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# Independent Fuzzy Events

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## 1. INTRODUCTION

The epoch making paper of Zadeh [3] provided a mathematical theory for the development of models in many areas of sicnetific research in which fuzziness is a pervasive phenomenon [cf. 2, 5]. In 1968, Zadeh [4] introduced the concept of a fuzzy event and studied its basic properties. This was the first attempt at providing a mathematical theory that accounts for the two fundamental modes of uncertainty inherent in most practical problems, i.e., statistical uncertainty and fuzziness or ambiguity.

This paper deals with independent fuzzy events. In Section 2 some results which are basically of a set theoretic nature are presented. Section 3 contains the basis elementary results on independent fuzzy events. Finally in Section 4 analogues of some of the important Zero–One laws of prbability theory for fuzzy events are established.

## 2. PRELIMINARIES

Let $\Omega$ be a set, $\mathscr{F}(\Omega)$ be the lattice of fuzzy subsets of $\Omega$, and for each $A \in \mathscr{F}(\Omega)$, $\mu_A$ be the membership function of $A$, i.e., $\mu_A: \Omega \to [0, 1]$. The concepts of a monotone sequence in $\mathscr{F}(\Omega)$ and its limit are defined in exactly the same way as in elementary set theory. The following lemmas present the basic results on monotone sequences in $\mathscr{F}(\Omega)$. The proofs are straightforward and will not be presented here.

LEMMA 2.1. *Let $\langle A_n \rangle$ be a monotone sequence in $\mathscr{F}(\Omega)$ and $A = \lim_{n\to\infty} A_n$. Then $\forall \omega \in \Omega$, $\mu_A(\omega) = \lim_{n\to\infty} \mu_{A_n}(\omega)$.*

LEMMA 2.2. *Let $\langle A_n \rangle$ be a monotone sequence in $\mathscr{F}(\Omega)$ and $A = \lim_{n\to\infty} A_n$. Then*

(a) *$\forall B \in \mathscr{F}(\Omega)$, $\langle A_n \backslash B \rangle$ is a monotone sequence in $\mathscr{F}(\Omega)$ and $\lim_{n\to\infty}(A_n \backslash B) = A \backslash B$;*



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(b) $\forall B \in \mathscr{F}(\Omega)$, $\langle B \backslash A_n \rangle$ *is a monotone sequence in* $\mathscr{F}(\Omega)$ *and* $\lim_{n\to\infty}(B\backslash A_n) = B\backslash A$;

(c) $\forall B \in \mathscr{F}(\Omega)$, $\langle B \cup A_n \rangle$ *is a monotone sequence in* $\mathscr{F}(\Omega)$ *and* $\lim_{n\to\infty}(B \cup A_n) = B \cup A$.

The concepts of lim sup and lim inf of sequences of fuzzy subsets of $\Omega$ are defined in exactly the same way as in elementary set theory. Using the definitions of union and intersection of fuzzy subsets of $\Omega$ we immediately get the following result.

LEMMA 2.3. *Let* $\langle A_n \rangle$ *be a sequence in* $\mathscr{F}(\Omega)$ *and* $A = \limsup_{n\to\infty} A_n$, $B = \liminf_{n\to\infty} A_n$. *Then* $\forall \omega \in \Omega$.

$$\mu_A(\omega) = \limsup_{n\to\infty} \mu_{A_n}(\omega)$$

$$\mu_B(\omega) = \liminf_{n\to\infty} \mu_{A_n}(\omega).$$

Using elementary properties of lim sup and lim inf of real numbers we can easily prove the following two lemmas.

LEMMA 2.4. *Let* $\langle A_n \rangle$ *be a sequence in* $\mathscr{F}(\Omega)$. *Then*

(a) $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$;

(b) $\liminf_{n\to\infty} A_n = (\limsup_{n\to\infty} A_n{}^c)^c$, *where c denotes complementation in* $\Omega$.

LEMMA 2.5. *Let* $\langle A_n \rangle$ *and* $\langle B_n \rangle$ *be sequences in* $\mathscr{F}(\Omega)$. *Then*

$$\limsup_{n\to\infty} A_n B_n \subseteq (\limsup_{n\to\infty} A_n)\,(\limsup_{n\to\infty} B_n).$$

We next consider subsets of $\mathscr{F}(\Omega)$ which satisfy certain algebraic closure properties.

