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Analysis of nonlinear oscillations of a punctual charge in the electric field of a charged ring via a Hamiltonian approach and the energy balance method

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ABSTRACT

The method of Hamiltonian approach and the energy balance method are applied to obtain the periodic solutions of nonlinear oscillations of a punctual charge in the electric field of charged ring. The obtained approximate frequencies are accurate for the entire range of oscillation amplitudes. A good agreement of the approximate frequencies and periodic solutions with the exact ones are demonstrated and discussed. It is also proved that the results of the energy balance method are better than the Hamiltonian approach for solving this equation.

1. Introduction

In recent years, many papers are devoted to the analysis of nonlinear differential equation. Nonlinear vibration has attracted the attention of many researchers. This might be due to its application in mechanics, electronics and physics. Currently, a variety of approaches have been invented and modified for solving nonlinear equations, such as the homotopy perturbation [1–4], variational approach [5–8], energy balance method [9–13], Hamiltonian approach [14,15], and other methods [16–19]. The Tanh method is an effective approach for solving the nonlinear equations. This method was applied for solving the Kolmogorov–Petrovski–Piskunov equation [20] and $(3+1)$-dimensional Kadomtsev–Petviashvili equations [21]. The transformed rational function method provides an analytical approach for solving the nonlinear partial differential equations. This method appropriately deals with the $(3 + 1)$-dimensional Jimbo–Miwa equation [22]. The multiple exp function method was proposed for exact multiple wave solutions of nonlinear partial differential equations where the $(3+1)$-dimensional potential Yu–Toda–Sasa–Fukuyama equation was investigated by this method [23]. This method is an efficient approach if computer algebra systems are adopted for use. Hirota's bilinear technique is a useful approach especially for solving bilinear problems. Indeed, bilinear equations are almost close to linear equations in which some properties are likely same as linear equations [24]. Recently, the Hamiltonian approach has been invented by He [15]. By means of He's new method, a large number of well-known nonlinear equations were solved. Yildirim et al. [25] solved three kinds of nonlinear equations by this method.

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1.1. Nonlinear oscillator with rational force

$$\ddot{x} + \frac{x^3}{x^2+1} = 0 \tag{1}$$

$$\ddot{x} + \frac{x}{x^2+1} = 0. \tag{2}$$

1.2. Nonlinear oscillator with irrational elastic force

$$\ddot{x} + x - \frac{\lambda x}{\sqrt{x^2+1}} = 0. \tag{3}$$

A generalized nonlinear equation was analyzed by Cveticanin et al. by means of it. Cveticanin et al. [26] solved this equation and investigated the frequency responses of the system for some special cases.

$$\ddot{x} + x\,|x|^{\alpha-1} = 0. \tag{4}$$

Nonlinear Vibration of a rigid rod on a circular surface was investigated by Khan et al. [27] with the Hamiltonian approach. In the referred papers, the Hamiltonian approach has valid solutions and simple procedure. The energy balance method is more well known and has been applied in various types of nonlinear equations; for instance, relativistic oscillator [28], nonlinear oscillator with fractional elastic force [12], harmonic Duffing oscillator [10] and so on [11,13]. This method was primarily introduced by He [9] and then developed by Sfahani et al. [29] and Younesian et al. [12]. Although the application of this method is so simple but the reliability and accuracy of this method is high in comparison with perturbative approaches. In this paper, the nonlinear oscillations of a punctual charge in the electric field of charged ring [30–32] are analyzed by means of the Hamiltonian approach and the energy balance method. According to the solution procedure, it is also demonstrated that the energy balance method is much simpler than the the Hamiltonian approach for solving this equation and even the accuracy of EBM is higher than HA for the first approximation. It is remarkable that EBM without any requirement to the difficult mathematical procedure can solve this equation with a high reliability.

