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Uniformly Elliptic Operators with Measurable Coefficients

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1. INTRODUCTION

Over the last ten years there have been substantial developments in the spectral thery of second order elliptic operators with measurable coefficients, largely as a result of the proof of strong pointwise upper and lower bounds on the heat kernels of such operators. These developments are well documented in [7, 24]. However, for elliptic operators of higher order, most of the literature applies only to operators whose highest order coefficients are differentiable to some order, or uniformly continuous at least. See [2, 12, 14, 17–19] for accounts of this theory. Gurarie [15, 16] discusses the L^p spectral thery of elliptic operators whose principal symbol is allowed to have uniformly small discontinuities, but our goal is to allow discontinuities of any size in the principal symbol, subject to an overall uniform ellipticity condition. Moreover we wish to consider the L^p spectral theory for $p=1$, which necessitates stronger control than has previously been possible for operators with discontinuous coefficients.

It is obviously the case that simple discontinuities of the coefficients inside a domain can be handled by imposing suitable internal boundary conditions. However, this method is not appropriate for a medium containing many small irregular and randomly distributed subregions with different properties. It is well-known even for second order operators that the determination of an equivalent homogenized operator corresponding to a uniform medium with intermediate properties is a highly non-trivial matter.

In order to achieve our stated goal we consider a general class of self-adjoint uniformly elliptic operators H of order $2m$, which satisfy certain quadratic form inequalities, of a type used in [7, 9]. The main new problem is that for $m>1$ there is no a priori reason why the semigroup e^{-Ht} initially defined on $L^2(\mathbf{R}^N)$ should be extendable to $L^1(\mathbf{R}^N)$. For $m=1$ this is a consequence of the applicability of the theory of symmetric Markov semigroups, which is definitely not relevant if $m>1$. The Harnack inequality cannot be

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used to prove pointwise bounds, and Sobolev embedding theorems are also of limited use when the coefficients of the operators concerned are measurable.

In spite of all of these problems we have been able to obtain pointwise upper bounds on the fundamental solution $K(t, x, y)$ of the evolution equation

$$\frac{df}{dt} = -Hf \tag{1}$$

under certain conditions. These bounds are of the same character as those already known for uniformly elliptic operators with smooth highest order coefficients [12, 17, 19]. The bounds can be used to prove that the operators e^{-Ht} extend to strongly continuous holomorphic one-parameter semigroups acting on L^p. We then use a new abstract functional calculus due to the author [10, 11] to prove that the L^p spectrum of H is independent of p for a certain range of values of p, and for all $1 \leqslant p < \infty$ in some case.

EXAMPLES. A typical example of an operator which our methods can handle is

$$Hf(x) := \sum_{r=1}^{N} \frac{\partial^2}{\partial x_r^2} \left\{ a_r(x) \frac{\partial^2 f}{\partial x_r^2} \right\},$$

where $c^{-1} \leqslant a_r(x) \leqslant c$ for some $c > 0$, all $1 \leqslant r \leqslant N$ and all $x \in \mathbf{R}^N$. A second example is

$$Hf(x) := \Delta\{a(x)\, \Delta f(x)\},$$

where $c^{-1} \leqslant a(x) \leqslant c$ for all $x \in \mathbf{R}^N$. Further examples relavant to the transmission problem between two homogeneous media are described in Section 10. The L^p spectral theory of such operators seems not to have previously been elucidated.

We consider elliptic operators of order $2m$ on $L^2(\mathbf{R}^N)$ which satisfy two abstract inequalities, which we describe next. The modifications of the inequalities to Riemannian manifolds are described in Section 9. A particular important class of operators which satisfy the conditions below is described in the next section. Let H_0 be the operator $H_0 := (-\Delta)^m$ on $L^2(\mathbf{R}^N)$ and let Q_0 be the associated quadratic form

$$Q_0(f) := \|H_0^{m/2} f\|_2^2 = \int_{\mathbf{R}^N} |\xi|^{2m}\, |\hat{f}(\xi)|^2\, d^N\xi$$

with domain the Sobolev space $W^{m,2}$. We have

$$c^{-1}\,\|f\|_{m,2}^2 \leqslant Q_0(f) + \|f\|_2^2 \leqslant c\,\|f\|_{m,2}^2$$

for some $c>0$ and all $f \in W^{m,2}$, where $\|\ \|_{m,2}$ is the standard norm on $W^{m,2}$.

We start with a sesquilinear form $Q(f,g)$ defined for $f, g \in C_c^\infty := C_c^\infty(\mathbf{R}^N)$ which satisfies the Gårding inequality

$$\tfrac{1}{2}Q_0(f) \leqslant Q(f) \leqslant cQ_0(f) + c\,\|f\|_2^2 \tag{2}$$

for some $c>0$ and all $f \in C_c^\infty$, where $Q(f) := Q(f,f)$. This inequality implies that Q is closable and that the closure has domain $W^{m,2}$ on which (2) is still valid. It is standard that the closure is associated with a non-negative self-adjoint operator H for which $\mathrm{Dom}(H^{1/2}) = W^{m,2}(\mathbf{R}^N)$. This is the operator which we study in the paper. We shall avoid identifying the domain of H.

Our second assumption involves bounds on a twisted form which is defined in terms of a certain class of multipliers. Let α, β denote multi-indices of the usual type, and D^α the corresponding partial differentiation operators, defined on $\mathbf{C}^\infty(\mathbf{R}^N)$. We use the standard notation $|\alpha| := \alpha_1 + \cdots + \alpha_N$. Let $\mathscr{E}_m$ denote the set of all bounded real-valued C^∞ functions ϕ on $\mathbf{R}^N$ such that $\|D^\alpha\phi\|_\infty \leqslant 1$ for all α such that $1 \leqslant |\alpha| \leqslant m$. We make no assumption on the size of $\|\phi\|_\infty$. Given $\lambda \in \mathbf{R}$ and $\phi \in \mathscr{E}_m$ the functions $e^{\lambda\phi}$ may be regarded as bounded invertible multiplication operators on L^2 and also on $W^{m,2}$. We define the complex-valued function $Q_{\lambda\phi}$ by

$$Q_{\lambda\phi}(f) := Q(e^{-\lambda\phi}f, e^{\lambda\phi}f)$$

for all $f \in W^{m,2}$. We assume that

$$|Q_{\lambda\phi}(f) - Q(f)| \leqslant \tfrac{1}{4}Q(f) + k(1+\lambda^{2m})\,\|f\|_2^2 \tag{3}$$

for some $k \geqslant 0$ and all $f \in W^{m,2}$. The constant $\frac{1}{4}$ may be replaced by another value less than 1.

The above two assumptions are the only ones needed for our theoretical development. We mention that a self-adjoint uniformly strongly elliptic form whose highest order coefficients are uniformly continuous satisfies Gårding's inequality; see [1, p. 78]. The proof of (3) for such forms is similar to that for the forms of the next section.

2. Superelliptic Operators

In this section we describe one important class of elliptic operators which satisfy the above assumptions. Another class is described in Section 10, while the modifications of the theory needed to treat operators on manifolds are discussed in Section 9. We shall consider symmetric operators given formally by

$$Hf(x) := \sum_{\substack{|\alpha| \leqslant m \\ |\beta| \leqslant m}} D^\alpha\{a_{\alpha,\beta}(x)\, D^\beta f(x)\},$$

where $a_{\alpha,\beta}(x) = \overline{a_{\beta,\alpha}(x)}$ are complex-valued bounded measurable functions on $\mathbf{R}^N$ for all α, β. It is clear that C_c^∞ need not be contained in the domain of such operators. We therefore start from the quadratic form Q defined on C_c^∞ by

$$Q(f) := \int_{\mathbf{R}^N} \sum_{\substack{|\alpha| \leqslant m \\ |\beta| \leqslant m}} a_{\alpha,\beta}(x)\, D^\beta f(x)\, \overline{D^\alpha f(x)}\, d^N x.$$

The order of any term in either of the above two equations in defined to be $|\alpha| + |\beta|$. We shall call the operators superelliptic when they satisfy the following further restrictions.

We assume that $Q = Q_0 + Q_1 + Q_2$ where each of the terms is of the above form, but with the following extra conditions. Q_0 is a constant coefficient homogeneous elliptic form of order $2m$ in the usual sense. Q_1 is also homogeneous of order $2m$ and is positive in the rather strong sense that

$$\sum_{\substack{|\alpha| = m \\ |\beta| = m}} a_{1,\beta}(x)\, v_\beta \overline{v_\alpha} \geqslant 0 \tag{4}$$

for all complex-valued vectors v and all $x \in \mathbf{R}^N$. Finally Q_2 is of order less than $2m$ and is handled by perturbative methods.

