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Tiling pictures of the plane with dominoes

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Abstract

We consider the problem of tiling with dominoes pictures of the plane, in theoretical and algorithmic aspects. For generalities and other tiling problems, see for example Refs. Beauquier et al. (1995), Conway and Lagarias (1990), Kannan and Soroker (1992), Kenyon (1992), and Beauquier (1991). The pictures which are considered here may have holes, but uniquely *balanced* holes, that is every hole, if chessboard-like coloured, has an equal number of black squares and of white ones. We give an algorithmic characterization of tilable pictures and a canonical decomposition into 'strongly' tilable subpictures. The given algorithm is linear as far the considered pictures have a finite number of (balanced) holes. In the same hypothesis there is a good parallel algorithm (in class NC). Graphical extension of the used method (*heights' method*) is applied to a class of bipartite planar graphs. The particular case of without holes pictures is developed in Fournier (1996).

As far as I know, the results in this paper are new, except the notions and the theorem in Section 2, which are substantially present in Thurston (1990).

1. The problem

We first consider the problem of tiling with dominoes *without holes* pictures of the plane. Here, a picture is a subset of unit squares of the plane (considered as $Z \times Z$), see. for example, Fig. 2(b).

It is easy to see that a necessary condition for the existence of a tiling for a chessboard-like coloured picture is that there is in the picture an equal number of black squares and of white ones (*balanced* picture).

This condition is not sufficient as we can see, while analysing the picture in Fig. 1. More deeply, the impossibility of tiling this picture is due to the existence of the line *abc* which delimits a subpicture, under *abc*, with the following properties:

(1) there are more black squares than white ones,

(2) the line *abc* runs alongside white squares of the subpicture.

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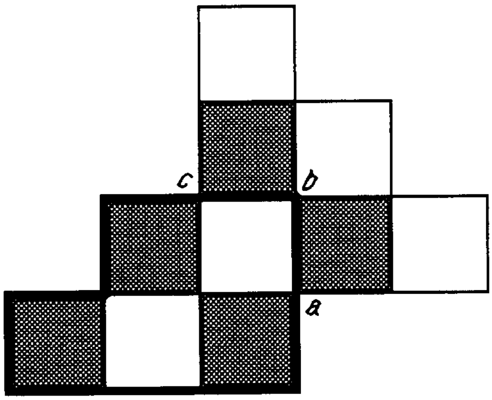


Fig. 1. Non tilable balanced picture and obstruction to tiling.

Such a subpicture is called an *obstruction to tiling*. If a picture has no obstruction to tiling, then it is tilable. In fact, this condition corresponds to the classical Hall's condition for matchings in bipartite graphs. We shall express this condition in another way.

2. Characterization of tilable without holes pictures

The boundary cycle of a without holes picture F determines a function h on its vertices, up to constants: if $(s_0, s_1, \ldots, s_p)$ is the sequence of vertices of the boundary cycle, with $s_p = s_0$, let us define:
— $h(s_0) = k$, where k is some integer,
— for $i = 1, \ldots, p$:

$$h(s_i) = \begin{cases} h(s_{i-1}) + 1 & \text{if the edge } s_{i-1}s_i \text{ delimits a white square of } F, \\ h(s_{i-1}) - 1 & \text{otherwise.} \end{cases}$$

Such a function will be called a *height function on the boundary of the picture.*

Let F be a picture and C its boundary. We associate with F the *white on the left*-directed graph G in the following way: the vertices of G are the vertices of F and the arcs of G correspond to edges of F not in C and oriented in such a way that a white square is always on the left-hand side of each arc. We denote $d(s,t)$ the distance from a vertex s to a vertex t in G (the length of a shortest path from s to t in G).

