The degree of an eight-dimensional real quadratic division algebra is 1, 3, or 5

Ernst Dieterich *, Ryszard Rubinsztein

Matematiska Institutionen, Uppsala Universitet, Box 480, SE-751 06 Uppsala, Sweden

Received 26 October 2009
Available online 31 October 2009

Abstract

A celebrated theorem of Hopf (1940) [11], Bott and Milnor (1958) [1], and Kervaire (1958) [12] states that every finite-dimensional real division algebra has dimension 1, 2, 4, or 8. While the real division algebras of dimension 1 or 2 and the real quadratic division algebras of dimension 4 have been classified (Dieterich (2005) [6], Dieterich (1998) [3], Dieterich and Öhman (2002) [9]), the problem of classifying all 8-dimensional real quadratic division algebras is still open. We contribute to a solution of that problem by proving that every 8-dimensional real quadratic division algebra has degree 1, 3, or 5. This statement is sharp. It was conjectured in Dieterich et al. (2006) [7].

© 2009 Elsevier Masson SAS. All rights reserved.

MSC: 17A35; 17A45; 55P91

Keywords: Real quadratic division algebra; Degree; Real projective space; Fundamental group; Liftings

1. Introduction

Let A be an algebra over a field k, i.e. a vector space over k equipped with a k-bilinear multiplication \( A \times A \to A \), \((x, y) \mapsto xy\). Every element \( a \in A \) determines k-linear operators \( L_a : A \to A, x \mapsto ax \) and \( R_a : A \to A, x \mapsto xa \). A division algebra over \( k \) is a non-zero \( k \)-algebra \( A \) such that \( L_a \) and \( R_a \) are bijective for all \( a \in A \setminus \{0\} \). A quadratic algebra over \( k \) is a non-zero \( k \)-algebra \( A \) with unity 1, such that for all \( x \in A \) the sequence 1, \( x \), \( x^2 \) is \( k \)-linearly dependent.

Here we focus on quadratic division algebras which are real (i.e. \( k = \mathbb{R} \)) and finite-dimensional. They form a category \( \mathcal{D}_q \) whose morphisms \( f : A \to B \) are non-zero linear maps

* Corresponding author.
E-mail addresses: Ernst.Dieterich@math.uu.se (E. Dieterich), Ryszard.Rubinsztein@math.uu.se (R. Rubinsztein).

0007-4497/$ – see front matter © 2009 Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.bulsci.2009.10.001
satisfying \( f(xy) = f(x)f(y) \) for all \( x, y \in A \). For any positive integer \( n \), the class of all \( n \)-dimensional objects in \( \mathcal{D}^q \) forms a full subcategory \( \mathcal{D}^q_n \) of \( \mathcal{D}^q \). The (1,2,4,8)-theorem for finite-dimensional real division algebras [11,1,12] implies that \( \mathcal{D}^q = \mathcal{D}^q_1 \cup \mathcal{D}^q_2 \cup \mathcal{D}^q_4 \cup \mathcal{D}^q_8 \). It is well known (see e.g. [9]) that both \( \mathcal{D}^q_1 \) and \( \mathcal{D}^q_2 \) consist of one isoclass only, represented by \( \mathbb{R} \) and \( \mathbb{C} \), respectively. (Even the categories \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) of all 1-dimensional and 2-dimensional real division algebras respectively are classified. While \( \mathcal{D}_1 = \mathcal{D}^q_1 \) [9], \( \mathcal{D}_2 \) consists of a 4-parameter family of isoclasses [6].) The category \( \mathcal{D}^q_2 \) is no longer trivial, but still its structure is well understood and its objects are classified by a 9-parameter family [3,5,9]. In contrast, the category \( \mathcal{D}^q_8 \) seems to be much more difficult to approach. The problem of understanding its structure appears to have a very hard core, which still is far from a finishing solution. Among the insight so far obtained, the degree of an 8-dimensional real quadratic division algebra is notable. Following [7] it is a natural number \( \text{deg}(A) \) associated with any \( A \in \mathcal{D}^q_8 \), which is invariant under isomorphisms and satisfies the estimate \( 1 \leq \text{deg}(A) \leq 5 \). Examples of algebras \( A \in \mathcal{D}^q_8 \) having degree 1, 3, and 5 respectively are also given in [7]. For every \( d \in \{1,\ldots,5\} \), the class of all algebras in \( \mathcal{D}^q_8 \) having degree \( d \) forms a full subcategory \( \mathcal{D}^{qd}_8 \) of \( \mathcal{D}^q_8 \).