Standard elliptic estimates show that

$$|Q_1(f)| \leqslant c Q_0(f)$$

for all $f \in C_c^\infty$ and that

$$|Q_2(f)| \leqslant \varepsilon Q_0(f) + c_\varepsilon \|f\|_2^2 \tag{5}$$

for all $\varepsilon > 0$, some $c_\varepsilon < \infty$ and all $f \in C_c^\infty$. It is quite possible (compare [15, 16, 21]) for (5) to be valid even if the coefficients of Q_2 lie in certain L^p spaces, but we shall not pursue this except in Section 8. The assumption (4) implies that

$$Q_1(f) \geqslant 0$$

for all $f \in C_c^\infty$. Therefore

$$\tfrac{1}{2}Q_0(f) - c\,\|f\|_2^2 \leq Q(f) \leq cQ_0(f) + c\,\|f\|_2^2$$

for some $c > 0$ and all $f \in C_c^\infty$. We add a large enough constant to H to ensure that (2) is satisfied.

If $\phi \in \mathscr{E}_m$ then we define

$$H_{\lambda\phi} := e^{\lambda\phi} H e^{-\lambda\phi} \tag{6}$$

so that

$$\mathrm{Dom}(H_{\lambda\phi}) = \{f \in L^2 : e^{-\lambda\phi} f \in \mathrm{Dom}(H)\}$$

and

$$\mathrm{Spec}(H_{\lambda\phi}) = \mathrm{Spec}(H).$$

The asociated complex-valued form $Q_{\lambda\phi}$ on $W^{m,2}$ may be written in the form

$$Q_{\lambda\phi}(f) := \int_{\mathbf{R}^N} \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} a_{\alpha,\beta}(x)\{e^{\lambda\phi} D^\beta e^{-\lambda\phi} f(x)\}\{e^{-\lambda\phi} D^\alpha e^{\lambda\phi} \overline{f(x)}\}\, d^N x.$$

In our next lemma $\sum'$ stands for the sum over multi-indices α, β and non-negative integers p, q satisfying $p + |\alpha| \leq m$, $q + |\beta| \leq m$ and $|\alpha| + |\beta| \leq 2m - 1$.

LEMMA 1. *If $f \in C_c^\infty$ then*

$$Q_{\lambda\phi}(f) - Q(f) = \int_{\mathbf{R}^N} \sum{}' c_{\alpha,\beta,p,q}\, \lambda^{p+q} D^\beta f\, \overline{D^\alpha f}\, d^N x,$$

where the functions $c_{\alpha,\beta,p,q}$ on $\mathbf{R}^N$ have uniform bounds independent of f, ϕ, and λ.

Proof. Each term in $Q_{\lambda\phi}(f)$ is expanded using formulae of the type

$$e^{\lambda\phi} D^\alpha e^{-\lambda\phi} f = D^\alpha f + \sum_{\substack{|\beta| < |\alpha| \\ q \leq |\alpha| - |\beta|}} b_{\alpha,\beta}\, \lambda^q D^\beta f,$$

where the functions $b_{q,\beta}$ are combinations of derivatives of ϕ and so are bounded uniformly independently of the particular choice of $\phi \in \mathscr{E}_m$. The decomposition of the theorem follows. The restriction $0 \leq |\alpha| + |\beta| \leq 2m - 1$ arises from the fact that the top order terms are all combined in $Q(f)$.

LEMMA 2. *There exists $k \geqslant 0$ such that*

$$|Q_{\lambda\phi}(f) - Q(f)| \leqslant \tfrac{1}{4} Q(f) + k(1+\lambda^{2m}) \|f\|_2^2,$$

for all $f \in W^{m,2}$.

Proof. Because of the density of C_c^∞ in $W^{m,2}$ it is sufficient to prove the inequality for all $f \in C_c^\infty$. The last lemma yields

$$\text{LHS} \leqslant c \sum{}' |\lambda|^{p+q} \|D^\alpha f\|_2 \|D^\beta f\|_2.$$

If we put $s := p + |\alpha| \leqslant m$ then

$$\begin{aligned} \|\lambda^p D^\alpha f\|_2^2 &\leqslant \int_{\mathbf{R}^N} (c_1 |\xi|^{2s} + c_2 |\lambda|^{2s}) |\hat{f}(\xi)|^2 \, d^N\xi \\ &\leqslant c_1 \|f\|_{m,2}^2 + c_2(1+\lambda^{2m}) \|f\|_2^2, \end{aligned}$$

where c_1 can be chosen to be arbitrarily small unless $p = 0$ and $|\alpha| = m$. Each term in $\sum{}'$ has at least one of $|\alpha| < m$ and $|\beta| < m$ and so is dominated by

$$c_1 \|f\|_{m,2}^2 + c_2(1+\lambda^{2m}) \|f\|_2^2,$$

where c_1 may be taken to be arbitrarily small.

3. An Abstract Spectral Invariance Theorem

In this section we prove an abstract functional analytic theorem which gives conditions under which two unbounded operators acting on different but related Banach spaces have the same spectrum.

Let $\mathscr{B}$ be a complex vector space provided with two norms $\| \;\|_1$ and $\| \;\|_2$. Suppose that $\mathscr{B}$ is a Banach space with respect to the norm

$$\|f\| := \|f\|_1 + \|f\|_2$$

and let $\mathscr{B}_i$ be the completions of $\mathscr{B}$ with respect to the norms $\| \;\|_i$ for $i = 1, 2$. We say that two bounded operators $A_i : \mathscr{B}_i \to \mathscr{B}_i$ are consistent if $A_1 f = A_2 f$ for all $f \in \mathscr{B} = \mathscr{B}_1 \cap \mathscr{B}_2$. Let $T_i(z)$ be two consistent strongly continuous holomorphic semigroups of bounded operators acting on $\mathscr{B}_i$ for $i = 1, 2$ and defined for all $z = re^{i\theta}$ such that $r > 0$ and $|\theta| < \pi/2$. Let their infinitesimal generators be $-H_i$.

THEOREM 3. *If both T_1 and T_2 satisfy estimates of the form*

$$\|T_i(z)\| \leqslant c(\cos\theta)^{-M} \exp[kr\cos\theta]$$

for some positive constants c, M, k, all $r>0$ and all $|\theta|<\pi/2$ then

$$\mathrm{Spec}(H_1) = \mathrm{Spec}(H_2) \subseteq [-k, \infty).$$

Proof. Our assumptions imply that $e^{-kz}T_i(z)$ are bounded holomorphic semigroups with angle $\pi/2$ in the sense of [6, p. 59]. It is immediate from [6, Theorem 2.33] that the spectrum of H_i is contained in $[-k, \infty)$ for $i=1, 2$. The resolvents $(z-H_i)^{-1}$ are now defined for all $z \notin \mathbf{R}$ by norm convergent integrals of the semigroups and are therefore consistent. As in [10] one sees that they satisfy bounds of the form

$$\|(z-H_i)^{-1}\| \leqslant \frac{c_1\langle z\rangle^M}{|\mathrm{Im}(z)|^{M+1}}.$$

The functional calculus of [10] now allows us to define

$$f(H_i) := -\frac{1}{\pi}\int_{\mathbf{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z-H_i)^{-1}\, dx\, dy$$

for all $f \in C_c^\infty(\mathbf{R})$. Here $\tilde{f} \in C_c^\infty(\mathbf{C})$ is an almost analytic extension of f and the integral is norm convergent. The operators $f(H_i)$ are consistent because of their mode of definition. Lemma 4 of [11] states that if $\lambda \in \mathbf{R}$ then $\lambda \notin \mathrm{Spec}(H_i)$ if and only if there exists $f \in C_c^\infty$ such that $f(x)=1$ for all x in some interval around λ but $f(H_i)=0$.

If $\lambda \notin \mathrm{Spec}(H_1)$ then $f(H_1)\phi = 0$ for such an f and for all $\phi \in B_1 \cap \mathscr{B}_2$. This implies that $f(H_2)\phi = 0$ by consistency. The set $B_1 \cap \mathscr{B}_2$ is dense in $\mathscr{B}_2$ so $f(H_2)=0$ and $\lambda \notin \mathrm{Spec}(H_2)$. The converse is similar.

4. L^2 OFF-DIAGONAL BOUNDS

We now return to the general situation described at the end of Section 1, in which (2) and (3) are the only assumptions. Our first task is to obtain L^2 off-diagonal upper bounds on the fundamental solution of the evolution equation (1). In later sections these will be converted into pointwise or L^p upper bounds. The starting point is a definition of distance between two compact sets.