DEFINITION 2.6. Let $\mathscr{A} \subseteq \mathscr{F}(\Omega)$. Then we say that

(a) $\mathscr{A}$ is an *F-algebra* iff (i) $\Omega \in \mathscr{A}$; (ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$; (iii) $A_1, A_2, \ldots, A_n \in \mathscr{A} \Rightarrow \bigcup_{i=1}^n A_i \in \mathscr{A}$.

(b) $\mathscr{A}$ is an *F — σ-algebra* iff (i) and (ii) as in (a) and (iii) $\langle A_i \rangle$ a sequence in $\mathscr{A} \Rightarrow \bigcup_{i=1}^\infty A_i \in \mathscr{A}$.

(c) $\mathscr{A}$ is an *F-monotone class* iff $\langle A_i \rangle$ a monotone sequence in $\mathscr{A} \Rightarrow \lim_{i\to\infty} A_i \in \mathscr{A}$.

THEOREM 2.7. *Let* $\mathscr{A} \subseteq \mathscr{F}(\Omega)$. *Then* $\exists$ *a smallest F-algebra alg.*$(\mathscr{A})$, *a smallest F–σ-algebra σ-alg.*$(\mathscr{A})$, *and a smallest F-monotone class M.C.*$(\mathscr{A})$ *containing* $\mathscr{A}$.

*Proof.* Identical to the classical case [cf. 1, p. 17].

DEFINITION 2.8. alg.($\mathscr{A}$), $\sigma$-alg.($\mathscr{A}$), and M.C.($\mathscr{A}$) given by the above theorem are called *the F-algebra, the F–σ-algebra,* and *the F-monotone class generated by* $\mathscr{A}$ respectively.

THEOREM 2.9. *An F-algebra is an F–σ-algebra iff it is an F-monotone class.*

Proof is straightforward and will net be presented here.

THEOREM 2.10. *Let* $\mathscr{A}$ *be an F-algebra. Then* $\sigma$*-alg.*($\mathscr{A}$) = *M.C.*($\mathscr{A}$).

*Proof.* Let $\forall A \in \mathscr{F}(\Omega)$,

$$\mathscr{M}_A = \{B\colon B \in \mathscr{F}(\Omega) \text{ and } A\backslash B,\ B\backslash A,\ A \cup B \in \text{M.C.}(\mathscr{A})\}.$$

Using Lemma 2.2 one can easily show that $\forall A \in \mathscr{F}(\Omega)$, $\mathscr{M}_A$ is an $F$-monotone class. Since $\mathscr{A}$ is an $F$-algebra and M.C.($\mathscr{A}$) is the smallest $F$-monotone class containing $\mathscr{A}$, we have

$$\forall A \in \mathscr{A}, \qquad \text{M.C.}(\mathscr{A}) \subseteq \mathscr{M}_A .$$

Using the symmetry in the definition of $\mathscr{M}_A$ one can easily show that

$$\forall A \in \text{M.C.}(\mathscr{A}), \qquad \text{M.C.}(\mathscr{A}) \subseteq \mathscr{M}_A .$$

Using this result and the fact that $\mathscr{A}$ is an $F$-algebra it is trivial to show that M.C.($\mathscr{A}$) is an $F$-algebra. Therefore by Theorem 2.9, M.C.($\mathscr{A}$) is an $F$–$\sigma$-algebra. Hence $\sigma$-alg.($\mathscr{A}$) $\subseteq$ M.C.($\mathscr{A}$). Since $\sigma$-alg.($\mathscr{A}$) is clearly an $F$-monotone class, M.C.($\mathscr{A}$) $\subseteq$ $\sigma$-alg.($\mathscr{A}$). Therefore $\sigma$-alg.($\mathscr{A}$) = M.C.($\mathscr{A}$).