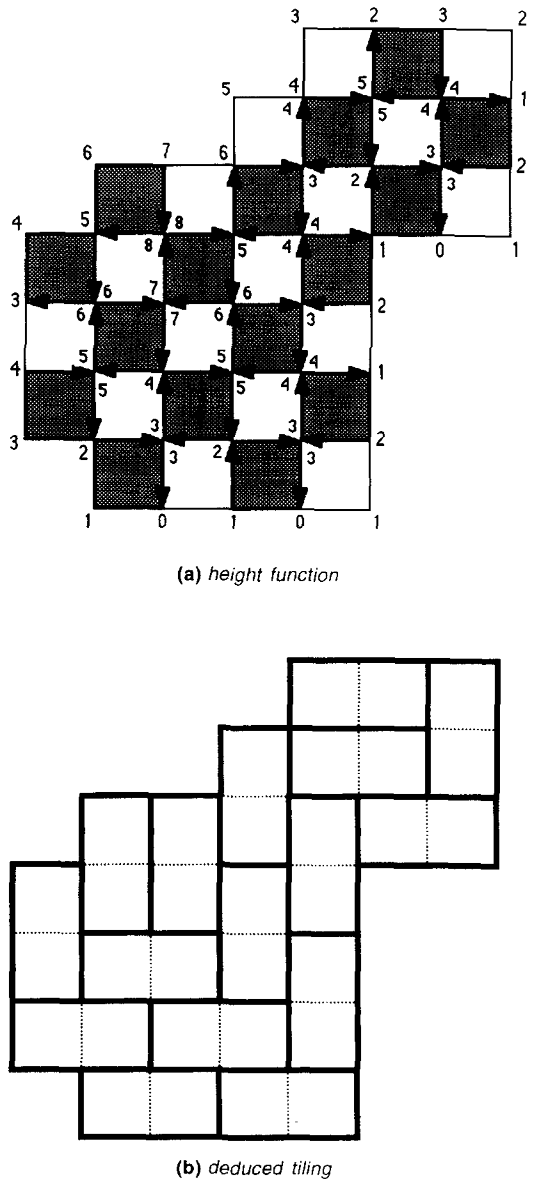
We define a function h on the vertices of F, associated with a height function g on the boundary C of F, in the following way:

For every vertex t of F, $h(t) = \min\{g(s) + d(s,t)/s \text{ vertex of } C\}$. Such a function is called a *height function on the picture.*

Note that for every vertex t of C we have $h(t) \leqslant g(t)$. The inequality $h(t) < g(t)$ is possible and it corresponds to the existence of an obstruction to tiling in the picture. The main result is:

Theorem. *Let F be a picture, C the boundary of F, g a height function on C, h a height function on F associated with g. The picture is tilable if and only if for every $s \in C$, we have $h(s) = g(s)$. Moreover, if F is tilable, the edges tt' such that* $|h(t) - h(t')| = 1$ define a tiling of F.

Example. See Fig. 2.



(a) *height function*

(b) *deduced tiling*

Fig. 2. Tilable picture.

This theorem is informally given in [12], with a proof based on combinatorial group theory. We give in [5] a different proof, based on Hall's condition for matchings in bipartite graphs.

3. Algorithmic point of view

This result allows us to find again the linear Thurston's algorithm [12].

In fact, the height function is equal to a distance in an *extension* of the white on the left graph (for details on this extension, see [5]). It is then possible to calculate the height function h on F, and so to calculate a tiling of F, if the condition $h(s) = g(s)$ for every $s \in C$ is true, by a classical Breadth First Search algorithm (note that the height function g on C is easy to calculate). The whole algorithm is linear.

More interesting is the possibility of a parallelization of this algorithm, by means of a parallelization of Breadth first search, which is the main part of the algorithm. A classical parallel algorithm views the graph as an incidence matrix M and repeatedly squares M, returning M^n and so the BFS numbering of the graph. This yields an algorithm using $O(\log n)$ time and n^3 processors on the concurrent-read concurrent-write (CRCW) parallel random access machine (PRAM). An improved parallel algorithm that computes the BFS numbering of a directed graph is given in [6]. Finally, we have an efficient algorithm for tiling pictures without holes (this problem is in the class NC). This last result is rather unexpected, apart from the new point of view given here.