If E, F are disjoint compact subsets of $\mathbf{R}^N$ we define

$$d(E, F) := \sup_{\phi \in \mathscr{E}_m} [\inf\{\phi(x) - \phi(y) : x \in E,\ y \in F\}].$$

LEMMA 4. *If E and F are disjoint compact convex subsets of $\mathbf{R}^N$ then*

$$\tilde{d}(E, F) \leqslant d(E, F) \leqslant N^{1/2}\, \tilde{d}(E, F),$$

where $\tilde{d}(E, F)$ is the Euclidean distance between E and F.

Proof. Every $\phi \in \mathscr{E}_m$ has $|\nabla\phi(x)| \leqslant N^{1/2}$ for all $x \in \mathbf{R}^N$. Therefore $|\phi(x) - \phi(y)| \leqslant N^{1/2}\, |x - y|$ for all $x \in E$, $y \in F$. This implies the second inequality of the lemma. There exists a linear function ϕ on $\mathbf{R}^N$ with gradient equal to 1 such that

$$\tilde{d}(E, F) = \inf\{\phi(x) - \phi(y) : x \in E,\ y \in F\}.$$

This satisfies all of the conditions for $\mathscr{E}_m$ except that it is not bounded. Indeed $D^\alpha \phi = 0$ for all $|\alpha| > 1$. Given $\delta > 0$ we define a bounded approximation to ϕ as follows. Let $h \in C_c^\infty(\mathbf{R})$ satisfy $0 \leqslant h(u) \leqslant 1$ for all $u \in \mathbf{R}$, $h(u) = 1$ if $|u| \leqslant 1$, and $h(u) = 0$ if $|u| \geqslant 2$. Then put

$$k_\delta(v) := \int_0^v h(\delta u)\, du$$

for all $v \in \mathbf{R}$. It is clear that $k_\delta \in \mathbf{C}^\infty$ is bounded and that $k_\delta(v) = v$ if $|v| \leqslant \delta^{-1}$. We now define $\phi_\delta(x) := k_\delta(\phi(x))$. Given $1 \leqslant |\alpha| \leqslant m$ a routine calculation establishes that $|D^\alpha \phi_\delta(x)| \leqslant 1$ for all $x \in \mathbf{R}^N$, provided $\delta > 0$ is small enough. Therefore $\phi_\delta \in \mathscr{E}_m$ provided δ is small enough. This yields

$$(1 + \varepsilon)^{-1}\, \tilde{d}(E, F) \leqslant d(E, F)$$

for all $\varepsilon > 0$.

LEMMA 5. *We have*

$$\mathrm{Dom}(H_{\lambda\phi}) \subseteq W^{m,2}$$

and

$$\langle H_{\lambda\phi} f, f\rangle = Q_{\lambda\phi}(f)$$

for all $f \in \mathrm{Dom}(H_{\lambda\phi})$. There exists a constant k such that

$$\mathrm{Re}(\langle H_{\lambda\phi} f, f\rangle) \geqslant -k(1 + \lambda^{2m})\, \|f\|_2^2$$

for all $f \in \mathrm{Dom}(H_{\lambda\phi})$, all $\lambda \in \mathbf{R}$ and all $\phi \in \mathscr{E}_m$.

Proof. If $f \in \mathrm{Dom}(H_{\lambda\phi})$ then

$$e^{-\lambda\phi} f \in \mathrm{Dom}(H) \subseteq \mathrm{Dom}(H^{1/2}) = W^{m,2}.$$

An easy calculation now establishes that $f \in W^{m,2}$. Therefore

$$\begin{aligned}\langle H_{\lambda\phi} f, f\rangle &= \langle H(e^{-\lambda\phi} f), e^{\lambda\phi} f\rangle \\ &= \langle H^{1/2}(e^{-\lambda\phi} f), H^{1/2}(e^{\lambda\phi} f)\rangle \\ &= Q_{\lambda\phi}(f).\end{aligned}$$

The final statement of the lemma follows immediately from (3).

The next lemma is due to Gaffney [13] and was used in a context similar to the present one in [9].

LEMMA 6. *There exists a constant $k \geqslant 0$ such that*

$$\|e^{-H_{\lambda\phi} t}\| \leqslant \exp[k(1+\lambda^{2m})t]$$

for all $t > 0$, $\lambda \in \mathbf{R}$ and $\phi \in \mathscr{E}_m$.

Proof. It follows directly from the definition (6) of $H_{\lambda\phi}$ that $f_t := e^{-H_{\lambda\phi} t} f$ lies in $\mathrm{Dom}(H_{\lambda\phi})$ for all $f \in L^2$ and all $t > 0$. Moreover,

$$\begin{aligned}\frac{d}{dt}\|f_t\|_2^2 &= -\langle H_{\lambda\phi} f_t, f_t\rangle - \langle f_t, H_{\lambda\phi} f_t\rangle \\ &\leqslant 2k(1+\lambda^{2m})\,\|f_t\|_2^2.\end{aligned}$$

The bound follows by solving this differential inequality.

LEMMA 7. *There exist $c, k > 0$ such that*

$$\|H_{\lambda\phi}\, e^{-H_{\lambda\phi} t}\| \leqslant \frac{c}{t}\exp[4k(1+\lambda^{2m})t]$$

for all $t > 0$, $\lambda \in \mathbf{R}$ and $\phi \in \mathscr{E}_m$.

Proof. It follows from (6) that

$$\|H_{\lambda\phi}\, e^{-H_{\lambda\phi} t}\| \leqslant t^{-1}\exp[2\,|\lambda|\,\|\phi\|_\infty] < \infty$$

for all $t > 0$. The improved estimate of the lemma is obtained by a different method. Given $f \in L^2$, $r > 0$ and $|\theta| \leqslant \pi/3$ put

$$f_r := \exp[-H_{\lambda\phi} r e^{i\theta}] f.$$

Then $f_r \in \mathrm{Dom}(H_{\lambda\phi})$ and

$$\frac{d}{dr}\|f_r\|_2^2 = -e^{i\theta}\langle H_{\lambda\phi} f_r, f_r\rangle - e^{-i\theta}\langle f_r, H_{\lambda\phi} f_r\rangle$$
$$= -e^{i\theta} Q_{\lambda\theta}(f_r) - e^{-i\theta}\overline{Q_{\lambda\phi}(f_r)}$$
$$= -(e^{i\theta} + e^{-i\theta})\, Q(f_r) + D_r,$$

where an application of (3) implies

$$|D_r| \leqslant \tfrac{1}{2} Q(f_r) + 2k(1+\lambda^{2m})\, \|f_r\|_2^2.$$

Now $|\theta| \leqslant \pi/3$ implies $e^{i\theta} + e^{-i\theta} \geqslant 1$. Therefore

$$\frac{d}{dr}\|f_r\|_2^2 \leqslant 2k(1+\lambda^{2m})\, \|f_r\|_2^2.$$

Solving this differential inequality yields

$$\|f_r\|_2 \leqslant \exp[k(1+\lambda^{2m})r]\, \|f\|_2.$$

It follows that

$$\|\exp[-H_{\lambda\phi} z - 2k(1+\lambda^{2m})z]\| \leqslant 1$$

for all $z = re^{i\theta}$ with $r > 0$ and $|\theta| \leqslant \pi/3$. The theory of bounded holomorphic semigroups now yields

$$\|(H_{\lambda\phi} + 2k(1+\lambda^{2m}))\exp[-H_{\lambda\phi} t - 2k(1+\lambda^{2m})t]\| \leqslant c_0/t$$

for all $t > 0$. Therefore

$$\|H_{\lambda\phi}\exp[-H_{\lambda\phi} t]\|$$
$$\leqslant \frac{c_0}{t}\exp[2k(1+\lambda^{2m})t] + 2k(1+\lambda^{12m})\exp[3k(1+\lambda^{2m})t]$$
$$\leqslant \frac{c}{t}\exp[4k(1+\lambda^{2m})t].$$

We write P_E to denote the projection on L^2 obtained by multiplying by the characteristic function of a closed subset E of $\mathbf{R}^N$.

THEOREM 8. *There exist positive constants* c_1, c_2 *such that*

$$\|P_E e^{-Ht} P_F\| \leqslant \exp[-c_1\, d(E, F)^{2m/(2m-1)}\, t^{-1/(2m-1)} + c_2 t]$$

for all disjoint compact subsets $E, F \in \mathbf{R}^N$ *and all* $t > 0$.

Note. Given E, F the above theorem provides useful information for small $t > 0$. Given $t > 0$ it only provides useful information if $d(E, F)$ is large.