COROLLARY 2.11. *If an F-monotone class contains an F-algebra, then it contains the F–σ-algebra generated by that F-algebra.*

## 3. INDEPENDENT FUZZY EVENTS

The concept of a fuzzy event ($F$-event) was first introduce by Zadeh [4]. In this section we will present a number of elementary results on independent $F$-events.

Let $(\Omega, \Sigma, P)$ be a probability space and $Bl([0, 1])$ be the collection of Borel subsets of [0, 1]. Let $\mathscr{B} = \{B\colon B \in \mathscr{F}(\Omega) \text{ and } \mu_B \text{ is } \Sigma, Bl([0, 1]) \text{ measurable}\}$. It is trivial to show that $\mathscr{B}$ is an $F$–$\sigma$-algebra.

DEFINITION 3.1. (a) We say that $B$ is a *fuzzy event* iff $B \in \mathscr{B}$; (b) $\forall B \in \mathscr{B}$, $\tilde{P}(B) = E[\mu_B]$, where $E$ is the expectation with respect to $P$.

Following Zadeh [4], in defining independence of $F$-events we employ products rather than intersection. Recall that $\forall A, B \in \mathscr{F}(\Omega)$, $\mu_{AB}(\cdot) = \mu_A(\cdot)\,\mu_B(\cdot)$.

DEFINITION 3.2. Let $n \geqslant 1$, and $A_1, A_2, \ldots, A_n \in \mathscr{B}$. Then we say that $A_1, A_2, \ldots, A_n$ are *mutually F-independent* iff $i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, n\} \Rightarrow \tilde{P}[\prod_{j=1}^{m} A_{i_j}] = \prod_{j=1}^{m} \tilde{P}(A_{i_j})$.

DEFINITION 3.3. Let $\Lambda$ be a set and $\forall \lambda \in \Lambda$, $A_\lambda \in \mathscr{B}$. Then we say that (a) $\{A_\lambda : \lambda \in \Lambda\}$ is a ***mutually F-independent collection*** iff all finite subcollections of it are mutually $F$-independent; (b) $\{A_\lambda : \lambda \in \Lambda\}$ is a *pairwise F-independent collection* iff $\lambda_1, \lambda_2 \in \Lambda \Rightarrow \tilde{P}(A_{\lambda_1} A_{\lambda_2}) = \tilde{P}(A_{\lambda_1})\, \tilde{P}(A_{\lambda_2})$.

DEFINITION 3.4. Let $\Lambda$ be a set and $\forall \lambda \in \Lambda$, $\mathscr{A}_\lambda \subseteq \mathscr{B}$. Then we say that $\{\mathscr{A}_\lambda : \lambda \in \Lambda\}$ is a ***mutually (pairwise) F-independent family*** iff $\{A_\lambda : A_\lambda \in \mathscr{A}_\lambda, \lambda \in \Lambda\}$ is a mutually (pairwise) $F$-independent collection.

Clearly mutual $F$-independence implies pairwise $F$-independence. When only two $F$-events are involved, the two concepts obviously coincide and in that case we will simply say that two events are $F$-independent.

The following theorem presents the fundamental properties of $F$-independent events.

THEOREM 3.5. (a) *$\forall A \in \mathscr{B}$, $A$ and $\Omega$ are F-independent;*

(b) *$\forall A, N \in \mathscr{B}$ such that $\tilde{P}(N) = 0$, $A$ and $N$ are F-independent;*

(c) *Let $A \in \mathscr{B}$. Then $\{\{A\}, \mathscr{B}\}$ is an F-independent family iff $\exists \alpha \in [0, 1]$ such that $\mu_A(\omega) = \alpha$ a.e. $(P)$;*

(d) *$A, B, N \in \mathscr{B}$, $\tilde{P}(N) = 0$, and $A$ and $B$ F-independent $\Rightarrow A \cup N$ and $B$ are F-independent and $A \backslash N$ and $B$ are F-independent.*

(e) *$\exists A_1, A_2, A_3 \in \mathscr{B}$ which are mutually F-independent but $A_1 \cup A_2$ and $A_3$ are not F-independent;*

(f) *$A_1, A_2, A_3 \in \mathscr{B}$, $A_1, A_3$ and $A_2, A_3$ F-independent, $A_1 \cap A_2 = \varnothing \Rightarrow A_1 \cup A_2$ and $A_3$ are F-independent.*

*Proof.* (a) Immediate consequence of the fact that $\mu_\Omega(\omega) = 1$, $\forall \omega \in \Omega$.

