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Deterministic sampling of sparse trigonometric polynomials[☆]

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ABSTRACT

One can recover sparse multivariate trigonometric polynomials from a few randomly taken samples with high probability (as shown by Kunis and Rauhut). We give a deterministic sampling of multivariate trigonometric polynomials inspired by Weil's exponential sum. Our sampling can produce a deterministic matrix satisfying the statistical restricted isometry property, and also nearly optimal Grassmannian frames. We show that one can exactly reconstruct every M -sparse multivariate trigonometric polynomial with fixed degree and of length D from the determinant sampling X , using the orthogonal matching pursuit, and with $|X|$ a prime number greater than $(M \log D)^2$. This result is optimal within the $(\log D)^2$ factor. The simulations show that the deterministic sampling can offer reconstruction performance similar to the random sampling.

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1. Introduction

We investigate the problem of reconstructing sparse multivariate trigonometric polynomials from a few samples on $[0, 1]^d$. Let Π_q^d denote the space of all trigonometric polynomials of maximal order $q \in \mathbb{N}$ in dimension d . An element f of Π_q^d is of the form

$$f(x) = \sum_{k \in [-q, q]^d \cap \mathbb{Z}^d} c_k e^{2\pi i k \cdot x}, \quad x \in [0, 1]^d,$$

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where $c_k \in \mathbb{C}$. The dimension of Π_q^d will be denoted by $D := (2q + 1)^d$. We denote the support of the sequence of coefficients of c_k by T , i.e.,

$$T := \{k : c_k \neq 0\}.$$

Throughout this paper, we will mainly deal with “sparse” trigonometric polynomials, i.e., we assume that $|T|$ is much smaller than the dimension of D of Π_q^d . We set

$$\Pi_q^d(M) := \bigcup_{\substack{T \subset [-q, q]^d \cap \mathbb{Z}^d \\ |T| \leq M}} \Pi_T,$$

where Π_T denotes the space of all trigonometric polynomials whose coefficients are supported on T . Note that the set $\Pi_q^d(M)$ is the union of linear spaces and consists of all trigonometric polynomials whose Fourier coefficients are supported on a set $T \subset [-q, q]^d \cap \mathbb{Z}^d$ satisfying $|T| \leq M$. The aim of the paper is to sample a trigonometric polynomial $f \in \Pi_q^d(M)$ at N points and try to reconstruct f from these samples. We denote the sampling set

$$X := \{x_1, \dots, x_N\}.$$

We would like to reconstruct $f \in \Pi_q^d(M)$ from its sample values

$$y = f(x), \quad x \in X.$$

We use a decoder Δ that maps from \mathbb{C}^N to Π_q^d , and the role of Δ is to provide an approximation to f . The previous work concerns randomly taken samples. In [3,4], the authors choose X by taking samples randomly on a lattice and use the *basis pursuit* (BP) as the decoder Δ . In [16], the result is generalized for the case of x_1, \dots, x_N being the uniform distribution on $[0, 1]^d$. We state the result as follows:

Theorem 1.1 ([16]). *Assume $f \in \Pi_q^d(M)$ for some sparsity $M \in \mathbb{N}$. Let $x_1, \dots, x_N \in [0, 1]^d$ be independent random variables having a uniform distribution on $[0, 1]^d$. If for some $\epsilon > 0$ it holds that*

$$N \geq CM \log(D/\epsilon)$$

then with probability at least $1 - \epsilon$ the trigonometric polynomial f can be recovered from its sample values $f(x_j), j = 1, \dots, N$, by basis pursuit. The constant C is absolute.

In [14], one chooses Δ as the orthogonal matching pursuit (Algorithm 1) and X is chosen according to one of two probability models:

- (1) The sampling points x_1, \dots, x_N are independent random variables having a uniform distribution on $[0, 1]^d$.
- (2) The sampling points x_1, \dots, x_N have a uniform distribution on the finite set $2\pi\mathbb{Z}_m^d/m$ for some $m \in \mathbb{N} \setminus \{1\}$.

Theorem 1.2 ([14]). *Let $X = (x_1, \dots, x_N)$ be chosen according to one of the two probability models. Suppose that*

$$N \geq CM^2 \log(D/\epsilon).$$

Then, with probability at least $1 - \epsilon$, OMP recovers every M -sparse trigonometric polynomial. The constant C is absolute.