In the present article we resume the investigation of \( \text{deg}(A) \). In Section 2 we apply topological arguments to prove that \( \text{deg}(A) \) always is odd. It follows that the category \( \mathcal{D}^q_8 \) decomposes into three non-empty blocks \( \mathcal{D}^{q1}_8 \), \( \mathcal{D}^{q3}_8 \), and \( \mathcal{D}^{q5}_8 \). In Section 3 we summarize our structural insight into these three blocks, concluding that the problem of understanding the structure of \( \mathcal{D}^q_8 \) has been reduced to the problem of understanding the structure of the category of all 7-dimensional dissident algebras having degree 3 or 5.

2. Main result

Towards our main result we need to recall a few facts related to the notion of a dissident map. Let \( V \) be a vector space over a field \( k \). Following [8], a dissident map on \( V \) is a \( k \)-linear map \( \eta : V \times V \to V \) such that \( v, w, \eta(v \wedge w) \) are \( k \)-linearly independent whenever \( v, w \) are. It is well known that a finite-dimensional real vector space \( V \) admits a dissident map if and only if \( \dim(V) \in \{0,1,3,7\} \) [4, Proposition 7].

Let \( \eta : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7 \) be a dissident map. We equip \( \mathbb{R}^7 \) with the standard scalar product \( \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}, \ (v,w) \mapsto v \cdot w \). For every \( v \in \mathbb{R}^7 \setminus \{0\} \), the subspace \( \eta(v \wedge v^\perp) = \{\eta(v \wedge w) \mid w \in v^\perp\} \) in \( \mathbb{R}^7 \) is a hyperplane, which only depends on the line \( [v] \) spanned by \( v \). Thus \( \eta \) induces a map
\[
\eta_\mathbb{P} : \mathbb{P}(\mathbb{R}^7) \to \mathbb{P}(\mathbb{R}^7), \quad \eta_\mathbb{P}(\{v\}) = (\eta(v \wedge v^\perp))^\perp,
\]
which actually is bijective [8, Proposition 2.2]. Following [7], a lifting of \( \eta_\mathbb{P} \) is a map \( \Phi : \mathbb{R}^7 \to \mathbb{R}^7, \ \Phi(v) = (\varphi_1(v),\ldots,\varphi_7(v)) \), satisfying the following three conditions: (a) all component maps \( \varphi_1,\ldots,\varphi_7 \) are homogeneous real polynomials of common degree \( d \geq 1 \); (b) if \( v \in \mathbb{R}^7 \setminus \{0\} \), then \( \Phi(v) \neq 0 \) and \( [\Phi(v)] = \eta_\mathbb{P}([v]) \); (c) the polynomials \( \varphi_1,\ldots,\varphi_7 \) are relatively prime.

According to [7, Theorem 2.4] a lifting \( \Phi \) of \( \eta_\mathbb{P} \) exists, is unique up to non-zero real multiples, and satisfies \( 1 \leq d \leq 5 \). It is therefore justified to define the degree of a dissident map \( \eta \) on \( \mathbb{R}^7 \) by \( \text{deg}(\eta) := \text{deg}(\Phi) := d \).

**Proposition 2.1.** The degree of any dissident map on \( \mathbb{R}^7 \) is odd.

**Proof.** Our proof of Proposition 2.1 is topological in nature and, in particular, uses the fundamental group of the projective space \( \mathbb{P}(\mathbb{R}^7) \).
Let $\pi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{P}^{7}$ be the quotient map mapping a vector $v \in \mathbb{R}^7 \setminus \{0\}$ to the line $[v] \in \mathbb{P}(\mathbb{R}^7)$ spanned by $v$. We equip $\mathbb{R}^7 \setminus \{0\}$ with its standard Euclidean topology and $\mathbb{P}(\mathbb{R}^7)$ with the quotient topology. For a point $z \in \mathbb{P}(\mathbb{R}^7)$ let $\pi_1(\mathbb{P}(\mathbb{R}^7), z)$ be the fundamental group of $\mathbb{P}(\mathbb{R}^7)$ based at $z$. We recall that

$$
\pi_1(\mathbb{P}(\mathbb{R}^7), z) = \mathbb{Z}_2, \\
\mathbb{Z}_2 = \{0, 1\} being the group of integers modulo 2 [2, Section III.5].
$$