(b) Follows immediately from the fact that $\mu_N(\omega) = 0$ a.e. $(P)$.

(c) If $\exists \alpha \in [0, 1]$ such that $\mu_A(\omega) = \alpha$ a.e. $(P)$, then clearly $\forall B \in \mathscr{B}$, $A$ and $B$ are $F$-independent.

Conversely suppose $\{\{A\}, \mathscr{B}\}$ is an $F$-independent family. Then $A$ is $F$-independent of $A$. Hence

$$E[\mu_A^2] = \tilde{P}[AA] = \tilde{P}[A]\, \tilde{P}[A] = \{E[\mu_A]\}^2.$$

Therefore $\mathrm{Var}(\mu_A) = 0$. Hence $\exists \alpha \in [0, 1]$ such that $\mu_A(\omega) = \alpha$ a.e. $(P)$.

(d) Immediate consequence of the fact that $\mu_N(\omega) = 0$ a.e. $(P)$ and therefore $\mu_{N^c}(\omega) = 1$ a.e. $(P)$.

(e) Let $\Omega = [0, 1]$, $\Sigma =$ Borel family of $[0, 1]$, and $P =$ Restriction of Lebesgue measure to $[0, 1]$. Let

$$\begin{aligned} \mu_{A_1}(\omega) &= \tfrac{1}{2} \quad \omega \leqslant \tfrac{1}{2} \qquad \mu_{A_2}(\omega) = \tfrac{7}{8}, \qquad \forall \omega \in \Omega, \\ &= 1 \quad \omega > \tfrac{1}{2}, \end{aligned}$$

$$\begin{aligned} \mu_{A_3}(\omega) &= 0 \quad \omega \leqslant \tfrac{1}{4} \\ &= \tfrac{3}{4} \quad \tfrac{1}{4} < \omega \leqslant \tfrac{3}{4}. \\ &= 0 \quad \omega > \tfrac{3}{4} \end{aligned}$$

It is a trivial exercise to verify that $A_1$, $A_2$, $A_3$ are mutually $F$-independent, but $A_1 \cup A_2$, $A_3$ are not $F$-independent.

(f) Since $A_1 \cap A_2 = \varnothing$, $\forall \omega \in \Omega$, $\mu_{A_1}(\omega) \wedge \mu_{A_2}(\omega) = 0$. Therefore $\forall \omega \in \Omega$, $\exists i = 1, 2$ such that $\mu_{A_i}(\omega) = 0$. Let $S_i = \{\omega\colon \omega \in \Omega \text{ and } \mu_{A_i}(\omega) > 0\}$, $i = 1, 2$. Then $\forall \omega \in \Omega$

$$\begin{aligned} \mu_{A_1 \cup A_2}(\omega) &= \mu_{A_1}(\omega) \vee \mu_{A_2}(\omega) \\ &= \mu_{A_1}(\omega)\, \chi_{S_1}(\omega) + \mu_{A_2}(\omega)\, \chi_{S_2}(\omega). \end{aligned}$$

Hence

$$\begin{aligned} \tilde{P}[(A_1 \cup A_2)\, A_3] &= E[\mu_{A_1}\chi_{S_1}\mu_{A_3} + \mu_{A_2}\chi_{S_2}\mu_{A_3}] \\ &= E[\mu_{A_1}\chi_{S_1}\mu_{A_3}] + E[\mu_{A_2}\chi_{S_2}\mu_{A_3}] \\ &= E[\mu_{A_1}\mu_{A_3}] + E[\mu_{A_2}\mu_{A_3}] \\ &= E[\mu_{A_3}]\, \{E[\mu_{A_1}] + E[\mu_{A_2}]\} \\ &= \tilde{P}(A_3)\, \tilde{P}(A_1 \cup A_2). \end{aligned}$$

THEOREM 3.6. *If we replace an arbitrary finite or infinite collection of F-events $A_n$ in a sequence of mutually (pairwise) F-independent events $\langle A_n \rangle$ by their complements, the new sequence of F-events will also be mutually (pairwise) F-independent.*

Proof involves simple computation and will not be presented here.

