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Note

# A note on vertex-transitive non-Cayley graphs from Cayley graphs generated by involutions

Jana Tomanová

*Comenius University, Bratislava, Slovakia*

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ABSTRACT

We show that the result of Watkins (1990) [19] on constructing vertex-transitive non-Cayley graphs from line graphs yields a simple method that produces infinite families of vertex-transitive non-Cayley graphs from Cayley graphs generated by involutions. We also prove that the graphs arising this way are hamiltonian provided that their valency is at least six.

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## 1. Introduction

The growing interest in constructions of vertex-transitive non-Cayley graphs (for which we will be using the acronym VTNCGs throughout) goes, to a large extent, back to Marušič's call [12] for a characterization of the set of all possible orders of VTNCGs. Proving that a particular vertex-transitive graph is not a Cayley graph may be hard. Two types of general methods for this purpose have crystallized in the past two decades. *Direct* methods are based on Sabidussi's characterization of Cayley graphs [16]. Their essence lies in showing that the automorphism group of a given graph has no subgroup acting regularly on vertices of the graph, see [5,14,15]. *Indirect* methods try to identify structural properties of Cayley graphs which they do not share with vertex-transitive graphs in general. Examples include residue classes (modulo the valency) of the number of closed walks at a fixed vertex [6], configurations fixed by involutions [13], and relations between a graph and its line graph [19].

The aim of this note is to show that the 'line-graph' method of [19] can be used to produce VTNCGs from Cayley graphs generated by involutions. The construction (with a number of examples) is presented in Section 2 and further extended in Section 3. The final section contains a remark on hamiltonicity of the resulting VTNCGs in the case of valency at least six.

## 2. The construction

Our method of constructing VTNCGs from Cayley graphs generated by involutions is based on a straightforward application of the following interesting result of Watkins [19] which seems to have been somewhat neglected in the literature.

**Theorem A** ([*19*]). *Let $\Gamma$ be a graph on at least five vertices that is edge-transitive, non-bipartite and whose vertices have odd valency. Then the line graph of $\Gamma$ is a VTNCG.*

*E-mail address:* tomanova@fmph.uniba.sk.

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Recall that for a given (finite) group $G$ and a generating set $X$ for $G$ such that $1 \notin X$ and $X = X^{-1}$, the *Cayley graph* $C(G, X)$ has vertex set $G$, and two vertices $g, h \in G$ are adjacent if $gh^{-1} \in X$, which is equivalent to $hg^{-1} \in X$. In this setting, right multiplication by any fixed element of $G$ induces an automorphism of the Cayley graph and in this way $G$ acts regularly on the vertices of $C(G, X)$. The valency of $C(G, X)$ is simply $|X|$, the number of elements in the generating set. If $|X|$ is odd, then due to the fact that $X$ is closed under taking inverses, $X$ contains an odd number of involutions. The extreme case when all elements in $X$ are involutions can be used for constructing VTNCGs using automorphisms of the group $G$. Any automorphism of $G$ that preserves $X$ setwise is a graph automorphism of $C(G, X)$. Moreover, assume that $X$ is an orbit of a subgroup of $Aut(G)$ such that $|X|$ is odd. Then, because any group automorphism preserves the order of elements, all the elements of an orbit must be of the same order, and since $|X|$ is odd, and as such has to contain at least one involution, all the elements in $X$ have to be involutions. This is the reason why the all-involution assumption has been included in the statement of the following consequence of Theorem A.

**Corollary 1.** *Let $C(G, X)$ be a non-bipartite Cayley graph of order at least five and let $X$ consist of an odd number of involutions. If $X$ is an orbit of some subgroup of $Aut(G)$, then the line graph of $C(G, X)$ is a VTNCG.*

**Proof.** Let $H$ be a subgroup of $Aut(G)$ such that $X$ is an orbit of $H$. The regular action of $G$ on $C(G, X)$ combined with the action of $H$ guarantees that the group $Aut(C(G, X))$ acts transitively on the arcs (and hence on the edges) of $C(G, X)$. By Theorem A, the line graph of $C(G, X)$ is a VTNCG. $\square$

The simplest way to construct a set $X$ satisfying the assumption of our Corollary is to take a conjugacy class of a single involution $x$ by an element $y$ of odd order. To make sure that the graph $C(G, X)$ is not bipartite, it is sufficient to consider groups $G$ containing no subgroups of index two. Indeed, if $C(G, X)$ is bipartite, then the subgroup $H$ generated by all the products $ab$ where $a, b \in X$ is a subgroup of index two in $G$. This leads us to the following application of Corollary 1.