The map $\pi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{P}(\mathbb{R}^7)$ is a locally trivial fibration with the fibre homeomorphic to $\mathbb{R} \setminus \{0\}$. We choose a point $z \in \mathbb{P}(\mathbb{R}^7)$ and a vector $\tilde{z} \in \mathbb{R}^7 \setminus \{0\}$ such that $\pi(\tilde{z}) = z$. Then every vector $w$ belonging to the fibre $\pi^{-1}(z)$ is of the form $w = q\tilde{z}$ with $q \in \mathbb{R}$, $q \neq 0$.

Denote by $I$ the unit interval $[0, 1]$.

Let $\sigma : I \to \mathbb{P}(\mathbb{R}^7)$ be a loop in $\mathbb{P}(\mathbb{R}^7)$ at the point $z$ i.e. $\sigma$ is a continuous mapping such that $\sigma(0) = \sigma(1) = z$. Since $\pi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{P}(\mathbb{R}^7)$ is a fibration, it follows from the Homotopy Lifting Theorem, [2, Theorem VII.6.4], that the loop $\sigma : I \to \mathbb{P}(\mathbb{R}^7)$ can be lifted to a continuous mapping $\tilde{\sigma} : I \to \mathbb{R}^7 \setminus \{0\}$ such that $\pi \circ \tilde{\sigma} = \sigma$ and $\tilde{\sigma}(0) = \tilde{z}$. (Observe that, in general, $\tilde{\sigma}$ will not be a loop but just a path.) Since $\sigma(1) = z$, it follows that $\tilde{\sigma}(1) \in \pi^{-1}(z)$ and, hence, that $\tilde{\sigma}(1) = q\tilde{z}$ for some $q = q(\sigma) \in \mathbb{R}$, $q \neq 0$. (Actually, the real number $q$ depends not only on the loop $\sigma$ but also on the choice of the lifting $\tilde{\sigma}$ which is not unique.) The next lemma is rather obvious. We include a proof for the convenience of the reader.

**Lemma 2.2.** The loop $\sigma : I \to \mathbb{P}(\mathbb{R}^7)$ represents the trivial element in $\pi_1(\mathbb{P}(\mathbb{R}^7), z) = \mathbb{Z}_2$ if and only if $q > 0$.

**Proof.** ($\Leftarrow)$ Suppose that $q > 0$. Then there is a line segment $\tilde{r}$ in $\pi^{-1}(z)$ from $q\tilde{z}$ to $\tilde{z}$ consisting of points of the form $s\tilde{z}$ with $s$ between 1 and $q$. The path product $\tilde{\sigma} = \tilde{\sigma} * \tilde{r}$ is now a loop in $\mathbb{R}^7 \setminus \{0\}$ starting and ending at $\tilde{z}$. The space $\mathbb{R}^7 \setminus \{0\}$ is homotopy equivalent to the 6-dimensional sphere $S^6$ and hence the fundamental group $\pi_1(\mathbb{R}^7 \setminus \{0\}, \tilde{z}) = 0$ is trivial. Therefore the loop $\sigma = \tilde{\sigma} * \tilde{r}$ is null-homotopic in $\mathbb{R}^7 \setminus \{0\}$. It follows that the loop $\pi \circ \tilde{\sigma} = (\pi \circ \tilde{\sigma}) * (\pi \circ \tilde{r}) = \sigma * (\pi \circ \tilde{r})$ is null-homotopic in $\mathbb{P}(\mathbb{R}^7)$. Since $\pi \circ \tilde{r}$ is a constant loop, the loops $\sigma * (\pi \circ \tilde{r})$ and $\sigma$ are homotopic. Therefore the loop $\sigma$ is null-homotopic in $\mathbb{P}(\mathbb{R}^7)$ and represents the trivial element in $\pi_1(\mathbb{P}(\mathbb{R}^7), z)$.