The following lemma will be needed in the sequel.

LEMMA 3.7. *Let $\langle A_n \rangle$ be a monotone sequence of F-events. Then* $\tilde{P}(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \tilde{P}(A_n)$.

*Proof.* Immediate consequence of Lemma 2.1 and the Bounded Convergence Theorem.

THEOREM 3.8. *Let $\Lambda$ be a set and $(\mathscr{A}_\lambda: \lambda \in \Lambda)$ be a mutually F-independent family of F-algebras. Then $\{\sigma\text{-alg.}(\mathscr{A}_\lambda): \lambda \in \Lambda\}$ is a mutually F-independent family.*

*Proof.* Let $\lambda_0 \in \Lambda$ and

$$\mathscr{M}_{\lambda_0} = \{A: A \in \mathscr{B} \text{ and } \{\{A\}, \mathscr{A}_\lambda, \lambda \neq \lambda_0\} \text{ is a mutually } F\text{-independent family}\}.$$

*Assertion.* $\mathscr{M}_{\lambda_0}$ is an $F$-monotone family.

*Proof.* Let $\langle A_n \rangle$ be a monotone sequence in $\mathscr{M}_{\lambda_0}$, $A = \lim_{n\to\infty} A_n$, and $A_{\lambda_i} \in \mathscr{A}_{\lambda_i}$, $i = 1, 2, \dots, m$, $\lambda_i \neq \lambda_0$. Using Lemmas 2.1, 3.7 and the Bounded Convergence Theorem we get

$$\begin{aligned}
\tilde{P}\left[A \prod_{i=1}^{m} A_{\lambda_i}\right] &= E\left[(\mu_A) \prod_{i=1}^{m} \mu_{A_{\lambda_i}}\right] \\
&= E\left[\left(\lim_{n\to\infty} \mu_{A_n}\right)\left(\prod_{i=1}^{m} \mu_{A_{\lambda_i}}\right)\right] \\
&= E\left[\lim_{n\to\infty} \left\{(\mu_{A_n}) \prod_{i=1}^{m} \mu_{A_{\lambda_i}}\right\}\right] \\
&= \lim_{n\to\infty} E\left[(\mu_{A_n}) \prod_{i=1}^{m} \mu_{A_{\lambda_i}}\right] \\
&= \lim_{n\to\infty} \tilde{P}\left[A_n \prod_{i=1}^{m} A_{\lambda_i}\right] \\
&= \lim_{n\to\infty} \tilde{P}(A_n) \prod_{i=1}^{m} \tilde{P}(A_{\lambda_i}) \\
&= \tilde{P}\left(\lim_{n\to\infty} A_n\right) \prod_{i=1}^{m} \tilde{P}(A_{\lambda_i}).
\end{aligned}$$

Hence $\lim_{n\to\infty} A_n \in \mathscr{M}_{\lambda_0}$. Therefore $\mathscr{M}_{\lambda_0}$ is an $F$-monotone class.

Clearly $\mathscr{A}_{\lambda_0} \subseteq \mathscr{M}_{\lambda_0}$. Since $\mathscr{M}_{\lambda_0}$ is an $F$-monotone class, by Corollary 2.11, $\sigma\text{-alg.}(\mathscr{A}_{\lambda_0}) \subseteq \mathscr{M}_{\lambda_0}$.

## 4. FUZZY SETS WITH CONSTANT MEMBERSHIP FUNCTIONS

In the study of stochastically independent events, null sets and their complements play a special role. They are the only sets that are stochastically independent of themselves. Theorem 3.5(c) states that in the theory of $F$-events this role is played by $F$-sets whose membership functions are constant a.e. $(P)$. In this section extensions of some of the basic Zero–One laws of the theory of probability to $F$-events are presented.

Let $(\Omega, \Sigma, P)$ be a probability space, $\mathscr{B}$ be the collection of $F$-events and $\forall B \in \mathscr{B}$, $\tilde{P}(B) = E[\mu_B]$, where $E$ denotes the expectation with respect to $P$.