The aim of this paper is twofold. First, we present a deterministic sampling X and show that OMP can recover every $f \in \Pi_q^d(M)$ exactly using X with $|X|$ being a prime number greater than $C(M \log D)^2$, provided q is fixed. So, we improve Theorem 1.2. Second, we construct a StRIP matrix with large range size. In particular, we exploit the connections between the exponential sum and the RIP matrix.

We now discuss the organization of this paper and we summarize its main contributions. In Section 2, we introduce the deterministic sampling and show that the coherence of the corresponding sampling matrix is less than or equal to $(d - 1)/\sqrt{N}$ provided $N \geq 2q + 1$ and N is a prime number. As a conclusion, we find that the OMP algorithm can recover $f \in \Pi_q^d(M)$ exactly when

Algorithm 1 The orthogonal matching pursuit.

Input: sampling matrix \mathcal{F}_X , sampling vector $y = (f(x_j))_{j=1}^N$, maximum allowed sparsity M or residual tolerance ϵ
Output: the Fourier coefficient c and its support T .
Initialize: $r^0 = y, c^0 = 0, \Lambda^0 = \emptyset, \ell = 0$.
while $\|r^\ell\|_2 > \epsilon$ or $\ell < M$ **do**
 match: $h^\ell = \mathcal{F}_X^T r^\ell$
 identity: $\Lambda^{\ell+1} = \Lambda^\ell \cup \{\operatorname{argmax}_j |h^\ell(j)|\}$
 update: $c^{\ell+1} = \operatorname{argmin}_{z: \operatorname{supp}(z) \subset \Lambda^{\ell+1}} \|y - \mathcal{F}_X z\|_2$
 $r^{\ell+1} = y - \mathcal{F}_X c^{\ell+1}$
 $\ell = \ell + 1$
end while

$N \geq (d-1)^2(2M-1)^2+1$ with the additional assumption of N being a prime number not less than $2q+1$. It has been shown that if $N \leq CM^{3/2}$, then with high probability there exists an M -sparse coefficient vector c depending on the sampling set such that OMP fails (it is also believed that the bound can be improved to $O(M^2)$; see [15]). So, if one requires exact recovery of all sparse trigonometric polynomials from a single sampling set X , within the factor $\log^2 D$, our deterministic sampling size almost meets the optimal bound for $f \in \Pi_q^d(M)$ where q is fixed and d is variable. In Section 3, we show that the $N \times D$ deterministic sampling matrix obeys the statistical restricted isometry property (StRIP) of order $O(N(\log(D/N))^2/(\log D(\log N)^2))$. Though there are many deterministic sensing matrices satisfying the StRIP [5,8], the restriction on the size of our matrices is light compared with those on the others. Moreover, the StRIP matrix also implies nearly optimal Grassmannian frames. In Section 4, we show that the deterministic sampling can provide reconstruction performance similar to that of the random ones, by numerical experiments.

2. Deterministic sampling

2.1. Exponential sums and sampling matrices

We first introduce a result in number theory, which plays a central role in our determinant sampling.

Theorem 2.1 ([20]). Suppose that p is a prime number. Suppose $f(x) = m_1x + \dots + m_dx^d$ and there is a $j, 1 \leq j \leq d$, such that $p \nmid m_j$. Then

$$\left| \sum_{x=1}^p e^{\frac{2\pi i f(x)}{p}} \right| \leq (d-1)\sqrt{p}.$$

We furthermore introduce the definition of *mutual incoherence*. Let matrix $A = (a_1, \dots, a_D) \in \mathbb{C}^{N \times D}$, where $N \leq D$ and $\|a_i\|_2 = 1$. The mutual incoherence of A is defined by

$$\mathcal{M}(A) := \max_{i \neq j} \langle a_i, a_j \rangle.$$

The lower bound of $\mathcal{M}(A)$, which is also called *Welch's bound*, is given in [21]:

$$\mathcal{M}(A) \geq \sqrt{\frac{D-N}{(N-1)D}}. \tag{1}$$

If the equality holds, we call the A *optimal Grassmannian frames*. As shown in [7], the equality can hold only if $D \leq N^2$ (see also [19]). Suppose that $N \geq 2q+1$ is a prime number. We choose the sampling set $X = \{x_1, \dots, x_N\}$ with

$$x_j = (j, j^2, \dots, j^d)/N \bmod 1, \quad j = 1, \dots, N, \tag{2}$$

and denote by \mathcal{F}_X the $N \times D$ sampling matrix with entries

$$(\mathcal{F}_X)_{j,k} = \exp(2\pi i k \cdot x_j), \quad j = 1, \dots, N, k \in [-q, q]^d.$$