($\Rightarrow$) Suppose that the loop $\sigma : I \to \mathbb{P}(\mathbb{R}^7)$ represents the trivial element in $\pi_1(\mathbb{P}(\mathbb{R}^7), z)$. Thus there exists a continuous mapping (homotopy) $H : I \times I \to \mathbb{P}(\mathbb{R}^7)$ such that $H(t, 0) = \sigma(t)$ and $H(0, s) = H(1, s) = H(t, 1) = z$ for all $t, s \in I$. Again, according to the Homotopy Lifting Theorem there exists a continuous mapping (lift) $\tilde{H} : I \times I \to \mathbb{R}^7 \setminus \{0\}$ such that $\pi \circ \tilde{H} = H$ and $\tilde{H}(t, 0) = \tilde{\sigma}(t)$, $\tilde{H}(0, s) = \tilde{z}$ and $\tilde{H}(1, s) = \tilde{\sigma}(1)$ for all $t, s \in I$. It follows, in particular, that $\tilde{H}(t, 1) \in \pi^{-1}(z)$ for all $t \in I$ and that $\tilde{H}(0, 1) = \tilde{z}$ while $\tilde{H}(1, 1) = \tilde{\sigma}(1) = q\tilde{z}$. In other words, $\tilde{H}(t, 1)$, $t \in I$, is an arc in $\pi^{-1}(z)$ from $\tilde{z}$ to $\tilde{\sigma}(1) = q\tilde{z}$. As $\pi^{-1}(z)$ consists of points of the form $r\tilde{z}$ with $r \in \mathbb{R} \setminus \{0\}$ it follows that $q > 0$.

That completes the proof of Lemma 2.1. □

The diagram

\[
\begin{array}{ccc}
\mathbb{R}^7 \setminus \{0\} & \xrightarrow{\phi} & \mathbb{R}^7 \setminus \{0\} \\
\downarrow \pi & & \downarrow \pi \\
\mathbb{P}(\mathbb{R}^7) & \xrightarrow{\eta_\pi} & \mathbb{P}(\mathbb{R}^7)
\end{array}
\]
commutes. Since the components $\varphi_1, \ldots, \varphi_7$ of $\Phi$ are polynomials, the map $\Phi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{R}^7 \setminus \{0\}$ is continuous. It follows that the map $\eta_\Phi : \mathbb{P}(\mathbb{R}^7) \to \mathbb{P}(\mathbb{R}^7)$ is also continuous. The map $\eta_\Phi$ is bijective [8, Proposition 2.2] and the projective space $\mathbb{P}(\mathbb{R}^7)$ is compact and Hausdorff. Therefore $\eta_\Phi$ is a homeomorphism. Let us choose a point $z_0 \in \mathbb{P}(\mathbb{R}^7)$ and let us denote by $z_1$ the point $\eta_\Phi(z_0) \in \mathbb{P}(\mathbb{R}^7)$. The homeomorphism $\eta_\Phi$ induces a group isomorphism

$$\eta_{\Phi_*} : \pi_1(\mathbb{P}(\mathbb{R}^7), z_0) \xrightarrow{\approx} \pi_1(\mathbb{P}(\mathbb{R}^7), z_1).$$

Let us choose a point $\tilde{z}_0 \in \pi^{-1}(z_0) \subset \mathbb{R}^7 \setminus \{0\}$. Let us denote by $\tilde{z}_1$ the point $\Phi(\tilde{z}_0) \in \mathbb{R}^7 \setminus \{0\}$. Then $\tilde{z}_1 = \Phi(\tilde{z}_0) \in \pi^{-1}(z_1)$.

