules. So if we classify all objects in $\mathcal{X}$ with a certain homology, then via Koszul duality we classify all weakly Koszul Λ-modules with a certain adjusted associated graded module.

Let $\Gamma = \bigoplus_{i \geqslant 0} \Gamma_i$ be a graded algebra. We consider Γ as an A_∞-algebra concentrated in degree 0. The ordinary grading of Γ gives an additional structure which is also inherited by our modules. So what we really are considering are *graded* A_∞-modules. This extra grading can be introduced more formally by considering A_∞-algebras (and their A_∞-modules) over the monoidal base category of graded vector spaces, but we will not take this approach here. For definitions we follow [L] and [K2].

For us (*graded*) A_∞*-module* over Γ is a bigraded space

$$N = \bigoplus_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} N^i_j$$

with maps

$$m_n : \Gamma^{\otimes n-1} \otimes N \to N, \quad n \geqslant 1$$

of bidegree $(n-2, 0)$ satisfying the rules

$$m_1 m_1 = 0,$$
$$m_1 m_2 = m_2(\mathbb{1} \otimes m_1)$$

and for $n \geqslant 3$

$$\sum_{i=1}^{n} (-1)^{i(n-1)} m_{n-i+1}\big(\mathbb{1}^{\otimes n-i} \otimes m_i\big) = \sum_{j=1}^{n-2} (-1)^{j-1} m_{n-1}\big(\mathbb{1}^{\otimes n-j-2} \otimes m \otimes \mathbb{1}^{\otimes j}\big)$$

where $\mathbb{1}$ is the identity map and m is the multiplication of Γ. Some terms are omitted from the usual definition since there are no higher multiplications in Γ.

We only consider *strictly unital* modules, that is modules N such that for all $a \in N$, we have $m_2(1, a) = a$ and $m_n(\gamma_1, \dots, \gamma_{n-1}, a) = 0$ if $n \geqslant 3$ and $1 \in \{\gamma_1, \dots, \gamma_{n-1}\}$.

Note that if we let $N^i = \bigoplus_{j\in\mathbb{Z}} N^i_j$, then the N^i together with m_1 form a complex $(N^\bullet, m_1)$ of graded k-modules.

Two special cases are important. The first is when $m_n = 0$ for $n \geqslant 3$, the other is when $m_1 = 0$. In the second case each N^i is a graded Γ-module, not only a graded k-module. In the first case N is essentially a complex of graded Γ-modules, and we view complexes of graded Γ-modules in this way.

A *morphism* $f : L \to N$ between two A_∞-modules L and N is given by a family of maps

$$f_n : \Gamma^{\otimes n-1} \otimes L \to N, \quad n \geqslant 1$$

of bidegree $(n-1, 0)$ satisfying the rules

$$f_1 m_1 = m_1 f_1,$$
$$f_1 m_2 - f_2(\mathbb{1} \otimes m_1) = m_2(\mathbb{1} \otimes f_1) + m_1 f_2$$

and for $n \geqslant 3$

$$\sum_{i=1}^{n}(-1)^{i(n-1)} f_{n-i+1}\big(1^{\otimes n-i}\otimes m_i\big)+\sum_{j=1}^{n-2}(-1)^{j} f_{n-1}\big(1^{\otimes n-j-2}\otimes m\otimes 1^{\otimes j}\big)$$
$$=\sum_{r=1}^{n}(-1)^{(r+1)n} m_{n-r+1}\big(1^{\otimes n-r}\otimes f_r\big).$$

Note in particular that f_1 is a chain map $f_1:(L^\bullet, m_1)\to(N^\bullet, m_1)$ between complexes of graded k-modules.

We only consider *strictly unital* morphisms, that is morphisms f such that $f_n(\lambda_1,\dots,\lambda_{n-1},a)=0$ whenever $n\geqslant 2$ and $1\in\{\lambda_1,\dots,\lambda_{n-1}\}$.

The *identity morphism* $f:N\to N$ is given by $f_1=1$ and $f_i=0$ for all $i>0$. The composition $fg:N\to M$ of two morphisms $f:L\to M$ and $g:N\to L$ is given by the rule

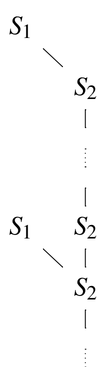
$$(fg)_n=\sum_{i=1}^{n} f_{n-i+1}\big(1^{\otimes n-i}\otimes g_i\big).$$

We say that f is a *quasi-isomorphism* if f_1 is a quasi-isomorphism. With the definitions we have made, the quasi-isomorphism classes we get do not differ significantly from the ones we have for complexes of graded Γ-modules. Each quasi-isomorphism class of A_∞-modules over Γ corresponds to (has as a subclass) exactly one quasi-isomorphism class of complexes of graded Γ-modules.