Proof. Let $\phi \in \mathscr{E}_m$ satisfy $\phi|_E \geqslant 0$ and $\phi|_F \leqslant -d(E, F)/(1+\varepsilon)$. Then

$$\begin{aligned}\|P_E e^{-Ht} P_F\| &= \|e^{-\lambda\phi} P_E e^{\lambda\phi} e^{-Ht} e^{-\lambda\phi} P_F e^{\lambda\phi}\| \\ &\leqslant \|e^{-\lambda\phi} P_E\| \, \|e^{-H_{\lambda\phi}t}\| \, \|P_F e^{\lambda\phi}\| \\ &\leqslant \exp\left[k(1+\lambda^{2m})t - \lambda \frac{d(E, F)}{1+\varepsilon} \right].\end{aligned}$$

The proof is completed by putting

$$\lambda := c d(E, F)^{1/(2m-1)} t^{-1/(2m-1)}$$

for a suitable value of c.

We will need to extend the above theorem to complex times. This can be achieved by using a lemma from analytic function theory, of a type which has already proved useful in the theory of second order elliptic operators [7, 8]. In this lemma the constants are specified precisely.

LEMMA 9. *Let F be analytic on $\mathbf{C}^+ := \{z : \mathrm{Re}(z) > 0\}$ and suppose that it satisfies the bounds*

$$|F(re^{i\theta})| \leqslant a_1 (r\cos\theta)^{-\beta}$$
$$|F(r)| \leqslant a_1 r^{-\beta} \exp[-a_2 r^{-\alpha}]$$

for some $a_1 > 0$, $a_2 > 0$, $\beta \geqslant 0$, $0 < \alpha < 1$, all $r > 0$ and all $|\theta| < \pi/2$. Then

$$|F(re^{i\theta})| \leqslant a_1 2^{\beta} (r\cos\theta)^{-\beta} \exp[-\tfrac{1}{2} a_2 \alpha r^{-\alpha} \cos\theta]$$

for all $r > 0$ and all $|\theta| < \pi/2$.

Proof. Let $0 < \gamma < \pi/2$ and define

$$G(z) := r^{-\beta} F(z^{-1}) \exp[a_2 e^{i(\pi/2 - \gamma\alpha)} z^{\alpha}/\sin(\alpha\gamma)].$$

The assumptions of the lemma yield

$$\begin{aligned}|G(r)| &= r^{-\beta} |F(r^{-1})| \exp[a_2 r^{\alpha}] \\ &\leqslant a_1\end{aligned}$$

and

$$|G(re^{i\gamma})| = r^{-\beta} |F(r^{-1} e^{-i\gamma})| \exp[a_2 \operatorname{Re}(e^{i(\pi/2 - \gamma\alpha)} r^{\alpha} e^{i\gamma\alpha})/\sin(\alpha\gamma)]$$
$$\leqslant a_1(\cos \gamma)^{-\beta}$$

for all $r > 0$. The three lines lemma implies that

$$|G(re^{i\theta})| \leqslant a_1(\cos \gamma)^{-\beta\theta/\gamma} \leqslant a_1(\cos \gamma)^{-\beta}$$

for all $r > 0$ and $0 \leqslant \theta \leqslant \gamma$. Combining this with a similar estimate for negative θ we obtain

$$|F(re^{i\theta})| \leqslant a_1 r^{-\beta}(\cos \gamma)^{-\beta} \exp[-a_2 r^{-\alpha} \sin((\gamma - |\theta|)\alpha)/\sin(\alpha\gamma)] \quad (7)$$

for all $r > 0$ and $|\theta| \leqslant \gamma$. We now put $\gamma := \pi/4 + |\theta|/2$, so that $|\theta| < \gamma < \pi/2$. The bound

$$\cos \gamma = \sin(\pi/4 - |\theta|/2) \geqslant \tfrac{1}{2} \sin(\pi/2 - |\theta|) = \tfrac{1}{2} \cos \theta$$

implies

$$(\cos \gamma)^{-\beta} \leqslant 2^{\beta}(\cos \theta)^{-\beta}.$$

Since $0 < \alpha \leqslant 1$ we also have

$$\begin{aligned} \frac{\sin((\gamma - |\theta|)\alpha)}{\sin(\gamma\alpha)} &\geqslant \sin((\gamma - |\theta|)\alpha) \\ &= \sin\left(\left(\frac{\pi}{2} - |\theta|\right)\frac{\alpha}{2}\right) \\ &\geqslant \frac{\alpha}{2} \sin\left(\frac{\pi}{2} - |\theta|\right) \\ &= \frac{\alpha}{2} \cos \theta. \end{aligned}$$

The lemma follows by substituting the last two inequalities into (7).

THEOREM 10. *There exist positive constants c_2, c_3 such that*

$$\|P_E \exp[-Hre^{i\theta}] P_F\|$$
$$\leqslant \exp[-c_3\, d(E, F)^{2m/(2m-1)} r^{-1/(2m-1)} \cos \theta + c_2 r \cos \theta]$$

for all disjoint compact subsets $E, F \in \mathbf{R}^N$, all $r > 0$ and all $|\theta| < \pi/2$.

Proof. Given $f, g \in L^2$ put

$$F(z) := \langle P_E e^{-(H+c_2)z} P_F f, g \rangle,$$

where c_2 is given in Theorem 8. We then apply Lemma 9 with $a_1 := \|f\|_2 \|g\|_2$, $\beta := 0$ and $\alpha := 1/(2m-1)$. The first bound assumed in Lemma 9 follows by the self-adjointness of H and the second was proved in Theorem 8.

5. Spectral Theory in $l^p(L^2)$

The off-diagonal decay proved in the last section may be used to transfer the semigroup e^{-Ht} from L^2 to other Banach spaces. We emphasize that there are two distinct issues, local regularization and global off-diagonal decay, which are usually treated simultaneously in this context. The identity operator, associated with the choice $H=0$, has optimal off-diagonal decay and no local regularisation; similar but less extreme behaviour occurs for other sub-elliptic operators H. In this section we only consider off-diagonal decay. Our main result, Corollary 13, is a spectral invariance theorem. Later sections discuss the question of local regularization.

The spaces $l^p(L^2)$ are defined for $1 \leqslant p < \infty$ as follows [5, 22]. Given $m \in \mathbf{Z}^N$ let C_m be the cube with centre m and edges oriented parallel to the axes and of length 1. If $f \in L^2_{\text{loc}}$ and $1 \leqslant p < \infty$ we say that $f \in l^p(L^2)$ if the norm

$$|||f|||_{p,2} := \left(\sum_{m \in \mathbf{Z}^N} \|f|_{C_m}\|_2^p \right)^{1/p}$$

is finite. The definition of $l^\infty(L^2)$ is similar. Clearly $l^2(L^2) = L^2$ and $l^p(L^2) \subseteq L^2$ if $1 \leqslant p \leqslant 2$. Moreover, $l^p(L^2)^* = l^q(L^2)$ in a natural sense if $1 \leqslant p < \infty$ and $p^{-1} + q^{-1} = 1$.

Theorem 11. *The semigroup e^{-Ht} restricts (resp. extends) to a semigroup $T_p(t)$ on $l^p(L^2)$ if $1 \leqslant p < 2$ (resp. $2 < p \leqslant \infty$). The induced semigroup on $l^p(L^2)$ is strongly continuous if $1 \leqslant p < \infty$.*

Proof. Let $p = 1$ and let $f \in L^2$ have support in C_n for some $n \in \mathbf{Z}^N$. If C_r equals or is a neighbour of C_n then $\|e^{-Ht}f|_{C_r}\|_2 \leqslant \|f\|_2$. In all other cases

$$d(C_r, C_n) \geqslant c\,|r-n|$$

for some $c > 0$, by Lemma 4, and

$$\|e^{-Ht}f|_{C_r}\|_2 \leqslant \exp[-c_1\,|r-n|^{2m/(2m-1)}\,t^{-1/(2m-1)} + c_2 t]\,\|f\|_2$$

by Theorem 8. Therefore if $0 < t \leqslant 1$ we have

$$\begin{aligned}|||e^{-Ht}f|||_{1,2} &\leqslant c\,\|f\|_2 + \sum_{r\neq n} \exp[-c_1\,|r-n|^{2m/(2m-1)} + c_2]\,\|f\|_2 \\ &\leqslant c'\,\|f\|_2.\end{aligned}$$

For general $f \in l^1(L^2)$ we have

$$\begin{aligned}|||e^{-Ht}f|||_{1,2} &\leqslant \sum_{n\in\mathbf{Z}^N} |||e^{-Ht}(f\,|_{C_n})|||_{1,2} \\ &\leqslant \sum_{n\in\mathbf{Z}^N} c'\,\|f\,|_{C_n}\|_2 \\ &= c'\,|||f|||_{1,2}\end{aligned}$$

again assuming that $0 < t \leqslant 1$. The same holds for all $1 \leqslant p < 2$ by interpolation and for $2 < p \leqslant \infty$ by duality. An application of the semigroup law then yields

$$|||e^{-Ht}f|||_{p,2} \leqslant ce^{at}\,|||f|||_{p,2}$$

for all $t \geqslant 0$ and all $f \in l^p(L^2)$, $1 \leqslant p \leqslant \infty$.