**Corollary 2.** *Let $G$ be a group with generating set $\{x, y\}$ where $x$ is an involution and $y$ is of odd order $d \geq 3$, and let $X = \{y^i x y^{-i} : 0 \leq i < d\}$. If $G$ does not contain proper normal subgroups of index less than or equal to $d$, then the line graph of $C(G, X)$ is a connected VTNCG.*

**Proof.** In view of the facts listed above it is sufficient to show that $X$ is a generating set for $G$. Observe that the subgroup $\langle X\rangle$ of $G = \langle x, y\rangle$ is invariant under conjugation by both $y$ and $x$ and hence is normal in $G$. We show that $G/\langle X\rangle$ is a cyclic group of order a divisor of $d$, and hence the index of $\langle X\rangle$ in $G$ is a divisor of $d$ as well. Our assumption $G = \langle x, y\rangle$ ensures that $G/\langle X\rangle = \langle\langle X\rangle, y\langle X\rangle\rangle$. It is now clear that if $y$ belongs to $\langle X\rangle$ then $G = \langle X\rangle$ and we are done. Assume that $y \notin \langle X\rangle$. Then $G/\langle X\rangle$ is a cyclic group whose order is the least integer $m$ such that $y^m \in \langle X\rangle$ and $1 < m < d$. But then $1 = (y^m)^s = (y^s)^m$, $s$ being the order of $y^m$ in $G$, which implies that $m$ is a divisor of $d$. Thus the index of $\langle X\rangle$ in $G$ is a divisor of $d$ as well. By our assumptions, however, $G$ has no such proper normal subgroup and therefore $\langle X\rangle = G$. The rest follows from Corollary 1. $\square$

We note that the assumption that $G$ does not contain normal subgroups of index at most $d$ can be replaced with the assumption '*$G$ has no subgroup of index 2 and $x$ is not contained in any proper normal subgroup of $G$*'.

Since simple groups automatically satisfy the last assumption of Corollary 2 (as well as its version described above), they are prime candidates for a possible application of this result. The hard part here is to find a generating set consisting of an involution and an element of odd order. Here we invoke a consequence of [8,9] by which all finite non-abelian simple groups except Suzuki groups, the symplectic groups $PSp_4(2^k)$ and $PSp_4(3^k)$, and a finite number of other exceptions are generated by an involution and an element of order 3. Corollary 2 then implies that for all such simple groups $G$ there exists a Cayley graph $C(G, X)$ of valency 3 such that its line graph is a connected VTNCG.

Another way to satisfy the assumptions of Corollary 1 is to take conjugacy classes that cannot be generated by conjugation by powers of a single element.

**Example 1.** Consider the alternating group $G = A_n$, where $n \geq 5$ and its subset $X = \{(ij)(kl) : |\{i, j, k, l\}| = 4 \text{ and } 1 \leq i, j, k, l \leq n\}$. It is well known that $A_n$ is, for all $n \geq 3$, generated by the set of all 3-cycles. Since any 3-cycle $(ijk)$ is a product $((ik)(i'k'))((i'k')(kj))$ of elements in $X$, where all $i, j, k, i', k'$ are pairwise distinct (note that we have assumed $n \geq 5$), it follows that $X$ generates $G$. It is clear that $|X| = 3\binom{n}{4}$ and therefore, $|X|$ is odd if and only if $n \equiv 4, 5, 6, 7 \pmod 8$.

For application of Corollary 1 we take the symmetric group $S_n$ as the subgroup of $Aut(G)$; as an aside, we mention the well known fact that $S_n$ is isomorphic to $Aut(A_n)$ for all $n$ except 2 and 6. Two permutations are conjugate in $S_n$ if and only if they have the same cycle type. This together with the fact that the set $X$ comprises all permutations of a given cycle type implies that the symmetric group $S_n$ fulfills the assumption of Corollary 1 including non-bipartiteness since $A_n$ is simple for $n \geq 5$. It follows that the line graph of $C(G, X)$ is a VTNCG.

Our next example is an illustration of a situation when one can take the generating set to be the set of *all* involutions of a group.

**Example 2.** Consider the projective special linear group $G = PSL(2, q)$, where $q > 3$ is a prime power, and the subset $X$ consisting of all involutions in $G$. We will show that the line graph of $C(G, X)$ is a VTNCG. Suppose first that $q$ is an odd number. Observe that $A \in G$ is an involution if and only if $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ where $a^2 + bc = -1$ and therefore, it is sufficient to determine the number of triples $(a, b, c)$ such that $a^2 + bc = -1$. Recalling that the multiplicative group of $GF(q)$ is a cyclic group of order $q-1$, it follows that $-1$ is a square in $GF(q)$ if and only if $q \equiv 1 (\mathrm{mod}\ 4)$. It is now clear that if $q \equiv 1 (\mathrm{mod}\ 4)$ then there are $q-2$ choices for $a$ such that $-1-a^2 \neq 0$ and for each non-zero $b$ we have a unique $c$ such that $bc = -1-a^2 \neq 0$. This gives $(q-2)(q-1)$ triples. Similarly, there are 2 choices for $a$ such that $-1-a^2 = 0$ and $2q-1$ choices for $b, c$ such that $bc = 0$. Since $A = -A$, we have $\frac{1}{2}((q-2)(q-1) + 2(2q-1)) = q\left(\frac{q+1}{2}\right)$ involutions in $G$. If $q \equiv 3 (\mathrm{mod}\ 4)$ then there are $q$ choices for $a$ such that $-1-a^2 \neq 0$ and for each non-zero $b$ we have a unique $c$. This gives $q\left(\frac{q-1}{2}\right)$ involutions. Obviously, the cardinality of $X$ is odd in the both cases $q \equiv 1, 3 (\mathrm{mod}\ 4)$. If $q$ is a power of two, then a similar but easier counting shows that $PSL(2, q) = SL(2, q)$ contains exactly $q^2 - 1$ elements of order two, an odd number again. For any two elements $g \in G$ and $x \in X$ we have $gxg^{-1} \in X$ and therefore, $\langle X \rangle$ is normal in $G$. By simplicity of $G$ it follows that $\langle X \rangle = G$ and $C(G, X)$ is a non-bipartite graph. Since any two involutions of $PSL(2, q)$ are conjugate in the group $PGL(2, q) < Aut(PSL(2, q))$ [4], the assertion follows from Corollary 1.