Since the space $\mathbb{R}^7 \setminus \{0\}$ is path-connected, we can find a path $\tilde{\alpha} : I \to \mathbb{R}^7 \setminus \{0\}$ such that $\tilde{\alpha}(0) = \tilde{z}_0$ and $\tilde{\alpha}(1) = -\tilde{z}_0$. Then the composition $\alpha = \pi \circ \tilde{\alpha} : I \to \mathbb{P}(\mathbb{R}^7)$ is a loop in $\mathbb{P}(\mathbb{R}^7)$ which starts and ends at $z_0 = \pi(\tilde{z}_0) = \pi(-\tilde{z}_0)$. The path $\tilde{\alpha}$ is a lift of the loop $\alpha$ to $\mathbb{R}^7 \setminus \{0\}$ which starts at $\tilde{z}_0$ and ends at $-\tilde{z}_0$. Thus, according to Lemma 2.1, the loop $\alpha$ represents the non-trivial element of $\pi_1(\mathbb{P}(\mathbb{R}^7), z_0)$.

Let us now consider the path $\tilde{\beta} = \Phi \circ \tilde{\alpha} : I \to \mathbb{R}^7 \setminus \{0\}$ in $\mathbb{R}^7 \setminus \{0\}$ starting at $\tilde{z}_1 = \Phi(\tilde{z}_0) \in \pi^{-1}(z_1)$ and ending at $\tilde{z}_2 = \Phi(-\tilde{z}_0) = \pi^{-1}(z_1)$. If $d = \deg(\Phi)$ is the degree of $\Phi$ then, by the definition of the degree, $\tilde{z}_2 = \Phi(-\tilde{z}_0) = (-1)^d \Phi(\tilde{z}_0) = (-1)^d \tilde{z}_1$.

The composition $\beta = \pi \circ \tilde{\beta} : I \to \mathbb{P}(\mathbb{R}^7)$ is a loop in $\mathbb{P}(\mathbb{R}^7)$ which starts and ends at the point $z_1 = \pi(\tilde{z}_1) = \pi(\tilde{z}_2)$. Moreover,

$$\beta = \pi \circ \tilde{\beta} = \pi \circ \Phi \circ \tilde{\alpha} = \eta_\Phi \circ \pi \circ \tilde{\alpha} = \eta_\Phi \circ \alpha.$$

Thus the homotopy class $[\beta]$ of the loop $\beta$ in $\pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$ is equal to $[\eta_\Phi \circ \alpha] = \eta_{\Phi_*}[\alpha]$. As the homotopy class $[\alpha]$ of $\alpha$ in $\pi_1(\mathbb{P}(\mathbb{R}^7), z_0)$ was non-trivial and $\eta_{\Phi_*} : \pi_1(\mathbb{P}(\mathbb{R}^7), z_0) \to \pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$ was an isomorphism, it follows that $\beta$ represents the non-trivial element of $\pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$.

On the other hand the path $\tilde{\beta} : I \to \mathbb{R}^7 \setminus \{0\}$ is a lifting of the loop $\beta$ to $\mathbb{R}^7 \setminus \{0\}$ which starts at the point $\tilde{z}_1 \in \mathbb{R}^7 \setminus \{0\}$ and ends at the point $\tilde{z}_2 = (-1)^d \tilde{z}_1$. Since $\beta$ represents the non-trivial element of $\pi_1(\mathbb{P}(\mathbb{R}^7), z_1)$, it follows from Lemma 2.1 that $(-1)^d < 0$ and, thus, the degree $d = \deg(\Phi)$ is odd.

That completes the proof of Proposition 2.1. \qed

**Remark 2.3.** The space $\mathbb{R}^7 \setminus \{0\}$ is homotopy equivalent to the 6-dimensional sphere $S^6$. Its homology group $H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z})$ with coefficients in $\mathbb{Z}$ in dimension 6 is isomorphic to the group of integers, $H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z}) \cong \mathbb{Z}$. The continuous mapping $\Phi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{R}^7 \setminus \{0\}$ induces a group homomorphism $\Phi_* : H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z}) \to H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z})$ which is given by multiplication by an integer usually called the (topological) degree of the map $\Phi$. Since the map $\eta_\Phi : \mathbb{P}(\mathbb{R}^7) \to \mathbb{P}(\mathbb{R}^7)$ is a homeomorphism and the algebraic degree $d = \deg(\Phi)$ in the sense of this paper is odd, it is rather easy to see that also the mapping $\Phi : \mathbb{R}^7 \setminus \{0\} \to \mathbb{R}^7 \setminus \{0\}$ is a homeomorphism. It follows that $\Phi_* : H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z}) \to H_6(\mathbb{R}^7 \setminus \{0\}, \mathbb{Z})$ is an isomorphism and, hence, its topological degree can only be equal to $\pm 1$. Thus the topological degree of the map $\Phi$ and its algebraic degree $d = \deg(\Phi)$ in the sense of [7] and of the present paper are different notions, at least when the algebraic degree $d = \deg(\Phi)$ is equal to 3 or 5.