Also, $f(x_j) = (\mathcal{F}_X c)_j$, where c is the vector of Fourier coefficients of f . We let ϕ_k denote the k th column of \mathcal{F}_X . A simple observation is that $\|\phi_k\|_2 = \sqrt{N}$. Set

$$\mu := \mathcal{M}(\mathcal{F}_X/\sqrt{N}).$$

Then we have:

Lemma 2.2.

$$\mu \leq (d - 1)/\sqrt{N}.$$

Proof. Recall that ϕ_k denotes the k th column of \mathcal{F}_X . Note that

$$|\langle \phi_m, \phi_k \rangle| = \left| \sum_{j=1}^N e^{2\pi i p(j)/N} \right|,$$

where p is a polynomial with degree d in the form of $(m - k) \cdot (j, \dots, j^d)$ with $m, k \in [-q, q]^d$. Then Theorem 2.1 implies that

$$|\langle \phi_m, \phi_k \rangle| \leq (d - 1)\sqrt{N}, \quad \text{for } m \neq k.$$

Hence,

$$\mu = \max_{m \neq k} \langle \phi_m, \phi_k \rangle / N \leq (d - 1)/\sqrt{N}. \quad \square$$

Let us consider recovery by OMP using the deterministic sampling X . We first recall the following theorem:

Theorem 2.3 ([14]). Assume $(2M - 1)\mu < 1$. Then OMP (and also BP) recovers every $f \in \Pi_q^d(M)$.

Combining Lemma 2.2 and Theorem 2.3, we have:

Theorem 2.4. Let the sampling set $X = \{x_1, \dots, x_N\}$ with

$$x_j = (j, j^2, \dots, j^d)/N \text{ mod } 1, \quad j = 1, \dots, N.$$

Suppose $N \geq \max\{2q + 1, (d - 1)^2(2M - 1)^2 + 1\}$ and N is a prime number. Then OMP (and also BP) recovers every M -sparse trigonometric polynomial exactly from the determinant sampling X .

2.2. Special sparsity patterns

For the deterministic sampling X , in Theorem 2.4, an additional assumption is that $N \geq 2q + 1 = D^{1/d}$. Hence, only when q is fixed and d is a variable can we say that $\max\{2q + 1, (M(d - 1))^2\} = O(M \log D)^2$. However, if there is prior information about the support of f , the restriction might be reduced. For $\Gamma \subset [-q, q]^d \cap \mathbb{Z}^d$, recall that we use Π_Γ to denote the space of all trigonometric polynomials in dimension d whose coefficients are supported on Γ . We let β_Γ be the minimum constant such that, for any $k_1, k_2 \in \Gamma$ with $k_1 \neq k_2$, there exists a non-zero entry of the vector $k_1 - k_2$, say $k_{1,j} - k_{2,j}$, such that $0 < |k_{1,j} - k_{2,j}| \leq \beta_\Gamma$. Then, for $f \in \Pi_\Gamma$, we can replace the condition $N \geq 2q + 1$ in Theorem 2.4 by $N \geq \beta_\Gamma + 1$. For example, we suppose that Γ_0 is a ‘curve’ in $[-q, q]^d$, which is defined by

$$\Gamma_0 := \left\{ \left(m, \left\lfloor \frac{m}{(2q + 1)^{1/d}} \right\rfloor, \dots, \left\lfloor \frac{m}{(2q + 1)^{(d-1)/d}} \right\rfloor \right) : -q \leq m \leq q \right\}.$$

Then a simple argument shows that $\beta_{\Gamma_0} \leq (2q + 1)^{1/d}$. Hence, OMP can recover M -sparse trigonometric polynomials $f \in \Pi_{\Gamma_0}$ exactly from determinant sampling X with $N \geq \max\{2q + 1, (d - 1)^2(2M - 1)^2 + 1\}$ and N being a prime number. In particular, if $d \geq \log_2(2q + 1)$, the condition $N \geq (2q + 1)^{1/d}$ is reduced to $N \geq 2$.