An important theorem [L, 3.3.1.7] states that for any strictly unital A_∞-module N, there is a A_∞-module structure on H^*N with $m_1=0$ and a strictly unital quasi-isomorphism between H^*N with this structure and N. If L and N are two modules, both with $m_1=0$, then each quasi-isomorphism between them is an isomorphism.

Therefore if we want to describe an object in $\mathcal{D}\,\mathrm{Gr}\,\Gamma$ (or strictly speaking an equivalent category), it suffices to specify its homology and an A_∞-module structure in its quasi-isomorphism class with $m_1=0$. If we want to classify all objects (up to isomorphism) in $\mathcal{X}$ with a certain homology, it suffices to classify all isomorphism classes of possible A_∞-module structures with $m_1=0$ on this homology.

Example 5.1. Let Λ and Γ be the same as in Example 4.1. Let $t\geqslant 1$ be a number and let $M_{(t)}$ denote the following infinite-dimensional Λ-module generated in degrees 0 and t.



In this example we find an A_∞-module structure (with $m_1 = 0$) for $GD(M_{(t)})$.

Since $\tilde{\mathfrak{gr}}(M_{(t)}) \simeq P_1 \oplus S_1\langle t\rangle$ is a Koszul module, the module $M_{(t)}$ is weakly Koszul. Therefore by Corollary 3.8 we have

$$H^0 GD(M_{(t)}) \simeq \check{E}(P_1) \simeq S_{1^*},$$
$$H^{-t} GD(M_{(t)}) \simeq \check{E}(S_1)\langle t\rangle \simeq P_{1^*}\langle t\rangle$$

and

$$H^n GD(M_{(t)}) \simeq 0$$

for $n \notin \{-t, 0\}$.

We now look for possible A_∞-structures (with $m_1 = 0$) on this homology. Fix a basis vector v for S_{1^*} and a basis vector w for the socle of $P_{1^*}\langle t\rangle$. Due to degree considerations (and the remark following this example), the only possibly non-zero higher structure is that $m_{t+2}(\beta, \beta, \dots, \beta, \alpha, v) = xw$ for some $x \in k$. All values of x give permissible A_∞-structures.

Again due to degree considerations, all quasi-isomorphisms between such structures must have $f_i = 0$ for $i \geqslant 2$. It is possible and easy to construct quasi-isomorphisms using only f_1 between structures with $x \neq 0$. We choose $x = 1$ as a representative for this orbit and denote by $N_{(t)}$ the corresponding object in $\mathcal{D}\,\mathrm{Gr}\,\Gamma$. The remaining case is $x = 0$ and corresponds to the object

$$P_{1^*}\langle t\rangle[t] \oplus S_{1^*} \simeq GD\big(P_1 \oplus S_1\langle t\rangle\big).$$

Since $M_{(t)} \not\simeq P_1 \oplus S_1\langle t\rangle$, we must have $GD(M_{(t)}) \simeq N_{(t)}$.

Therefore $GD(M_{(t)})$ can be described as the homology

$$H^0 GD(M_{(t)}) \simeq S_{1^*},$$
$$H^{-t} GD(M_{(t)}) \simeq P_{1^*}\langle t\rangle$$

with additional A_∞-structure

$$m_{t+2}(\beta, \beta, \dots, \beta, \alpha, v) = w.$$

Since t can be chosen arbitrarily large and $m_{t+2} \neq 0$, this example shows that arbitrarily high module structures are needed to describe all objects in $\mathcal{D}\,\mathrm{Gr}\,\Gamma$ in this way.

The weakly Koszul Λ-modules with adjusted associated graded module $P_1 \oplus S_1\langle t\rangle$ we have found to be $P_1 \oplus S_1\langle t\rangle$ and $M_{(t)}$.

Remark 5.2. We are assuming that the higher module structure maps m_n not only respect the ordinary grading, but also the grading given by the quiver. This can also be justified by a change of monoidal base category.

Example 5.3. In this example we show how to find all objects N in $\mathcal{D}\,\mathrm{Gr}\,\Gamma$ with the same homology as $GD(M)$ in Example 4.1. Each isomorphism class corresponds to a weakly Koszul module $\tilde{M}$ over Λ with $\tilde{\mathfrak{gr}}(\tilde{M}) \simeq \tilde{\mathfrak{gr}}(M)$. As a result of our computation we also find an A_∞-module structure for $GD(M)$.