Now let $A$ be an 8-dimensional real quadratic division algebra. Since $A$ is a quadratic algebra, Frobenius’s lemma [10,13] applies. It asserts that the set $V = \{ v \in A \setminus \mathbb{R}1 \mid v^2 \in \mathbb{R}1 \} \cup \{0\}$ of all purely imaginary elements in $A$ is a hyperplane in $A$, such that $A = \mathbb{R}1 \oplus V$. This Frobenius
decomposition of $A$ gives rise to the $\mathbb{R}$-linear maps $\varrho : A \to \mathbb{R}$ and $\iota : A \to V$ such that $a = \varrho(a)1 + \iota(a)$ for all $a \in A$. The induced algebra structure on $V$, i.e. the bilinear map $\eta : V \times V \to V$, $\eta(v, w) = \iota(vw)$, is anticommutative. Therefore it may be identified with the linear map $\eta : V \wedge V \to V$, $\eta(v \wedge w) = \iota(vw)$. Since $A$ is a division algebra, this linear map $\eta$ is dissident [14]. Any choice of a basis in $V$ identifies $\eta$ with a dissident map $\eta$ on $\mathbb{R}^2$, and the degree of $\eta$ does not depend on the chosen basis. It is therefore justified to define the degree of an 8-dimensional real quadratic division algebra $A$ by $\deg(A) := \deg(\eta)$.

**Corollary 2.4.** The degree of any 8-dimensional real quadratic division algebra is 1, 3, or 5.

**Proof.** Let $A \in \mathcal{D}_8^d$. Then $\deg(A) = \deg(\eta) = d$, where $1 \leq d \leq 5$ by [7, Theorem 2.4], and $\deg(\eta)$ is odd by Proposition 2.1.  

**Corollary 2.5.** The category $\mathcal{D}_8^d$ decomposes into its non-empty full subcategories $\mathcal{D}_8^1$, $\mathcal{D}_8^3$, and $\mathcal{D}_8^5$.

**Proof.** Corollary 2.4 states that the object class of $\mathcal{D}_8^d$ is the disjoint union of the object classes of $\mathcal{D}_8^1$, $\mathcal{D}_8^3$, and $\mathcal{D}_8^5$.

Let $f : A \to A'$ be a morphism in $\mathcal{D}_8^d$. The algebra structures on $A$ and $A'$ induce dissident maps $\eta$ and $\eta'$ on the purely imaginary hyperplanes $V$ and $V'$ of $A$ and $A'$, respectively. Now $f$ is injective because $A$ is a division algebra, and furthermore even bijective because $\dim(A) = \dim(A')$ is finite. So $f$ is an isomorphism of algebras. It induces an isomorphism of dissident maps $\sigma : \eta \to \eta'$, i.e. a linear bijection $\sigma : V \to V'$ satisfying $\sigma(\eta(v \wedge w)) = \eta'(\sigma(v) \wedge \sigma(w))$ for all $v, w \in V$. We conclude with [7, Proposition 3.1] that $\deg(\eta) = \deg(\eta')$, hence $\deg(A) = \deg(A')$. Thus $f$ is a morphism in $\mathcal{D}_8^d$ for some $d \in \{1, 3, 5\}$.

Altogether this proves the decomposition $\mathcal{D}_8^d = \mathcal{D}_8^1 \sqcup \mathcal{D}_8^3 \sqcup \mathcal{D}_8^5$ of the category $\mathcal{D}_8^d$. Non-emptiness of its blocks $\mathcal{D}_8^1$, $\mathcal{D}_8^3$, and $\mathcal{D}_8^5$ follows from [7, Section 6], where objects are constructed for each of them.  