THEOREM 4.1 (Extension of Kolomogrov's Zero–One Law). *Let* (i) $\langle A_n \rangle$ *be a sequence of pairwise F-independent events;*

(ii) $\forall n \geqslant 1$, $\mathscr{A}_n = \text{alg.}(A_1, A_2, \ldots, A_n)$, $\mathscr{A}^n = \text{alg.}(A_{n+1}, A_{n+1}, \ldots)$, *and* $\mathscr{B}^n = \sigma\text{-alg.}(A_{n+1}, A_{n+2}, \ldots)$;

(iii) $\forall n \geqslant 1$, $\mathscr{A}_n$ *and* $\mathscr{A}^n$ *be F-independent;*

(iv) $C \in \bigcap_{n=1}^{\infty} \mathscr{B}^n$. *Then* $\mu_C(\omega) =$ *constant a.e.* $(P)$.

*Proof.* Let $\forall n \geqslant 1, \mathscr{M}_n = \{B: B \in \mathscr{B} \text{ and } A \in \mathscr{A}_n \Rightarrow A, B \text{ are } F\text{-independent}\}$. Using the same argument as in the proof of the assertion in Theorem 3.8, one can easily show that $\forall n \geqslant 1$, $\mathscr{M}_n$ is an $F$-monotone class. Using hypothesis (iii) and Corollary 2.11, we get

$$\forall n \geqslant 1, \qquad \mathscr{B}^n \subseteq \mathscr{M}_n . \tag{1}$$

Let

$$\mathscr{M}_C = \{B: B \in \mathscr{B} \text{ and } B, C \text{ are } F\text{-independent}\}.$$

It is easy to show that $\mathscr{M}_C$ is an $F$-monotone class. Using (1) and the fact that $\forall n \geqslant 1$, $C \in \mathscr{B}^n$, we get

$$\bigcup_{n=1}^{\infty} \mathscr{A}_n \subseteq \mathscr{M}_C . \tag{2}$$

Since $\langle \mathscr{A}_n \rangle$ is a monotone increasing family of $F$-algebras, $\bigcup_{n=1}^{\infty} \mathscr{A}_n$ is clearly an $F$-algebra. Hence by Corollary 2.11

$$\sigma\text{–alg.}\left(\bigcup_{n=1}^{\infty} \mathscr{A}_n\right) \subseteq \mathscr{M}_C . \tag{3}$$

Since $\{A_1, A_2, \ldots\} \subseteq \bigcup_{n=1}^{\infty} \mathscr{A}_n$, we have

$$\mathscr{M}_C \supseteq \sigma\text{–alg.}\left(\bigcup_{n=1}^{\infty} \mathscr{A}_n\right) \supseteq \sigma\text{–alg.}(A_1, A_2, \ldots) \supseteq \bigcap_{n=1}^{\infty} \mathscr{B}^n.$$

Hence $C \in \mathscr{M}_C$. Therefore

$$\begin{aligned} \operatorname{Var}(\mu_C) &= E(\mu_C^2) - [E(\mu_C)]^2 \\ &= \tilde{P}(C^2) - [\tilde{P}(C)]^2 = 0 \end{aligned}$$

Hence $\mu_C(\omega) =$ constant a.e. $(P)$.

COROLLARY 4.2 (Extension of Borel's Zero–One Law). *Let* (i) $\langle A_n \rangle$ *be a sequence of pairwise F-independent events and* $A = \limsup_{n\to\infty} A_n$;

(ii) $\forall n \geq 1$, alg.$(A_1, A_2, \ldots, A_n)$ *and* alg.$(A_{n+1}, A_{n+2}, \ldots)$ *be F-independent. Then* $\mu_A(\omega) =$ *constant a.e.* $(P)$.

*Proof.* Let $\forall n \geq 1$, $\mathscr{B}^n = \sigma$-alg.$(A_{n+1}, A_{n+2}, \ldots)$. It is a trivial exercise to show that $A \in \bigcap_{n=1}^{\infty} \mathscr{B}^n$. Hence by Theorem 4.1, $\mu_A(\omega) =$ constant a.e. $(P)$.