2. Solution procedure

Assume a ring with radius R in which a charge $Q > 0$ is spread uniformly around that. The electric field E on the axis of ring is expressed by [30–32]

$$E(x) = \frac{1}{4\pi\varepsilon_0}\frac{Qx}{\left(R^2+x^2\right)^{3/2}} \tag{5}$$

where x denotes the distance along the axis. Consider a negative punctual charge $q = -|q|$ which is placed at a point on ring axis. A force $F(x)$ is exerted to the charge as follows:

$$F(x) = -\frac{1}{4\pi\varepsilon_0}\frac{|q|\,Qx}{\left(R^2+x^2\right)^{3/2}}. \tag{6}$$

Consequently, the equation of motion of the punctual charge with the mass m equals to the following nonlinear equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{1}{4\pi\varepsilon_0}\frac{|q|\,Qx}{\left(R^2+x^2\right)^{3/2}} = 0. \tag{7}$$

The initial conditions are

$$x(0) = x_0 \tag{8}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0. \tag{9}$$

Eq. (7) can be written as follows

$$\frac{1}{R}\frac{\mathrm{d}^2x}{\mathrm{d}x^2} + \omega_0^2\left(1+\frac{x^2}{R^2}\right)^{-3/2}\frac{x}{R} = 0 \tag{10}$$

where

$$\omega_0 = \sqrt{\frac{|q|\,Q}{4\pi\varepsilon_0 mR^3}}. \tag{11}$$

By defining two dimensionless variables y and τ as follows

$$x = Ry, \qquad t = \omega_0 \tau. \tag{12}$$

Eq. (9) is converted to

$$\frac{d^2y}{d\tau^2} + \frac{y}{\left(1+y^2\right)^{3/2}} = 0. \tag{13}$$

With initial conditions

$$y(0) = A, \qquad \frac{dy}{d\tau}(0) = 0. \tag{14}$$

2.1. Hamiltonian approach

The Hamiltonian form of Eq. (13) can be easily constructed as

$$H = \frac{1}{2}\dot{y}^2 - \frac{1}{\left(1+y^2\right)^{1/2}}. \tag{15}$$

By integrating Eq. (15) with respect to τ from 0 to $\frac{T}{4}$, we have

$$\tilde{H}(y) = \int_0^{\frac{T}{4}} \left[\frac{1}{2}\dot{y}^2 - \frac{1}{\left(1+y^2\right)^{1/2}}\right] d\tau. \tag{16}$$

We consider the solution $y(\tau) = A\cos(\omega\tau)$. Substituting it into Eq. (16), yields

$$\begin{aligned}\tilde{H}(A,\omega) &= \int_0^{\frac{T}{4}} \left[\frac{1}{2}A^2\omega^2\sin^2(\omega t) - \frac{1}{(1+A^2\cos^2(\omega\tau))^{1/2}}\right] d\tau \\ &= \int_0^{\frac{\pi}{2}} \left[\frac{1}{2}A^2\omega\sin^2\tau - \frac{1}{\omega\sqrt{1+A^2}}\frac{1}{\left(1-\frac{A^2}{1+A^2}\sin^2\tau\right)^{1/2}}\right] d\tau.\end{aligned} \tag{17}$$

Using the following formula of the first kind of elliptic integration

$$\int_0^{\frac{\pi}{2}} \frac{d\tau}{\left(1-m^2\sin^2\tau\right)^{1/2}} = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 m^2 + \left(\frac{1\times 3}{2\times 4}\right)^2 m^4 + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^2 m^6 + \cdots\right]. \tag{18}$$

We can reach to

$$\begin{aligned}\tilde{H}(A,\omega) = \frac{\pi}{4}A^2\omega - \frac{\pi}{2\omega\sqrt{1+A^2}} &\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{A^2}{1+A^2}\right)\right. \\ &\left. + \left(\frac{1\times 3}{2\times 4}\right)^2\left(\frac{A^2}{1+A^2}\right)^2 + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^2\left(\frac{A^2}{1+A^2}\right)^3 + \cdots\right\}.\end{aligned} \tag{19}$$

Setting

$$\begin{aligned}\frac{\partial}{\partial A}\left(\frac{\partial H(A,\omega)}{\partial\left(\frac{1}{\omega}\right)}\right) = -A\omega^2 - \frac{\partial}{\partial A}&\left(\frac{1}{\sqrt{1+A^2}}\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{A^2}{1+A^2}\right) + \left(\frac{1\times 3}{2\times 4}\right)^2\left(\frac{A^2}{1+A^2}\right)^2\right.\right. \\ &\left.\left. + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^2\left(\frac{A^2}{1+A^2}\right)^3 + \cdots\right\}\right) = 0.\end{aligned} \tag{20}$$