4. Canonical decomposition of tilings

Let F be a without holes tilable picture with boundary C. We define a *fracture path* of F as a path in the white on the left-directed graph from a vertex $s \in C$ to a vertex $s' \in C$ such that $g(s') = g(s) + l$, where l is the length of the path. A *fracture edge* is a picture's edge which is on a fracture path, and the *fracture graph* is the graph induced by fracture edges. It is to note that *a fracture edge is never covered by a domino in a picture's tiling.*

We define a *strongly tilable* picture as a tilable picture for which every edge, not on the boundary, can be covered in some tiling. Clearly, if a strongly tilable picture has a number of squares > 2, it is not tilable in an unique way.

The main result here is:

Theorem. *The fracture graph of a without holes tilable picture divides the picture into without holes and strongly tilable subpictures. Furthermore, each picture's tiling breaks up into subpicture's tilings.*

We deduce many corollaries of this result, in particular, a characterization of uniqueness case of tiling:

Corollary. *A without holes picture is uniquely tilable if and only if its fracture graph divides it into dominoes.*

There is an interesting property of strongly tilable pictures which comes from properties of 'elementary bipartite graphs' (given in [9]): *For any white case and any black square there exists a tiling of the picture which does not cover just these two cases.*

5. Case of pictures with holes

We can extend our algorithm to the case of pictures with *balanced* holes: that is, every hole, if chessboard-like coloured, has an equal number of black squares and of white ones. Let $C_0, C_1, \dots, C_p$ be the boundaries cycles of the picture, C_0 being the outer boundary cycle. For the calculation of a height function the inner boundaries cycles $C_1, \dots, C_p$ are supposed to be clockwise oriented and C_0 is supposed to be anticlockwise oriented.

The algorithm is based on two procedures.

— *Boundaries_fit*: For each boundary C_i with some given value on some of its vertices, this procedure calculates a height function on this boundary such that for every vertex $s \in C_i$ with a given value v, we have $h(s) \leqslant v$, and we have equality for some vertex of C_i (to do that, calculate any height function on C_i, and adjust it by means of a constant).

— *Inside_propagation*: This procedure calculates a height function on the picture as in the case of without holes pictures, that is from the values on the boundaries and in the white on the left-directed graph G, by means of the formula

$$h(t) = \min\{g(s) + d(s,t)/s \text{ vertex of } C\},$$

where d is the distance in G and $C = C_0 \cup C_1 \cup \dots \cup C_p$.

Algorithm

begin

fix some value on some vertex of C_0;

do p + 1 times

boundaries_fit;

inside_propagation

end do

end.

Result. *The picture is tilable if and only if the last iteration of the algorithm has no effect on the values of the height function. When the picture is tilable, we get a tiling as in the case of without holes pictures, that is with edges tt' such that $|h(t) - h(t')| = 1$, where h is the obtained height function on the figure.*

6. Weighted digraph associated with a picture

Let there be given a picture, we consider the white on the left digraph augmented by the oriented boundaries' edges, and then, we define for each arc a a weight

$$\mathrm{w(a)} = \begin{cases} +1 & \text{if } a \text{ is not on a boundary or if } a \text{ is on a boundary and} \\ & \text{has a white square of the picture on its left-hand side,} \\ -1 & \text{if } a \text{ is on some boundary and has a black square} \\ & \text{on its left-hand side.} \end{cases}$$

This digraph is called the *associated weighted digraph.*

Theorem. *A picture with balanced holes is tilable if and only if its associated weighted digraph does not have any negative cycle.*

Unfortunately, this result does not give an interesting algorithm: in fact, the best Algorithm I knows which can find a negative cycle in a directed graph is running in $\mathrm{O}(n, e)$ [11]. This algorithm gives here a complexity $\mathrm{O}(n^2)$.

Given a tilable picture with balanced holes, let us define the *fracture graph* as the graph induced by edges which are on some null cycle (cycle of length 0). We have the following extension of the decomposition theorem:

Theorem. *The fracture graph of a tilable picture with balanced holes divides the picture into subpictures with balanced holes which are strongly tilable. Furthermore, each of picture's tiling breaks up into subpictures' tilings.*

7. Extensions

The previous algorithm, which we call here the *heights' method*, also works for more general pictures: that is, the without holes pictures which are composed of d-regular cells and whose degrees of its inner vertices are even [3].