*Remark.* When dealing with stochastically independent events $\langle A_n \rangle$, the independence of alg.$(A_1, A_2, \ldots, A_n)$ and alg.$(A_{n+1}, A_{n+2}, \ldots)$ follows from the independence of the events $A_n$. However in view of Theorem 3.5(e) this is not true for $F$-events. The following example shows that assumption (ii) in Corollary 4.2 and consequently assumption (iii) in Theorem 4.1 can not be removed.

EXAMPLE. Let $\Omega = [0, 1]$, $\mathscr{B} =$ Borel family of $[0, 1]$, and $P =$ Restriction of Lebesgue measure to $\mathscr{B}$. Let $\forall n \geq 1$,

$$\begin{aligned} \mu_{A_n}(\omega) &= \tfrac{1}{2} + (\sin 4n\pi\omega)/2 \qquad 0 \leq \omega \leq \tfrac{1}{2} \\ &= \tfrac{1}{2} + (\sin 4n\pi\omega)/n \qquad \tfrac{1}{2} < \omega \leq 1. \end{aligned}$$

It is trivial to show that $\langle A_n \rangle$ is a sequence of pairwise $F$-independent events. Let $A = \limsup_{n\to\infty} A_n$. It is easy to show that $\mu_A(\omega) = \chi(\omega)_{[0,1/2)} + \frac{1}{2}\chi(\omega)_{(1/2,1]}$ a.e. $(P)$.

LEMMA 4.3. *Let* $\langle A_n \rangle$ *be a sequence of F-events. Then*

(a) $$\tilde{P}(\limsup_{n\to\infty} A_n) = \lim_{m\to\infty} \tilde{P}\left(\bigcup_{n=m}^{m} A_n\right)$$

(b) $$\tilde{P}(\liminf_{n\to\infty} A_n) = \lim_{m\to\infty} \tilde{P}\left(\bigcap_{n=m}^{\infty} A_n\right).$$

*Proof.* Immediate consequence of Lemma 2.1 and the Bounded Convergence Theorem.

THEOREM 4.4. *Let* $\langle A_n \rangle$ *be a sequence of F-events and* $A = \limsup A_n$. *Then*

$$\sum_{n=1}^{\infty} \tilde{P}(A_n) < \infty \Rightarrow \mu_A(\omega) = 0 \qquad \text{a.e.} \quad (P).$$

*Proof.* Since $\tilde{P}$ is subadditive [cf. 4] we have $m \geq 1$

$$\tilde{P}\left(\bigcup_{n=m}^{\infty} A_n\right) \leq \sum_{n=m}^{\infty} \tilde{P}(A_n).$$

Since

$$\sum_{n=m}^{\infty} \tilde{P}(A_n) < \infty, \qquad \lim_{m\to\infty} \sum_{n=m}^{\infty} \tilde{P}(A_n) = 0.$$

Hence

$$\lim_{m\to\infty} \tilde{P}\left(\bigcup_{n=m}^{\infty} A_n\right) = 0.$$

Using Lemma 4.3 we get

$$E[\mu_A] = \tilde{P}(A) = \lim_{m\to\infty} \tilde{P}\left(\bigcup_{n=m}^{\infty} A_n\right) = 0.$$

Therefore $\mu_A(\omega) = 0$ a.e. $(P)$.

*Remark.* One may ask if an analogue of Borel–Canelli lemma can be established for $F$-events. More fully if $\langle A_n \rangle$ is a sequence of mutually $F$-independent events $A = \limsup_{n\to\infty} A_n$ and $\sum_{n=1}^{\infty} \tilde{P}(A_n) = \infty$, does it follow that $\mu_A(\omega) =$ constant $\neq 0$ a.e. $(P)$. The answer is clearly no. In fact if $\mu_{A_n}(\omega) = 1/n$, $\forall n \geqslant 1$, $\forall \omega \in \Omega$, then $\langle A_n \rangle$ are mutually $F$-independent, $\sum_{n=1}^{\infty} \tilde{P}(A_n) = \infty$, but $\mu_A(\omega) = 0$ a.e. $(P)$.

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