The proof of strong continuity as $t \to 0$ is carried out first for $p = 1$ and for $f \in L^2$ which have support in some particular C_n. Interpolation allows one to extend the result to p such that $1 < p < 2$ and then to all $f \in l^p(L^2)$. If $2 < p < \infty$ then $l^p(L^2)$ is reflexive and strong continuity may be proved by the use of [6, Theorem 1.34]. We omit the details.

We need to extend the above bounds to complex times, for which purpose we use Theorem 10.

THEOREM 12. *The operators* $\exp[-Hre^{i\theta}]$ *induce bounded operators* $T_p(re^{i\theta})$ *on* $l^p(L^2)$ *for all* $1 \leqslant p \leqslant \infty$, *all* $r \geqslant 0$ *and all* $|\theta| < \pi/2$. *These operators satisfy the bounds*

$$\|T_p(re^{i\theta})\| \leqslant c\exp[kr\cos\theta](\cos\theta)^{-N}$$

for all $1 \leqslant p \leqslant \infty$. *In particular* $T_p(z)\,e^{-kz}$ *is a bounded holomorphic semigroup on* $l^p(L^2)$ *with angle* $\pi/2$ *for all* $1 \leqslant p < \infty$.

Proof. We consider only the case $p = 1$, the other cases being handled using interpolation and duality. The relevant bound is

$$
\begin{aligned}
&\|T_1(re^{i\theta})\| \\
&\quad \leqslant c + c \sum_{\substack{n \in \mathbf{Z}^N \\ |n| \geqslant 2}} \exp[-c_4 |n|^{2m/(2m-1)} r^{-1/(2m-1)} \cos\theta + c_2 r \cos\theta] \\
&\quad \leqslant c + c \int_{\mathbf{R}^N} \exp[-c_4 |x|^{2m/(2m-1)} r^{1/(2m-1)} \cos\theta + c_2 r \cos\theta]\, d^N x \\
&\quad \leqslant c + c \int_0^\infty \exp[-c_4 s^{2m/(2m-1)} r^{-1/(2m-1)} \cos\theta + c_2 r \cos\theta]\, s^{N-1}\, ds \\
&\quad \leqslant c + c r^{N/2m} (\cos\theta)^{-N(2m-1)/2m} \exp[c_2 r \cos\theta] \\
&\qquad \times \int_0^\infty \exp[-u^{2m/(2m-1)}]\, u^{N-1}\, du \\
&\quad \leqslant c(1 + r^{N/2m} (\cos\theta)^{N/2m})(\cos\theta)^{-N} \exp[c_2 r \cos\theta] \\
&\quad \leqslant c \exp[kr \cos\theta](\cos\theta)^{-N}.
\end{aligned}
$$

COROLLARY 13. *Let $(-H_p)$ denote the generator of the strongly continuous holomorphic semigroup $T_p(t)$ on $l^p(L^2)$ for $1 \leqslant p < \infty$. Then* $\mathrm{Spec}(H_p) \subseteq [0, \infty)$ *and is independent of p for $1 \leqslant p < \infty$.*

Proof. This is an application of Theorem 3.

6. L^p SPECTRAL THEORY WHEN $2m > N$

The L^p theory of the semigroup e^{-Ht} has different features depending upon the magnitude of m. The theory is most complete under the assumption $2m > N$, which is the case we consider in this section. We comment that this case includes the choice $m = 2$ and $N \leqslant 3$, which has relevance to the study of elastic vibrations of solids. We establish that e^{-Ht} then induces a c_0-semigroup $T_p(t)$ on L^p for all $1 \leqslant p < \infty$, and that one can obtain pointwise bounds on the fundamental solution which are of the same type as in the case in which H has coefficients satisfying some local regularity conditions [12, 14, 17, 19]. We then use these facts to show that the L^p spectrum of H is independent of p for $1 \leqslant p < \infty$. The case $2m < N$ is studied in the next section.

We assume in all of the theorems in this section that $2m > N$, without further mention.

LEMMA 14. *The operator $(H+1)^{-1/2}$ is bounded from L^2 to L^∞. Indeed there exists a continuous uniformly bounded function $\phi : \mathbf{R}^N \to L^2(\mathbf{R}^N)$ such that*

$$
\{(H+1)^{-1/2} f\}(x) = \langle f, \phi(x) \rangle \tag{8}
$$

for all $f \in L^2$.

Proof. The inequality

$$0 \leqslant 1 + (-\Delta)^m \leqslant c(1+H)$$

may be rewritten in the form

$$\int_{\mathbf{R}^N} (1+|\xi|^2)^m \, |\hat{g}(\xi)|^2 \, d^N\xi \leqslant c \, \|(1+H)^{1/2} g\|_2^2$$

for all $g \in \mathrm{Dom}(H^{1/2})$. We deduce that

$$\int_{\mathbf{R}^N} |\hat{g}(\xi)| \, (1+|\xi|)^{\varepsilon} \, d^N\xi \leqslant c \, \|(1+H)^{1/2} g\|_2 \tag{9}$$

for some $\varepsilon > 0$ and all $g \in \mathrm{Dom}(H^{1/2})$. We now define $\phi(x) \in L^2$ by (8). It follows from (9) both that $\|\phi(x)\|_2 \leqslant c$ for all $x \in \mathbf{R}^N$ but also that

$$\|\phi(x) - \phi(y)\|_2 \leqslant c \, |x-y|^{\varepsilon}$$

for all $x, y \in \mathbf{R}^N$.

COROLLARY 15. *There exists a jointly continuous function* $K: \mathbf{C}^+ \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{C}$ *such that*

$$\{e^{-Hz} f\}(x) = \int_{\mathbf{R}^N} K(z, x, y) \, f(y) \, d^N y$$

for all $f \in L^1 \cap L^2$. *Moreover K is analytic in z for every* $x, y \in \mathbf{R}^N$.

Proof. We write

$$e^{-Hz} = A \cdot B(z) \cdot C,$$

where $A = (1+H)^{-1/2} : L^2 \to L^\infty$ is bounded by Lemma 14, $C = (1+H)^{-1/2} : L^1 \to L^2$ is bounded by duality and $B(z) := (1+H) \, e^{-Hz}$ are bounded operators on L^2 depending norm analytically upon $z \in \mathbf{C}^+$. The kernel is then defined by

$$K(z, x, y) := \langle B(z) \, \phi(y), \phi(x) \rangle.$$

Our next three lemmas are devoted to obtaining effective uniform bounds on the kernel $K(z, x, y)$.

LEMMA 16. *If* $f \in W^{m,2}$ *then*

$$\|f\|_\infty \leqslant \|(-\Delta)^{m/2} f\|_2^{N/2m} \, \|f\|_2^{1-N/2m}.$$

Proof. If $\gamma > 0$ then

$$\|f\|_\infty^2 \leqslant c\left(\int_{\mathbf{R}^N} |\hat{f}(\xi)|\, d^N\xi\right)$$
$$\leqslant c \int_{\mathbf{R}^N} |\hat{f}(\xi)|^2(\gamma + |\xi|^2)^m\, d^N\xi \int_{\mathbf{R}^N} (\gamma + |\xi|^2)^{-m}\, d^N\xi$$
$$= c\, \|(\gamma - \Delta)^{m/2} f\|_2^2\, \gamma^{(N-2m)/2}$$
$$\leqslant c(\gamma^m \|f\|_2^2 + \|(-\Delta)^{m/2} f\|_2^2)\, \gamma^{(N-2m)/2}$$
$$= c\gamma^{N/2} + c\gamma^{(N-2m)/2}\, \|(-\Delta)^{m/2} f\|_2^2.$$

The lemma follows by minimizing the above with respect to γ.

LEMMA 17. *If $f \in L^2$ and $t > 0$ then*

$$\|e^{-Ht} f\|_\infty \leqslant c t^{-N/4m}\, e^{t/2}\, \|f\|_2.$$

Proof. If $f_t := e^{-Ht} f$ then

$$\|f_t\|_\infty^2 \leqslant \|(-\Delta)^{m/2} f_t\|_2^{N/m}\, \|f_t\|_2^{2-N/m}$$
$$\leqslant c\langle (H+1) f_t, f_t\rangle^{N/2m}\, \|f_t\|_2^{2-N/m}$$
$$\leqslant c((1+t^{-1})\, \|f\|_2^2)^{N/2m}\, \|f\|_2^{2-N/m}$$
$$\leqslant t^{-N/2m}\, e^t\, \|f\|_2^2.$$

In the ensuing calculations we shall frequently write $\|A\|_{q,p}$ to stand for the norm of an operator $A : L^p \to L^q$.