## 3. Extension to categorical product

Let us now consider the *categorical product* of graphs to generate VTNCGs. Recall that for given two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ the *categorical product* $\Gamma_1 \times \Gamma_2$ of $\Gamma_1$ and $\Gamma_2$ is the graph with the vertex set $V_1 \times V_2$, and two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are joined by an edge if and only if $\{u_1, v_1\} \in E_1$ and $\{u_2, v_2\} \in E_2$. Note, that if $\Gamma_1 = C(G_1, X_1)$ and $\Gamma_2 = C(G_2, X_2)$ then by definition of the adjacency relation in Cayley graphs, their categorical product is the Cayley graph on the group $G_1 \times G_2$ with respect to the generating set $X_1 \times X_2$. Assume that we are given two non-bipartite, edge-transitive graphs $\Gamma_1$ and $\Gamma_2$ of odd valency $d_1$ and $d_2$, respectively. Then immediately from the definition of the categorical product follows that the graph $\Gamma_1 \times \Gamma_2$ is edge-transitive and of odd valency $d = d_1 d_2$. To see that the graph $\Gamma_1 \times \Gamma_2$ is not bipartite, it is sufficient to observe that to any given odd cycles $C_1$ and $C_2$ of $\Gamma_1$ and $\Gamma_2$, respectively, there corresponds a closed walk of length equal to $lcm(|C_1|, |C_2|)$ in $\Gamma_1 \times \Gamma_2$.

We have proved that $\Gamma_1 \times \Gamma_2$ fulfills the assumptions of Theorem A and we may conclude that the line graph of $\Gamma_1 \times \Gamma_2$ is a VTNCG. Obviously, the above can be extended by induction to the categorical product of any finite family of appropriate graphs.

**Theorem 1.** *Let $\Gamma_1, \ldots, \Gamma_n$, $n \geq 2$, be edge-transitive, non-bipartite and odd valency graphs. Then the line graph of $\Gamma_1 \times \cdots \times \Gamma_n$ is a VTNCG.* □

We apply this construction to show that there are infinite families of *simple orbit graphs* that are VTNCGs. The *simple orbit graph* $O(G, X)$ associated with a (finite) group $G$ and a non-empty subset $X$ of $G$ such that $\langle x \rangle \cap \langle x' \rangle = \{1\}$ for any two distinct elements $x, x' \in X$ is defined as follows [18]. The vertex set of $O(G, X)$ is the union, over all $x \in X$, of orbits of left translations induced by $x$, and two distinct vertices $u$ and $v$ are joined by an edge if and only if the orbits they represent have a (unique) element in common. In the same paper it was shown that any simple orbit graph $O(G, X)$ admits a representation as a colour clique graph of a Cayley graph on the group $G$. This gives rise a natural question of characterizing simple orbit graphs that are VTNCGs, maybe, in terms of Cayley graphs. Since the line graph of $C(G, X)$ is the simple orbit graph $O(G, X)$ whenever $X$ consists of involutions [18], our examples together with their extensions in the spirit of Theorem 1 give a partial answer to the problem. We remark that (simple) orbit graphs are a special case of *G-graphs* introduced as a potential tool for group isomorphism testing in [1]. Also, simple orbit graphs can be regarded as a generalization of double-coset graphs introduced in [3] and studied in connection with constructions of edge- but not vertex-transitive graphs [7,11].

## 4. A remark on hamiltonicity

We conclude with addressing the question of whether or not every VTNCG that arises by our constructions contains a Hamiltonian cycle. First we recall that a connected, vertex-transitive graph of valency $k$ is $k$-edge-connected [10], and the line graph of a $k$-edge-connected graph is $k$-connected [2]. From this it follows that the line graph of a Cayley graph $C(G, X)$ is $|X|$-connected.

A well-known conjecture of Thomassen states that every 4-connected line graph is hamiltonian. Moreover, it was observed in [17] that if $\Gamma$ is a counterexample to this conjecture, then $\Gamma$ is the line graph of a graph that contains a vertex of valency at most 3. Combining all the above facts we obtain:

**Theorem 2.** *Let $\Gamma = C(G, X)$ be a connected Cayley graph. If $|X| > 3$, then the line graph of $\Gamma$ is hamiltonian.* □

From this it immediately follows that the VTNCGs we have considered are hamiltonian whenever their root Cayley graphs are of valency greater than three, in which case our VTNCGs have valency at least six.

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