Finally, the natural frequency of the system is obtained as

$$\omega_{HA} = \sqrt{-\frac{1}{A}\frac{\partial}{\partial A}\left(\frac{1}{\sqrt{1+A^2}}\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{A^2}{1+A^2}\right) + \left(\frac{1\times 3}{2\times 4}\right)^2\left(\frac{A^2}{1+A^2}\right)^2 + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^2\left(\frac{A^2}{1+A^2}\right)^3 + \cdots\right\}\right)}. \tag{21}$$

2.2. Energy balance method

The following Hamiltonian is established for Eq. (13) using the semi-inverse method

$$H = \frac{1}{2}\dot{y}^2 - \frac{1}{\left(1+y^2\right)^{1/2}} = -\frac{1}{\left(1+A^2\right)^{1/2}}. \tag{22}$$

The trial function $y = A\cos(\omega\tau)$ is then employed to determine the angular frequency. The following residual is consequently obtained as

$$R(\tau) = \frac{1}{2}A^2\omega^2\sin^2(\omega\tau) - \frac{1}{\left(1+A^2\cos^2(\omega\tau)\right)^{1/2}} + \frac{1}{\left(1+A^2\right)^{1/2}}. \tag{23}$$

Setting

$$\underset{\omega\tau\to\frac{\pi}{4}}{R(\tau)} = 0. \tag{24}$$

And solving the above equation leads to:

$$\omega_{EBM} = \frac{2}{A}\sqrt{\left(1+\frac{A^2}{2}\right)^{-1/2} - \left(1+A^2\right)^{-1/2}}. \tag{25}$$

3. Discussion and numerical results

In this part, the accuracy of the Hamiltonian approach and the energy balance method are discussed for large oscillation amplitude. The exact angular frequency [20] of Eq. (8) is

$$\omega_{ex} = 2\pi\left[\int_0^1 \frac{4A\mathrm{d}u}{\sqrt{(1+A^2u^2)^{-\frac{1}{2}} - (1+A^2)^{-\frac{1}{2}}}}\right]^{-1}. \tag{26}$$

We consider the expression for the exact frequency ω_e (Eq. (26)). For large amplitudes we obtain [20]:

$$\omega_e(A) \approx 2\pi\left[\int_0^1 \frac{2\sqrt{2u}}{\sqrt{1-u}}\frac{1}{A^{3/2}}\mathrm{d}u + \cdots\right]^{-1} = \frac{\sqrt{2}}{A^{3/2}} + \cdots. \tag{27}$$

The power-series expansion of the Hamiltonian approaches for the large oscillation amplitudes is

$$\omega_{HA}(A) = \frac{1}{A^{3/2}}\sqrt{1+\left(\frac{1}{2}\right)^2 + \left(\frac{1\times 3}{2\times 4}\right)^2 + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^2 + \cdots}. \tag{28}$$

The expression of frequency for large amplitudes by energy balance method is equal to

$$\omega_{EBM}(A) = \frac{2}{A^{3/2}}\sqrt{\sqrt{2}-1}. \tag{29}$$

From Eqs. (26)–(28), frequency ratios for large amplitudes are

$$\lim_{A\to\infty}\frac{\omega_{EBM}}{\omega_{ex}} = 0.9101797. \tag{30}$$

For three terms of the series in Eq. (23), we have

$$\lim_{A\to\infty}\frac{\omega_{HA}}{\omega_{ex}} = 0.833854. \tag{31}$$

Belendez applied the harmonic balance method [30] to this equation and obtained a first approximate to the following limit

$$\lim_{A\to\infty}\frac{\omega_{HBM}}{\omega_{ex}} = 0.822267. \tag{32}$$

According to Eqs. (30)–(32), we can conclude that the EBM is more accurate than HA and HBM in first approximation.

4. Conclusions

The Hamiltonian approach and the energy balance method were used to determine the frequency amplitude relationships for nonlinear oscillations of a punctual charge in the electric field of a charged ring. Validity and accuracy of HA and EBM were also discussed and demonstrated by comparing with the exact one. According to the results, we conclude that the EBM is more accurate than HA and HBM for this problem. We also found that for large amplitude of oscillation the EBM relative error is less than 9% and the relative error of HA is less than 18%.

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