We can even more extend the heights' method by considering the dual graph of the picture.

Let G be a bipartite graph, (X, Y) a bicoloration of its vertices, Δ its maximum degree.

We define the *Δ-augmented graph* of G as the graph G with a new vertex ω and for each vertex x of G, Δ-d(x) edges between ω and x (d(x) being the degree of x in G).

A *balanced splitting* of ω is a splitting of the vertex ω into vertices $\omega_1, \ldots, \omega_q$ $(q \geqslant 1)$, where these vertices share between themselves the edges of ω in a balanced way, that is each vertex ω_i gets an equal number of edges (ω, x) with $x \in X$ and edges (ω, y) with $y \in Y$.

Theorem. *The heights' method extends to bipartite planar graphs which admit a planar balanced splitting of its Δ-augmented graph.*

Consider a plane representation of a planar balanced splitting of the Δ-augmented graph. Each vertex ω_i, for $i = 1, \ldots, q$, defines a face which corresponds, by duality in the plane, to a balanced hole of a picture. One of these holes is in fact the exterior of the picture. In particular, the case $q = 1$ corresponds to without holes pictures.

Complexity of the obtained algorithm: $O(nq)$.

8. Open questions

— Known algorithms on matchings in graphs, in particular bipartite graphs, give algorithms of complexity $O(n^{1.5})$ for tiling any pictures.

We gave here a linear algorithm for the case of without holes pictures, or pictures with a *bounded* number of *balanced* holes.

Is there a linear algorithm for the general case? Or an algorithm with complexity between $O(n)$ and $O(n^{1.5})$?

— Instead of *perfect* matchings, we can consider *maximum* matchings: that is partial tilings with a maximum number of dominoes.

Is there a linear algorithm which can find a maximal matching in without holes pictures?

— It is natural to consider the following general problem: instead of considering dominoes, we consider 'bars' of some length, horizontal bars h_p of length p, and vertical bars v_q of length q.

Robson [10] proved that the question of tiling a picture by h_p and v_q is NP-complete when $p \geqslant 3$ or $q \geqslant 3$. Thus, the problem is closed... unless P = NP!

— Parallel algorithmic aspect gives other open questions, for instance:

Is the problem of finding a maximum tiling in class NC?

References

[1] D. Beauquier, éd., Polyominos et Pavages, Publication interne de l' université Paris 12 (Créteil, 1991).

[2] D. Beauquier, M. Nivat, E. Remila and M. Robson, Tiling figures of the plane with two bars, Comput. Geom. 5 (1995) 1–25.

[3] T. Chaboud, Pavages et graphes de Cayley planaires, Thèse E.N.S. Lyon, 1995.

[4] J.H. Conway and J.C. Lagarias, Tiling with Polyominoes and combinatorial group theory, J. Combin. Theory A 53 (1990) 183–208.

[5] J.C. Fournier, Pavage des figures planes sans trous par des dominos: Fondement graphique de l'algorithme de Thurston, Parallélisation, unicité et décomposition, Theoret. Comput. Sci. 159 (1996) 105–128.

[6] H. Gazit and G.L. Miller, An improved parallel algorithm that computes the BFS numbering of a directed graph, Inform. Process. Lett. 28 (1988) 61–65.

[7] S. Kannan and D. Soroker, Tiling polygons with parallelograms, Discrte Comput. Geom. 7 (1992) 175–188.
[8] C. Kenyon and K. Kenyon, Tiling a polygon with rectangles, FOCS 92 (1992) 610–619.
[9] L. Lovàsz, Combinatorial Problems and Exercices (North Holland, Amsterdam, 1979).
[10] J.M. Robson, Sur le pavage de figures du plan par des barres, in [1] (1991) 59–107.
[11] P. Spirakis and A. Tsakalidis, A very fast practical algorithm for finding a negative cycle in a digraph, Springer Lecture Notes in Comp. Sci., Vol. 226 (Springer, Berlin, 1986) 397–406.
[12] W.P. Thurston, Conway's tiling groups, Amer. Math. Monthly (1990) 757–773.