LEMMA 18. *We have*

$$|K(re^{i\theta}, x, y)| \leqslant c(r\cos\theta)^{-N/2m}$$

for all $r > 0$, $|\theta| < \pi/2$ and $x, y \in \mathbf{R}^N$.

Proof. Putting $r\cos\theta := t + is$ we may write

$$e^{-H(t+is)} = A \cdot B \cdot C,$$

where $A := e^{-Ht/2} : L^2 \to L^\infty$ is bounded using Lemma 17, $B := e^{-iHs} : L^2 \to L^2$ has norm 1 and $C : L^1 \to L^2$ is the adjoint of A. The continuity of the kernel K, proved in Corollary 15, allows us to deduce pointwise bounds from the norm bound

$$\|e^{-H(t+is)}\|_{\infty,1} \leqslant c t^{-N/2m}\, e^t.$$

We next turn to the question of finding off-diagonal bounds on $|K(t, x, y)|$ for all $t>0$.

LEMMA 19. *There exist positive constants* c_1, c_2, and k *such that*

$$|K(t, x, y)| \leqslant c_1 t^{-N/2m} \exp[-c_2 |x-y|^{2m/(2m-1)} t^{-1/(2m-1)} + kt]$$

for all $t>0$ *and* $x, y \in \mathbf{R}^N$.

Proof. Given $f \in L^2$, $\lambda \in \mathbf{R}$, $\phi \in \mathscr{E}_m$ and $t>0$ put

$$f_t := \exp[-H_{\lambda\phi} t] f.$$

It follows from Lemma 16 that

$$\begin{aligned}\|f_t\|_\infty &\leqslant c \,\|(-\Delta)^{m/2} f_t\|_2^{N/2m} \,\|f_t\|_2^{1-N/2m}\\ &\leqslant c Q(f_t)^{N/2m} \,\|f_t\|_2^{1-N/2m}.\end{aligned}$$

Assumption (3), Lemmas 6 and 7 now yield

$$\begin{aligned}\|f_t\|_\infty &\leqslant c\{\operatorname{Re} Q_{\lambda\phi}(f_t) + (1+\lambda^{2m}) \,\|f_t\|_2^2\}^{N/4m} \,\|f_t\|_2^{1-N/2m}\\ &\leqslant c\{\|H_{\lambda\phi} f_t\|_2 \,\|f_t\|_2 + (1+\lambda^{2m}) \,\|f_t\|_2^2\}^{N/4m} \,\|f_t\|_2^{1-N/2m}\\ &\leqslant c\{t^{-1} + (1+\lambda^{2m})\}^{N/4m} \exp[k(1+\lambda^{2m})t] \,\|f\|_2\\ &\leqslant c t^{-N/4m} \exp[(k+1)(1+\lambda^{2m})t] \,\|f\|_2.\end{aligned}$$

Therefore

$$\|\exp[-H_{\lambda\phi} t]\|_{\infty,2} \leqslant c t^{-N/4m} \exp[(k+1)(1+\lambda^{2m})t].$$

By duality we obtain the similar bound

$$\|\exp[-H_{\lambda\phi} t]\|_{2,1} \leqslant c t^{-N/4m} \exp[(k+1)(1+\lambda^{2m})t]$$

and these together yield

$$\|\exp[-H_{\lambda\phi} t]\|_{\infty,1} \leqslant c t^{-N/2m} \exp[(k+1)(1+\lambda^{2m})t]$$

by use of the semigroup property. But $\exp[-H_{\lambda\phi} t]$ has the kernel

$$K_{\lambda\phi}(t, x, y) = e^{\lambda\phi(x)} K(t, x, y)\, e^{-\lambda\phi(y)}$$

which is jointly continuous by Corollary 15, so

$$|e^{\lambda\phi(x)} K(t, x, y)\, e^{-\lambda\phi(y)}| \leqslant c t^{-N/2m} \exp[(k+1)(1+\lambda^{2m})t]$$

for all $t>0$ and $x, y \in \mathbf{R}^N$, or equivalently

$$|K(t, x, y)| \leqslant ct^{-N/2m} \exp[(\lambda\phi(y) - \lambda\phi(x)) + (k+1)(1+\lambda^{2m})t].$$

The lemma follows by optimizing with respect to $\phi \in \mathscr{E}_m$ and then $\lambda \in \mathbf{R}$.

THEOREM 20. *The operators e^{-Hz} may be extended to bounded operators $T_p(z)$ on L^p satisfying*

$$\|T_p(z)\| \leqslant c(\cos\theta)^{-2N|1/p-1/2|} e^{kr\cos\theta} \tag{10}$$

for all $1 \leqslant p \leqslant \infty$ and all $z = re^{i\theta}$ such that $r>0$ and $|\theta| < \pi/2$.

Proof. Lemmas 18 and 19 may be combined into the single bound

$$\begin{aligned} &|K(re^{i\theta}, x, y)| \\ &\quad\leqslant c_1(r\cos\theta)^{-N/2m} \\ &\qquad\times \exp[-c_2|x-y|^{2m/(2m-1)} r^{-1/(2m-1)} \cos\theta + kr\cos\theta] \end{aligned} \tag{11}$$

by applying Lemma 9 to the function

$$F(z) := K(z, x, y)\, e^{-kz}$$

for large enough $k \geqslant 0$.

We prove the bound (10) only for $p=1$. The other cases then follow by duality and interpolation. If $p=1$ then

$$\|T_1(z)\| = \sup_{y\in\mathbf{R}^N} \int_{\mathbf{R}^N} |K(z, x, y)|\, d^N x$$

by a standard theorem. This integral may be estimated yielding the bound

$$\|T_1(z)\| \leqslant c(\cos\theta)^{-N} e^{kr\cos\theta}.$$

THEOREM 21. *Let $-H_p$ be the generator of the holomorphic semigroup $T_p(z)$ defined on L^p for all $1 \leqslant p < \infty$ by*

$$\{T_p(z) f\}(x) := \int_{\mathbf{R}^N} K(z, x, y) f(y)\, d^N y.$$

Then the spectrum of H_p is independent of p.

Proof. Once again this is an application of Theorem 3.

7. L^p Spectral Theory when $2m < N$

If $2m < N$ then we are not able to prove that $\mathrm{Spec}(H_p)$ is independent of p for all $1 \leqslant p < \infty$. In this section we show independence for p in a certain interval around $p = 2$. In the next section we give an example which shows why a more complete result is not possible, unless stronger assumptions on the coefficients are made.

Our analysis depends upon the standard fact [1, p. 97] that

$$\|f\|_{p_c} \leqslant \|(-\Delta)^{p/2} f\|_2 \tag{12}$$

for all $f \in W^{m,2}$, where $p_c := 2N/(N-2m)$, so that $2 < p_c < \infty$. We put $q_c := 2N/(N+2m)$ so that $p_c^{-1} + q_c^{-1} = 1$, and only consider p in the range $[q_c, p_c]$.

Lemma 22. *There exist positive costants c_1 and c_2 such that*

$$\|\exp[H_{\lambda\phi} t]\|_{p_c, 2} \leqslant c_1 t^{-1/2} \exp[c_2(1+\lambda^{2m})t]$$

for all $\lambda \in \mathbf{R}$, $\phi \in \mathscr{E}_m$, and $t > 0$.

Proof. Given $f \in L^2$, $\lambda \in \mathbf{R}$, $\phi \in \mathscr{E}_m$, and $t > 0$ put

$$f_t := \exp[-H_{\lambda\phi} t] f.$$

By applications of (12), (3), Lemmas 6 and 7 we see that

$$\begin{aligned}
\|f_t\|_{p_c} &\leqslant c\ \|(-\Delta)^{m/2} f_t\|_2 \\
&\leqslant c\{Q(f_t) + \|f_t\|_2^2\}^{1/2} \\
&\leqslant c\{\mathrm{Re}\ Q_{\lambda\phi}(f_t) + (1+\lambda^{2m})\ \|f_t\|_2^2\}^{1/2} \\
&\leqslant c\{\|H_{\lambda\phi} f_t\|_2\ \|f_t\|_2 + (1+\lambda^{2m})\ \|f_t\|_2^2\}^{1/2} \\
&\leqslant c\{t^{-1} + (1+\lambda^{2m})\}^{1/2} \exp[k(1+\lambda^{2m})t]\ \|f\|_2 \\
&\leqslant c t^{-1/2} \exp[(k+1)(1+\lambda^{2m})t]\ \|f\|_2.
\end{aligned}$$

Lemma 23. *There exist positive constants c_1 and c_2 such that*

$$\|P_E \exp[-Hre^{i\theta}]\ P_F\|_{p_c, 2} \leqslant c_1(r\cos\theta)^{-1/2} e^{c_2 r \cos\theta}$$

for all $r > 0$, $|\theta| < \pi/2$ and all closed subsets E, F in $\mathbf{R}^N$.