The given homology is $H^0N \simeq S_{1^*}$, $H^{-1}N \simeq P_{1^*}\langle 1\rangle$, $H^{-2}N \simeq P_{1^*}\langle 2\rangle$ and $H^nN \simeq 0$ for $n \notin \{-2,-1,0\}$. In total this homology is 5-dimensional. We fix basis vectors

$$\begin{aligned} v_1 &\in \left(H^0N\right)_0, \\ v_2 &\in \left(H^{-1}N\right)_1, \\ v_3 &\in \left(H^{-1}N\right)_2, \\ v_4 &\in \left(H^{-2}N\right)_2, \\ v_5 &\in \left(H^{-2}N\right)_3. \end{aligned}$$

The possible higher products are $m_3(\beta,\alpha,v_1) = xv_3$, $m_3(\beta,\alpha,v_2) = yv_5$ and $m_4(\beta,\beta,\alpha,v_1) = zv_5$, where x, y and z are elements in k. All triples (x,y,z) give permissible A_∞-structures, so we have a 3-dimensional representation space. We now want to find the isomorphism classes. The possible quasi-isomorphisms are given by $f_1(v_i) = q_iv_i$, $1 \leqslant i \leqslant 5$ and $f_2(\beta,v_3) = \mu v_5$, where $q_i,\mu \in k$ and $q_i \neq 0$, $1 \leqslant i \leqslant 5$. It follows from the formulas that $q_2 = q_3$ and $q_4 = q_5$. If we let $a = q_2/q_1$, $b = q_4/q_2$ and $\rho = \mu/q_1$, then possible quasi-isomorphisms between triples (x,y,z) and (x',y',z') are given by the formulas

$$\begin{aligned} x' &= ax, \\ y' &= by + \rho, \\ z' &= abz - a\rho x. \end{aligned}$$

This divides the representation space into 4 orbits, namely

$$\begin{aligned} \mathcal{O}_1 &= \bigl\{(0,\rho,0) \bigm| \rho \in k\bigr\}, \\ \mathcal{O}_2 &= \bigl\{(a,\rho,-a\rho) \bigm| a,\rho \in k;\ a \neq 0\bigr\}, \\ \mathcal{O}_3 &= \bigl\{(0,\rho,c) \bigm| c,\rho \in k;\ c \neq 0\bigr\}, \\ \mathcal{O}_4 &= \bigl\{(a,\rho,c-a\rho) \bigm| a,c,\rho \in k;\ a,c \neq 0\bigr\}. \end{aligned}$$

We choose representatives $\mathbf{x}_1 = (0,0,0)$, $\mathbf{x}_2 = (1,0,0)$, $\mathbf{x}_3 = (0,0,1)$ and $\mathbf{x}_4 = (1,0,1)$ respectively. The object in $\mathcal{D}\operatorname{Gr}\Gamma$ corresponding to $\mathbf{x}_1$ is

$$S_{1^*} \oplus P_{1^*}\langle 1\rangle[1] \oplus P_{1^*}\langle 2\rangle[2] \simeq GD\bigl(P_1 \oplus S_1\langle 1\rangle \oplus S_1\langle 2\rangle\bigr).$$

From the previous example we recognise the object corresponding to $\mathbf{x}_2$ as

$$N_{(2)} \oplus P_{1^*}\langle 2\rangle[2] \simeq GD\bigl(M_{(2)} \oplus S_1\langle 2\rangle\bigr),$$

and the object corresponding to $\mathbf{x}_3$ as

$$N_{(3)} \oplus P_{1^*}\langle 1\rangle[1] \simeq GD\bigl(M_{(3)} \oplus S_1\langle 1\rangle\bigr).$$

Since $GD(M)$ is not isomorphic to any of these, it must correspond to $\mathbf{x}_4$.

So $GD(M)$ can be described as the homology

$$H^0GD(M) \simeq S_{1^*},$$
$$H^{-1}GD(M) \simeq P_{1^*}\langle 1\rangle,$$
$$H^{-2}GD(M) \simeq P_{1^*}\langle 2\rangle$$

with additional A_∞-structure

$$m_3(\beta, \alpha, v_1) = v_3,$$
$$m_4(\beta, \beta, \alpha, v_1) = v_5.$$

The weakly Koszul Λ-modules with adjusted associated graded module $P_1 \oplus S_1\langle 1\rangle \oplus S_1\langle 2\rangle$ we have found to be $P_1 \oplus S_1\langle 1\rangle \oplus S_1\langle 2\rangle$, $M_{(2)} \oplus S_1\langle 2\rangle$, $M_{(3)} \oplus S_1\langle 1\rangle$ and M.

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