**3. On the structure of $\mathcal{D}_8^1$, $\mathcal{D}_8^3$, and $\mathcal{D}_8^5$**

Corollary 2.5 reduces the problem of understanding the structure of $\mathcal{D}_8^d$ to the problem of understanding the structures of $\mathcal{D}_8^1$, $\mathcal{D}_8^3$, and $\mathcal{D}_8^5$. We proceed to summarize the present state of knowledge regarding the latter problem.

A **dissident triple** $(V, \xi, \eta)$ consists of a (finite-dimensional) Euclidean space $V = (V, \langle \cdot, \cdot \rangle)$, a linear form $\xi : V \wedge V \to \mathbb{R}$, and a dissident map $\eta : V \wedge V \to V$. The class of all dissident triples forms a category $\mathcal{Y}$ whose morphisms $\varphi : (V, \xi, \eta) \to (V', \xi', \eta')$ are orthogonal linear maps $\varphi : V \to V'$ satisfying $\xi = \xi' (\varphi \wedge \varphi)$ and $\varphi \eta = \eta' (\varphi \wedge \varphi)$. If $(V, \xi, \eta)$ is a dissident triple, then the vector space $\mathcal{F}(V, \xi, \eta) = \mathbb{R} \times V$, equipped with the multiplication

$$(\alpha, v)(\beta, w) = (\alpha \beta - \langle v, w \rangle + \xi(v \wedge w), \alpha v + \beta w + \eta(v \wedge w)),$$

is a real quadratic division algebra. If $\varphi : (V, \xi, \eta) \to (V', \xi', \eta')$ is a morphism of dissident triples, then the linear map

$$(\mathcal{F}(\varphi) : \mathcal{F}(V, \xi, \eta) \to \mathcal{F}(V', \xi', \eta'), \mathcal{F}(\varphi)(\alpha, v) = (\alpha, \varphi(v))$$
is a morphism of real quadratic division algebras. It is well known [5,9] that Osborn’s theorem [14] can be rephrased in the language of categories and functors as follows.

**Theorem 3.1.** The functor $\mathcal{F} : \mathcal{V} \rightarrow \mathcal{D}^q$ is an equivalence of categories.

For each $d \in \{1, 3, 5\}$ we denote by $\mathcal{V}^d$ the full subcategory of $\mathcal{V}$ formed by all dissident triples $(V, \xi, \eta)$ satisfying $\dim(V) = 7$ and $\deg(\eta) = d$. Then the equivalence of categories $\mathcal{F} : \mathcal{V} \rightarrow \mathcal{D}^q$ induces equivalences of categories $\mathcal{F}^d : \mathcal{V}^d \rightarrow \mathcal{D}^q_8$ for all $d \in \{1, 3, 5\}$. The category $\mathcal{V}^1_7$ admits an equivalent description entirely in terms of matrices, which we proceed to recall.

The octonion algebra $\mathbb{O}$ is well known to be quadratic. Hence it has Frobenius decomposition $\mathbb{O} = \mathbb{R}1 \oplus V$, and thereby it determines $\mathbb{R}$-linear maps $\varrho : \mathbb{O} \rightarrow \mathbb{R}$ and $\iota : \mathbb{O} \rightarrow V$ such that $a = \varrho(a)1 + \iota(a)$ for all $a \in \mathbb{O}$. The symmetric $\mathbb{R}$-bilinear form

$$\mathbb{O} \times \mathbb{O} \rightarrow \mathbb{R}, \quad (x, y) = 2\varrho(x)\varrho(y) - \frac{1}{2}\varrho(xy + yx)$$

is well known to be positive definite, thus equipping $\mathbb{O}$ with the structure of a Euclidean space. Every algebra automorphism $\alpha \in \text{Aut}(\mathbb{O})$ fixes the unity $1$ of $\mathbb{O}$ and is orthogonal. Since $1^1 = V$, it induces an orthogonal linear endomorphism $\alpha_V \in O(V)$. The map $\nu : \text{Aut}(\mathbb{O}) \rightarrow O(V), \nu(\alpha) = \alpha_V$ is an injective group homomorphism. Choosing an orthonormal basis in $V$, the subgroup $\nu(\text{Aut}(\mathbb{O})) < O(V)$ is identified with a subgroup of $O(7)$, which classically is denoted by $\mathbb{G}_2$. Simultaneously, the dissident map