Proof. If we put $\lambda = 0$ in Lemma 22 we obtain

$$\|e^{-Ht}\|_{p_c, 2} \leqslant c_1 t^{-1/2} e^{c_2 t}$$

for all $t > 0$. Writing $z := t + is$ we deduce that

$$\begin{aligned}\|e^{-H(t+is)}\|_{p_c,2} &\leq \|e^{-Ht}\|_{p_c,2}\, \|e^{-iHs}\|_{2,2} \\ &\leq c_1 t^{-1/2} e^{c_2 t}.\end{aligned}$$

This implies the stated estimate.

LEMMA 24. *There exist positive constants c_1, c_2 and k such that*

$$\begin{aligned}&\|P_E \exp[-Hre^{i\theta}]\, P_F\|_{p_c,2} \\ &\quad \leq c_1 (r\cos\theta)^{-1/2} \\ &\qquad \times \exp[-c_2\, d(E,F)^{2m/(2m-1)}\, r^{-1/(2m-1)} \cos\theta + kr\cos\theta]\end{aligned}$$

for all $r > 0$, $|\theta| < \pi/2$ and all disjoint compact convex subsets E, F in $\mathbf{R}^N$.

Proof. If $t > 0$ then it follows from Lemma 22 that

$$\|P_E e^{\lambda\phi} \exp[-Ht]\, e^{-\lambda\phi} P_F\|_{p_c,2} \leq c_1 t^{-1/2} \exp[c_2(1+\lambda^{2m})t].$$

If $\lambda \geq 0$ then

$$\|P_E e^{\lambda\phi} \exp[-Ht]\, e^{-\lambda\phi} P_F\|_{p_c,2} \geq e^{\lambda a(\phi)} \|P_E \exp[-Ht]\, P_F\|_{p_c,2},$$

where

$$a(\phi) := \inf\{\phi(x) : x \in E\} - \sup\{\phi(y) : y \in F\}.$$

Now

$$\sup\{a(\phi) : \phi \in \mathscr{E}_m\} = d(E,F)$$

by definition, so

$$\|P_E \exp[-Ht]\, P_F\|_{p_c,2} \leq c_1 t^{-1/2} \exp[-\lambda\, d(E,F) + c_2(1+\lambda^{2m})t].$$

On minimizing with respect to λ we obtain

$$\|P_E \exp[-Ht]\, P_F\|_{p_c,2} \leq c_1 t^{-1/2} \exp[-c_2\, d(E,F)^{2m/(2m-1)}\, t^{-1/2m} + c_3 t].$$

We now pass to complex times using Lemma 9. Given $f \in L^2$ and $g \in L^{q_c}$, we define

$$F(z) := \langle P_E e^{-Hz} P_F f, g\rangle\, e^{-kz}$$

for all $z \in \mathbf{C}^+$ and for large enough $k > 0$. The above calculation implies that

$$|F(r)| \leqslant c_1 r^{-1/2} \exp[-c_2\, d(E, F)^{2m/(2m-1)} r^{-1/2m}]$$

for all $r > 0$. Also Lemma 23 implies that

$$|F(r\, e^{i\theta})| \leqslant c_1 (r \cos\theta)^{-1/2}$$

for all $r > 0$ and $|\theta| < \pi/2$. The proof is completed by applying Lemma 9 and then taking the supremum over all $f \in L^2$ and $g \in L^{q_c}$.

THEOREM 25. *The operators* e^{-Hz} *on* $L^2 \cap L^p$ *extend to bounded operators* $T_p(z)$ *on* L^p *for all* $z \in \mathbf{C}$ *with* $\mathrm{Re}(z) > 0$ *and all* $q_c \leqslant p \leqslant p_c$. *Moreover* $P_p(z)$ *is strongly continuous holomorphic semigroup on* L^p *for all such* z *and* p. *There exist constants* c_1, c_2 *and* c_3 *such that*

$$\| T_p(r\, e^{i\theta}) \| \leqslant c_1 (\cos\theta)^{-c_3} \exp[c_2 r \cos\theta]$$

for all $q_c \leqslant p \leqslant p_c$, *all* $r > 0$ *and* $|\theta| < \pi/2$.

Proof. We only consider the case $p = q_c$ and omit the subscript c throughout the proof. The case $p = p_c$ is then obtained by duality, and all intermediate cases follow by interpolation. The main idea is to partition $\mathbf{R}^N$ into cubes whose sizes depend upon $t > 0$.

Let E and F be two cubes in $\mathbf{R}^N$ with edge length $t^{1/2m}$ and let $f \in L^q$. If

$$g := P_E \exp[Hr\, e^{i\theta}]\, P_F f$$

then by using Lemma 24 and duality we obtain

$$\|g\|_2 \leqslant A (r \cos\theta)^{-1/2}\, \|f\|_q$$

for all $r > 0$ and $|\theta| < \pi/2$, where

$$A := c_1 \exp[-c_2\, d(E, F)^{2m/(2m-1)}\, r^{-1/(2m-1)} \cos\theta + kr \cos\theta].$$

Put $a := (N + 2m)/N$ and $b := (N + 2m)/2m$, so that $a^{-1} + b^{-1} = 1$. Then

$$\begin{aligned} \|g\|_q^q &= \int_{\mathbf{R}^N} \chi_E\, |g|^q\, d^N x \\ &\leqslant \left(\int_{\mathbf{R}^N} |g|^{qa}\, d^N x \right)^{1/a} |E|^{1/b} \\ &= \|g\|_2^{2/a}\, |t|^{N/2mb}. \end{aligned}$$

Therefore

$$\|g\|_q \leqslant \|g\|_2 \, |t|^{N/2mhq} = \|g\|_2 \, |t|^{1/2} \leqslant A \, \|f\|_q.$$

This argument yields the bound

$$\begin{aligned}\|P_E \exp[-Hr\, e^{i\theta}] \, P_F\|_{q,q} \\ \leqslant c_1 \exp[-c_2\, d(E, F)^{2m/(2m-1)} \, r^{-1/(2m-1)} \cos\theta + kr\cos\theta]\end{aligned}$$

for all E and F of the stated type.

We now partition $\mathbf{R}^N$ into cubes C_n with centres $nt^{1/2m}$ and edge lengths $t^{1/2m}$, where $n \in \mathbf{Z}^N$. If E_r and E_n are neighbours then

$$\|P_{E_r} \exp[-Hr\, e^{i\theta}] \, P_{E_n}\|_{q,q} \leqslant c_1 \exp[kr\cos\theta]$$

but otherwise we have

$$\begin{aligned}\|P_{E_r} \exp[-Hr\, e^{i\theta}] \, P_{E_n}\|_{q,q} \\ \leqslant c_1 \exp[-c_4\, |r-n|^{2m/(2m-1)} \cos\theta + kr\cos\theta],\end{aligned}$$

where $c_4 > 0$. Now $L^q(\mathbf{R}^N) = l^q(L^q)$, where the latter space is defined in terms of our decomposition of $\mathbf{R}^N$ into cubes C_n. Therefore

$$\begin{aligned}\|e^{-Hz}\|_{q,q} &\leqslant \sup_{m\in Z^N} \sum_{n\in Z^N} \|P_{E_m} \exp[-Hr\, e^{i\theta}] \, P_{E_{m+n}}\|_{q,q} \\ &\leqslant c_1 (\cos\theta)^{-c_3} \exp[c_2 r\cos\theta],\end{aligned}$$

where $c_3 := ((2m-1)/2m)N$.

We omit proofs of the other statements of the theorem, which are all straightforward.

THEOREM 26. *The generator* $(-H_p)$ *of* $T_p(t)$ *has real spectrum independent of* p *for all* $q_c \leqslant p \leqslant p_c$.

Proof. This is another application of Theorem 3.

8. SCHRÖDINGER OPERATORS

In this section we consider a certain class of Schrödinger operators acting on $L^2(\mathbf{R}^N)$. These operators are defined by

$$H := -\Delta + V$$

interpreted in the quadratic form sense. However, our potentials do not lie in the Kato class K_N defined in [23], so much of the standard theory is not applicable. We assume merely that V has quadratic bound less than 1 with respect to H, in other words

$$\int_{\mathbf{R}^N} |V|\,|f|^2\,d^N x \leqslant c_1 \int_{\mathbf{R}^N} |\nabla f|^2\,d^N x + c_2 \int_{\mathbf{R}^N} |f|^2\,d^n x$$

for some $c_1 < 1$, $c_2 < \infty$ and all f for which the RHS is finite. Such Schrödinger operators have been studied by Schreieck and Voigt [20] who proved that the L^p spectrum is independent of p within a certain interval around $p = 2$. Similar results can also be obtained by the method of this paper, since it is not necessary for the coefficients of terms in Q_2 (which all have order less than $2m$) to be bounded in our general theory. As in [15, 16, 21] one only needs them to lie in some L^p or weak L^p space for which the relevant quadratic form inequality can be proved.