2.3. Related work

We are not the only ones seeking a deterministic Fourier sampling. We would especially like to note works by Iwen [11–13] and Bourgain et al. [2]. In [11], a deterministic Fourier sampling method is considered and a sublinear-time algorithm which recovers one-dimensional sparse trigonometric polynomials f is presented. However, the sampling in [11] needs a combinatorial structure, which seems non-trivial to construct. Moreover, the coherence of the sampling matrix in [11] is not small. In recent work [12,13], Iwen presents a binary matrix, say \mathcal{B} , with small coherence. By computing the product of \mathcal{B} and the discrete Fourier matrix Ψ , one can obtain a deterministic sampling matrix $\mathcal{B} \cdot \Psi$. However, the construction of \mathcal{B} requires $O(M \log D)$ large primes and one also needs a fast algorithm for computing the product of \mathcal{B} and Ψ .

Our work is in a different direction. The main concern of our methods is to present a deterministic sampling with an *analytic form* such that the coherence of the sampling matrix is as small as possible and hence the popular decoder algorithms, such as OMP and BP, can work well for the sampling. In fact, to construct our deterministic sampling, we only require a prime number N . The coherence of our deterministic sampling matrix almost meets Welch’s bound. Hence, we also present a nearly optimal harmonic Grassmannian frame, which may be of independent interest [19]. We also point out another difference between our study and that of Iwen. The deterministic sampling given in [11–13] is designed for trigonometric polynomials in a single variable. In [13], the author shows that one can deal with high dimensional trigonometric polynomials by a dimensionality reduction technique, but it requires the integer solutions of a linear equation and hence it is an indirect method. In contrast, our sampling is much more convenient for dealing with high dimensional trigonometric polynomials. Last but not least, it seems that in practice the algorithms given in [11,13] require many more samples than OMP and BP when the dimension d is large.

We next compare our study and that of Bourgain et al. In [2], for the case with $d = 1$, Bourgain et al. show a connection between deterministic Fourier sampling and Turán’s problem, and also present many possible approaches to constructing the sampling. Comparing with the approaches given in [2], we see that our sampling has the advantage of simplicity. Another important distinction is that our sampling strategy can be interpretable as a sampling strategy for recovering higher dimensional sparse trigonometric polynomials.

Finally, we would like to point out connections with chirp sensing codes. In [1], an $N \times N^2$ measurement matrix Θ is designed with chirp sequences forming the columns, i.e.,

$$(\Theta)_{\ell,k} = \exp(2\pi i r \ell^2 / N) \exp(2\pi i m \ell / N) \quad \text{with } 1 \leq \ell \leq N \text{ and } k = Nr + m, 1 \leq k \leq N^2.$$

If we take $d = 2$, then \mathcal{F}_X is reduced to the chirp sensing matrix Θ . Hence, the measurement matrix Θ can be considered as a special case of our determinant sampling matrix \mathcal{F}_X . Noting that the coherence of the matrix $\mathcal{M}(\mathcal{F}_X / \sqrt{N})$ is not more than $(d-1) / \sqrt{N}$, by an argument similar to one employed by DeVore in [6], we can show that \mathcal{F}_X / \sqrt{N} satisfies RIP of order k with constant $\delta = (k-1)(d-1) / \sqrt{N}$ (the definition of RIP is given in the next section). For fixed $0 < \delta < 1$, we can obtain the upper bound $k = O(\sqrt{N}/d)$. Using the method of additive combinatorics, in [2], Bourgain et al. investigate the RIP property of Θ and show that one can construct $N \times D$ RIP matrices of order k with $N = o(k^2)$, which implies a better upper bound of k . It will be a challenging problem to extend the result of [2] to the matrix \mathcal{F}_X .

3. RIP and StRIP

Given an $N \times D$ matrix Φ and any set T of column indices, we denote by Φ_T the $n \times |T|$ matrix composed of these columns. Similarly, for a vector $y \in \mathbb{C}^D$, we use y_T to denote the vector formed by the entries of y with indices from T . Following Candès and Tao, we say that Φ satisfies the restricted isometry property (RIP) of order k and constant $\delta \in (0, 1)$ if

$$(1 - \delta) \|y_T\|^2 \leq \|\Phi_T y_T\|^2 \leq (1 + \delta) \|y_T\|^2 \tag{3}$$

holds for all sets T with $|T| \leq k$. In fact, (3) is equivalent to requiring that the Grammian matrix $\Phi_T^T \Phi_T$ has all of its eigenvalues in $[1 - \delta, 1 + \delta]$ for all T with $|T| \leq k$. A fundamental question in

compressed sensing is that of the construction of a suitable RIP matrix Φ . In general, RIP matrices can be constructed using random variables such as Gaussian or Bernoulli ones as their entries. However, the construction of the deterministic RIP matrices is a challenging task (see also [2,6]). Given that there are no satisfactory deterministic constructions of RIP matrices, several authors suggest an alternative statistical version of RIP [5,10]:

Definition 3.1. We say that a deterministic matrix Φ satisfies the statistical restricted isometry property (StRIP) of order k with constant δ and probability $1 - \epsilon$ if

$$\Pr(|\|\Phi y\|^2 - \|y\|^2| \leq \delta \|y\|^2) \geq 1 - \epsilon,$$

with respect to a uniform distribution of the vectors y among all k -sparse vectors in \mathbb{R}^D of the same norm.

In [8], the authors derive the StRIP performance bound in terms of the mutual coherence of the sampling matrix and the sparsity level of the input signal, as summarized by the following theorem:

Theorem 3.2 ([8]). Let $\Phi = (\phi_1, \dots, \phi_D)$ be an $N \times D$ deterministic matrix, where each column has unit norm. Assume further that all row sums of Φ are equal to zero, and $\mathcal{M}(\Phi) \leq \alpha_1/\sqrt{N}$, where α_1 is some constant. Then Φ is a StRIP of order $k \leq c_0(\delta)N/\log D$ with constant δ and probability $1 - 1/D$, where $c_0(\delta) = \delta^2/(8\alpha_1^2)$.

As stated in [5,8,10], there are a large class of deterministic matrices which satisfy the StRIP. However, the restriction on the size (N and D) is heavy. For example, FZC codes [17] require that $D = N^2$ and N is a prime number, Gold/Gold-like codes [18] require that $D = (N + 1)^2$ and N is in the form of $2^p - 1$, $p \in \mathbb{Z}$, and so on. We next construct a deterministic StRIP matrix with the large range size on the basis of the method introduced in Section 2.

Suppose $N \in \mathbb{N}$ is a prime number. We suppose that $D \in \mathbb{N}$ can be written in the form of $D = p_1 \cdots p_d$, where $N = p_1 \geq p_2 \geq \dots \geq p_d \geq 2$ and p_1, \dots, p_d are prime numbers. For $t = 1, \dots, d$, we set

$$I_t := \begin{cases} [-(p_t - 1)/2, (p_t - 1)/2], & p_t \neq 2, \\ [0, 1], & p_t = 2. \end{cases}$$

Recall that

$$x_j = (j, j^2, \dots, j^d)/N \bmod 1, \quad j = 1, \dots, N,$$

and denote by $\tilde{\mathcal{F}}_X$ the $N \times D$ matrix with entries

$$(\tilde{\mathcal{F}}_X)_{j,k} = \exp(2\pi i k \cdot x_j), \quad j = 1, \dots, N, k \in I_1 \times I_2 \times \dots \times I_d.$$

Theorem 3.3. Suppose that $0 < \delta < 1$ is given. The matrix $\tilde{\Phi} := \tilde{\mathcal{F}}_X/\sqrt{N}$ is a StRIP of order $\delta^2 \frac{N}{8(\log D)} \left(\frac{\log(D/N)}{\log N} \right)^2$ with constant δ and probability $1 - 1/D$.

Proof. We denote the mutual coherence of $\tilde{\Phi}$ by $\tilde{\mu}$. Using the same method as the proof of Lemma 2.2, we obtain that $\tilde{\mu} \leq (d - 1)/\sqrt{N}$. Note that the all row sum of $\tilde{\mathcal{F}}_X$ is 0, and also that $(d - 1)^2 \geq \left(\frac{\log N}{\log(D/N)} \right)^2$. The conclusion follows from Theorem 3.2 directly. \square

Remark 3.4. When $D \gg N$, Welch's bound (1) is approximately equal to $1/\sqrt{N}$. Note that

$$\mathcal{M}(\tilde{\Phi}) \leq (d - 1)/\sqrt{N} \leq \log_2 D/\sqrt{N}.$$

The mutual incoherence of $\tilde{\Phi}$ almost meets Welch's bound, which implies that $\tilde{\Phi}$ is a nearly optimal harmonic Grassmannian frame (see [19]).