$$\eta : V \wedge V \rightarrow V, \quad \eta(v \wedge w) = \iota(vw)$$

is identified with a vector product map $\mathbb{R}^7 \wedge \mathbb{R}^7 \rightarrow \mathbb{R}^7, v \wedge w \mapsto v \times w$. Now denote by $\mathbb{R}^{7 \times 7}$ the set of all real $7 \times 7$-matrices, and by $\mathbb{R}^{7 \times 7}_{\text{ant}}, \mathbb{R}^{7 \times 7}_{\text{pds}}, \mathbb{R}^{7 \times 7}_{\text{spds}}$ the subsets of $\mathbb{R}^{7 \times 7}$ consisting of all matrices which are antisymmetric, positive definite symmetric, and positive definite symmetric of determinant $1$, respectively. We view the matrix quadruple set

$$\mathcal{Q} = \mathbb{R}^{7 \times 7}_{\text{ant}} \times \mathbb{R}^{7 \times 7}_{\text{ant}} \times \mathbb{R}^{7 \times 7}_{\text{pds}} \times \mathbb{R}^{7 \times 7}_{\text{spds}}$$

as the object set of a groupoid $\mathcal{Q}$ whose morphisms

$$S : (A, B, C, D) \rightarrow (A', B', C', D')$$

are the orthogonal matrices $S \in \mathbb{G}_2$ which satisfy

$$(SS^t, SBS^t, SCS^t, SDS^t) = (A', B', C', D').$$

Then the groupoid $\mathcal{Q}$ and the category $\mathcal{V}^1_7$ are related as follows. (For proofs see [7, Section 5].)

**Theorem 3.2.**

(i) If $(A, B, C, D) \in \mathcal{Q}$ then $\mathcal{G}(A, B, C, D) = (\mathbb{R}^7, \xi, \eta)$, with $\xi(v \wedge w) = v^tAw$ and $\eta(v \wedge w) = (B + C)D(Dv \times Dw)$, is in $\mathcal{V}^1_7$.

(ii) If $S : (A, B, C, D) \rightarrow (A', B', C', D')$ is a morphism in $\mathcal{Q}$ then $\mathcal{G}(S) : \mathcal{G}(A, B, C, D) \rightarrow \mathcal{G}(A', B', C', D')$, given by $\mathcal{G}(S)(v) = Sv$, is a morphism in $\mathcal{V}^1_7$.

(iii) The functor $\mathcal{G} : \mathcal{Q} \rightarrow \mathcal{V}^1_7$ is an equivalence of categories.
Composing the functors $\mathcal{F}$ and $\mathcal{F}_1$ to $\mathcal{H} = \mathcal{F}_1 \mathcal{F}$, we arrive at the following explicit description of the category $\mathcal{D}_8^{q1}$ entirely in terms of matrices.

**Corollary 3.3.** The functor $\mathcal{H} : \mathcal{D} \rightarrow \mathcal{D}_8^{q1}$ is an equivalence of categories. It is given on objects by $\mathcal{H}(A, B, C, D) = \mathbb{R} \times \mathbb{R}^7$, with multiplication $(\alpha, v)(\beta, w) = (\alpha \beta - v^t w + v^t A w, \alpha w + \beta v + (B + C)D(Dv \times Dw))$, and on morphisms by $\mathcal{H}(S)(\alpha, v) = (\alpha, Sv)$.

On the other hand, we do not know any description of the categories $\mathcal{V}_7^3$ or $\mathcal{V}_7^5$ entirely in terms of matrices. Indeed, the 7-dimensional dissident algebras $(V, \eta)$ of degree 3 or 5 which are inherent in the objects $(V, \xi, \eta)$ of $\mathcal{V}_7^3$ or $\mathcal{V}_7^5$ respectively seem hardly to be understood at present.

**References**