In this section we consider a simple example which demonstrates that it may be impossible to prove p-independence of the spectrum for large enough p. We start by defining the positive function

$$\psi(x) := |x|^{-\gamma} e^{-|x|}$$

on $\mathbf{R}^N$, where $N \geqslant 3$, and $\gamma > 0$ is to be determined. This function will turn out to be the ground state of a certain Schrödinger operator. We now define the potential V on $\mathbf{R}^N \backslash \{0\}$ by

$$V(x) := \Delta\psi(x)/\psi(x)$$

so that

$$-\Delta\psi + V\psi = 0$$

in the classical sense. An explicit computation yields

$$V(x) = -\gamma(N-2-\gamma)\ |x|^{-2} + (2\gamma - N + 1)\,|x|^{-1} + 1.$$

If $0 < \gamma < (N-2)/2$ then we see that $\psi \in W^{1,2}(\mathbf{R}^N)$. Moreover the negative part of V has form bound less than 1 with respect to $-\Delta$. The general theory of Schrödinger operators defined by quadratic form methods shows that there exist $\delta > 0$ and $c > 0$ such that

$$-\delta\Delta - c1 \leqslant H \leqslant -\delta^{-1}\Delta + c1$$

and that the essential spectrum of H equals $[1, \infty)$. The point here is that 0 is an isolated eigenvalue of H with multiplicity 1, the eigenfunction being

ψ, which is not bounded on $\mathbf{R}^N$. Further calculations reveal that $\psi \in L^p$ if and only if $1 \leqslant p < N/\gamma$.

THEOREM 27. *The operator H_p cannot have 0 as an isolated eigenvalue whose spectral projection is consistent with that of H_2 if $p \geqslant N/\gamma$ or $p \leqslant N/(N-\gamma)$.*

Proof. The spectral projection of H_2 corresponding to the eigenvalue 0 is

$$Pf := \|\psi\|_2^{-2} \langle f, \psi \rangle \psi.$$

This is only a bounded operator on L^p if $\psi \in L^p$ and also $\psi \in L^q$, where $p^{-1} + q^{-1} = 1$.

Another example, of a second-order elliptic operator for which the Cauchy problem is only well-posed in L^p for p in a small interval around $p = 2$, may be found in [19].

9. OPERATORS ON RIEMANNIAN MANIFOLDS

The description of the vibrations of elastic shells suggests that we should make a number of small changes to the theory described in the sections above. We outline these here. We first replace $\mathbf{R}^N$ by an N-dimensional Riemannian manifold M, which we take for simplicity to be compact and without boundary. Lebesgue measure is then replaced by the Riemannian measure of the manifold. We put $H_0 := (-\Delta)^m$, where $-\Delta$ is the Laplace–Beltrami operator on M, and take $W^{m,2}$ to be the usual Sobolev space on M. We then consider any operator H on $L^2(M)$ defined by a quadratic form Q which satisfies (1) and (2). This includes operators of superelliptic type with measurable coefficients described by local versions of the expressions in Section 2.

Since Lemma 14 uses the particular structure of $\mathbf{R}^N$, we need the following modification.

LEMMA 27. *If $2m > N$ then there is a norm continuous function $\phi : M \to L^2(M)$ such that*

$$\{(H+1)^{-1/2} f\}(x) = \langle f, \phi(x) \rangle$$

for all $f \in L^2$.

Proof. It is a standard fact that $W^{m,2}$ is compactly embedded into $C(M)$ if $2m > N$. This implies that if $B := \{f \in L^2 : \|f\|_2 \leqslant 1\}$ then

$(H+1)^{-1/2}B$ is an equicontinuous set in $C(M)$. Given $\varepsilon>0$ there exists $\delta>0$ such that if $x, y\in M$ and $d(x, y)<\delta$ then

$$|\langle f, \phi(x)\rangle - \langle f, \phi(y)\rangle| < \varepsilon$$

for all $f\in B$. Therefore $\|\phi(x)-\phi(y)\|_2<\varepsilon$.

If U is an open set in compact Riemannian manifold M then we may study the form Q defined on the closure $W_0^{m,2}$ of $C_c^\infty(U)$. The corresponding operator H on $L^2(U)$ is said to satisfy Dirichlet boundary conditions. If the assumptions (2) and (3) are valid in M then they are also valid in U subject to Dirichlet boundary conditions. The theory proceeds as before. It is worth commenting that if $2m>N$ then the kernel $K(t, x, y)$ is jointly continuous and vanishes uniformly as x or y converge to ∂U, without any boundary regularity assumptions.

10. Other Operators Satisfying the Hypotheses

Our basic hypotheses (2) and (3) are satisfied by superelliptic operators, and also by uniformly elliptic operators with smooth coefficients in the conventional sense. It is obvious that they are also satisfied by any sum of two such operators. In this section we describe two further classes of operators which satisfy (2) and (3). We do not however resolve the question of finding necessary and sufficient conditions on the coefficients of an elliptic operator for it to atisfy our hypotheses.

We start by considering elliptic operators of fourth order on $L^2(\mathbf{R}^N)$ associated with quadratic forms of the type

$$Q(f):=\int_{\mathbf{R}^N}\sum_{i,j}\{a_{i,j}(x)(D^{i,i}f(x))\overline{(D^{j,j}f(x))}+b_{i,j}(x)\,|D^{i,j}f(x)|^2\}\,d^Nx.$$

We assume that $b(x)$ and $a(x)$ are self-adjoint matrices for each $x\in\mathbf{R}^N$. We also suppose that $b_{i,i}(x)=0$ for all i, but do not assume that the coefficients $b_{i,j}(x)$ are non-negative. We also assume that all of the coefficients are bounded and measurable.

THEOREM 28. *Suppose that there exist $c_{i,j}\in W^{1,\infty}$ such that $b_{i,j}(x)\geqslant c_{i,j}(x)$ for all i, j and all $x\in\mathbf{R}^N$, and such that*

$$a(x)+c(x)\geqslant\varepsilon 1$$

in the sense of matrices for some $\varepsilon>0$ and all $x\in\mathbf{R}^N$. Then Q satisfies hypotheses (2) *and* (3).

Proof. If $f \in C_c^\infty$ then by integration by parts we see that

$$Q(f) \geqslant \int_{\mathbf{R}^N} \sum_{i,j} \{a_{i,j}(D^{i,i}f)\overline{(D^{j,j}f)} + c_{i,j} |(D^{i,j}f)|^2\} \, d^N x$$
$$= Q_0(f) + Q_1(f) + Q_2(f),$$

where

$$Q_0(f) := \varepsilon \int_{\mathbf{R}^N} \sum_i |D^{i,i}(f)|^2 \, d^N x$$

$$Q_1(f) := \int_{\mathbf{R}^N} \sum_{i,j} (a_{i,j}(x) + c_{i,j}(x) - \varepsilon\delta_{i,j})(D^{i,i}f)\overline{(D^{j,j}f)} \, d^N x$$

$$Q_2(f) := \int_{\mathbf{R}^N} \sum_{i,j} \left\{ \frac{\partial c_{i,j}}{\partial x_j} (D^j f)\overline{(D^{i,i}f)} - \frac{\partial c_{i,j}}{\partial x_i} (D^i f)\overline{(D^{i,j}f)} \right\} d^N x.$$

We deduce that

$$Q(f) \geqslant Q_0(f) + Q_2(f)$$

for all $f \in C_c^\infty$. The proof of the theorem now follows the methods of Section 2.

Example. We consider two homogeneous media each occupying a half of the complex plane. Let H be associated to the quadratic form

$$Q(f) := \int_{\mathbf{R}^2} \left\{ u \left| \frac{\partial^2 f}{\partial x^2} \right|^2 + v \left| \frac{\partial^2 f}{\partial y^2} \right|^2 + 2b \left| \frac{\partial^2 f}{\partial x \, \partial y} \right|^2 \right\} dx \, dy$$

acting on $C_c^\infty(\mathbf{R}^2)$, where

$$u(x, y) := \begin{cases} u_+ & \text{if } x > 0 \\ u_- & \text{if } x < 0 \end{cases}$$

with similar formulae for v and b. Then the condition of this section becomes

$$c > -(u_+ v_+)^{1/2}, \qquad c > -(u_- v_-)^{1/2}, \tag{13}$$

where $c = \min\{b_+, b_-\}$. This is weaker than the condition for Q to be the sum of a constant coefficient elliptic form and a superelliptic form.

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