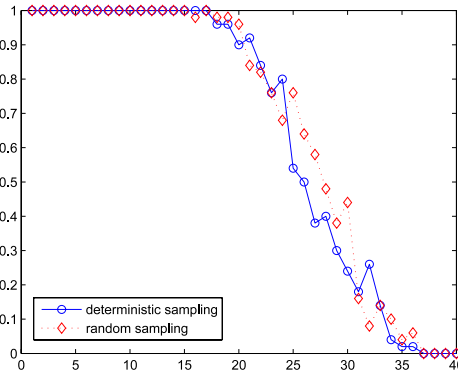


Fig. 1. Simulation results obtained using OMP.

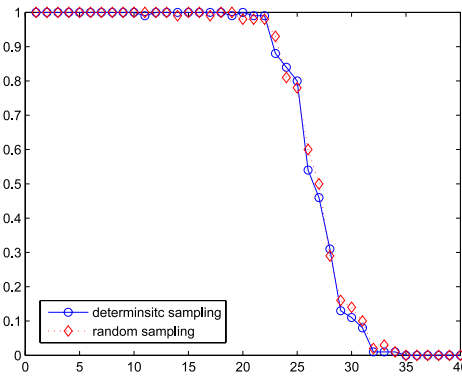


Fig. 2. Simulation results obtained using BP.

4. Numerical experiments

The purpose of the experiment is the comparison of the random sampling and the determinant sampling. Given the degree of the trigonometric polynomials $q \in \mathbb{N}$ and the number of variables $d \in \mathbb{N}$, the support set T is drawn from a uniform distribution over the set of all subsets of $[-q, q]^d$ of size M . The non-zero Fourier coefficients, and the real as well as the imaginary part of $c_j, j \in T$, have a Gaussian distribution with mean 0 and standard deviation 1. For the determinant sampling, we use the method introduced in Section 2 to produce the sampling points $x_j, j = 1, \dots, N$. Like [14], we choose the random sampling points on the basis of the continuous probability model.

Example 4.1. We take $q = 2, d = 5, D = 3125$ and draw a set T of size $M \in \{1, 2, \dots, 40\}$. The $N = 83$ deterministic sampling points are obtained by the method introduced in Section 2, and the random sampling points are produced by the continuous probability model. We reconstruct the Fourier coefficients using OMP and BP. In [14], the complexity of OMP is analyzed in detail. For BP, we use the optimization tools of CVX [9]. We repeat the experiment 100 times for each number M and calculate the success rate. Figs. 1 and 2 depict the numerical results obtained with the reconstructing algorithms OMP and BP, respectively. From these, one can observe that the performance of deterministic sampling is very similar to that of the random sampling. The numerical experiments indicate promise for the application of the deterministic sampling.

Example 4.2. As said before, RIP of order k with constant δ is equivalent to requiring that the Gramian matrix $\Phi_T^\top \Phi_T$ has all of its eigenvalues in $[1 - \delta, 1 + \delta]$ for all T with $|T| \leq k$. So, the purpose

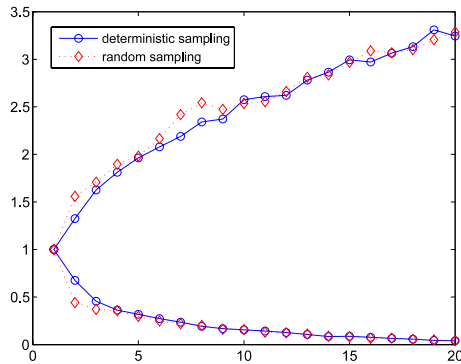


Fig. 3. Eigenvalue statistics of Gram matrices $\Phi_T^\top \Phi_T$ for deterministic sampling matrices and random sampling matrices.

of the second experiment is the comparison of the maximum and minimum eigenvalue statistics of Gram matrices $\Phi_T^\top \Phi_T$ of varying sparsity $M := |T|$ for a deterministic sampling matrix and a random sampling matrix. In order to achieve this, for every value M , sets T are drawn uniformly randomly over all sets and the statistics are accumulated from 50,000 samples. Fig. 3 shows the sample means of the maximum and minimum eigenvalues of $\Phi_T^\top \Phi_T$ for $M \in \{1, \dots, 20\}$.

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