An International Journal
computers & mathematics with applications
PERGAMON Computers and Mathematics with Applications 45 (2003) 1171–1179
www.elsevier.nl/locate/camwa

Discrete Semipositone Higher-Order Equations

R. P. AGARWAL
Department of Mathematical Sciences, Florida Institute of Technology
Melbourne, FL 32901, U.S.A.

S. R. GRACE
Department of Engineering Mathematics, Cairo University
Orman, Giza 12221, Egypt

D. O'REGAN
Department of Mathematics, National University of Ireland
Galway, Ireland

Abstract—This paper establishes existence for semipositone (n,p) and conjugate discrete boundary value problems. Our analysis relies on Krasnoselskii's fixed-point theorem in a cone. © 2003 Elsevier Science Ltd. All rights reserved.

Keywords—Semipositone, (n,p) and conjugate, Krasnoselskii's fixed-point theorem, Existence theory.

1. INTRODUCTION

This paper presents existence results for semipositone discrete higher-order problems. In particular, we discuss the (n,p), $n \geq 2$, discrete boundary value problem

$$\begin{aligned} \Delta^n y(k-n+1) + \mu f(k, y(k)) &= 0, & k &\in J_{n-1}, \\ \Delta^i y(0) &= 0, & 0 &\leq i \leq n-2, \\ \Delta^p y(T+n-p) &= 0, & 1 &\leq p \leq n-1 \quad (p \text{ is fixed}), \end{aligned} \tag{1.1}$$

where $\mu > 0$, $T \in \{1, 2, \dots\}$, $J_{n-1} = \{n-1, n, \dots, T+n-1\}$, and $y : I_n = \{0, \dots, T+n\} \to \mathbf{R}$. We look for nonnegative solutions to (1.1) in $C(I_n)$. Recall $C(I_n)$ denotes the class of maps w continuous on I_n (discrete topology) with norm $|w|_0 = \max_{k \in I_n} |w(k)|$. We note that throughout this paper our nonlinearity f may take *negative* values. Problems of this type are referred to as semipositone problems in the literature. The literature on positive solutions to higher-order difference equations (see [1–6] and the references therein) is almost totally devoted to the positone problem, i.e., to the problem when the nonlinearity takes only nonnegative values. To our knowledge only one paper [7] has partially discussed the semipositone problem in the discrete case.

The technique we supply in this paper will enable the reader to see that other boundary data, for example conjugate, focal, Sturm-Liouville, could also be discussed. To illustrate this point we

0898-1221/03/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0898-1221(03)00079-8

will in addition consider in this paper the n^{th}-order discrete conjugate boundary value problem

$$\begin{aligned} (-1)^{n-p}\Delta^n y(k-p) &= \mu f(k,y(k)), && k\in J_p,\\ \Delta^i y(0) &= 0, && 0\le i\le p-1,\\ \Delta^i y(T+n-i) &= 0, && 0\le i\le n-p-1, \end{aligned}$$

where $\mu > 0$, $T \in \{1,2,\dots\}$, $J_p = \{p, p+1, \dots, T+p\}$, $1 \le p \le n-1$, and $y : I_n \to \mathbf{R}$.

Existence in this paper will be established using Krasnoselskii's fixed-point theorem in a cone, which we state here for the convenience of the reader.

THEOREM 1.1. *Let $E = (E, \|\,.\,\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\bar\Omega_1 \subset \Omega_2$ and let $A : K \cap (\bar\Omega_2 \setminus \Omega_1) \to K$ be continuous and completely continuous. In addition, suppose either*

$$\|Au\| \le \|u\|, \quad \text{for } u \in K\cap\partial\Omega_1, \qquad \text{and} \qquad \|Au\| \ge \|u\|, \quad \text{for } u \in K\cap\partial\Omega_2,$$

or

$$\|Au\| \ge \|u\|, \quad \text{for } u \in K\cap\partial\Omega_1, \qquad \text{and} \qquad \|Au\| \le \|u\|, \quad \text{for } u \in K\cap\partial\Omega_2,$$

hold. Then A has a fixed point in $K \cap (\bar\Omega_2 \setminus \Omega_1)$.

2. SEMIPOSITONE PROBLEMS

In this section, we first discuss the discrete (n,p) boundary value problem

$$\begin{aligned} \Delta^n y(k-n+1) + \mu f(k,y(k)) &= 0, && k\in J_{n-1},\\ \Delta^i y(0) &= 0, && 0\le i\le n-2,\\ \Delta^p y(T+n-p) &= 0, && 1\le p\le n-1 \quad (p \text{ is fixed}), \end{aligned} \tag{2.1}$$

where $\mu > 0$, $T \in \{1,2,\dots\}$, $J_{n-1} = \{n-1, n, \dots, T+n-1\}$, and $y : I_n = \{0,\dots,T+n\} \to \mathbf{R}$. Of physical interest is the existence of solutions which are positive on J_{n-1}.

Before we prove our main result, we first recall two well-known results from the literature which will be used in our proof. The first lemma can be found in [2] and the second in [1, p. 773;2].

LEMMA 2.1. *Suppose $y : I_n \to \mathbf{R}$ satisfies*

$$\begin{aligned} \Delta^n y(k-n+1) &\le 0, && k\in J_{n-1},\\ \Delta^i y(0) &= 0, && 0\le i\le n-2,\\ \Delta^p y(T+n-p) &= 0, && 1\le p\le n-1. \end{aligned}$$

Then

$$y(k) \ge \frac{k^{(n-1)}}{(T+n)^{(n-1)}}\,|y|_0, \qquad \text{for } k \in I_n;$$

here $|y|_0 = \sup_{j\in I_n} |y(j)|$.

LEMMA 2.2. *The boundary value problem*

$$\begin{aligned} \Delta^n y(k-n+1) + 1 &= 0, && k\in J_{n-1},\\ \Delta^i y(0) &= 0, && 0\le i\le n-2,\\ \Delta^p y(T+n-p) &= 0, && 1\le p\le n-1, \end{aligned}$$

has a solution w with

$$w(k) \le \frac{(T+1)}{(n-p)\,(n-1)!}\,k^{(n-1)}, \qquad \text{for } k \in J_{n-1},$$

and

$$w(k) \le \frac{k^{(n-1)}}{(n-1)!}\left[\frac{(T+1)}{(n-p)} + \frac{n-1}{n}\right], \qquad \text{for } k \in I_n;$$

here

$$w(k) = \sum_{j=n-1}^{T+n-1} G(k,j) = \sum_{j=0}^{T} G_1(k,j) = \frac{k^{(n-1)}}{(n-1)!}\left[\frac{(T+1)}{(n-p)} - \frac{k-n+1}{n}\right],$$

for $k \in I_n$, where $G(k,j) = G_1(k, j-n+1)$ and G_1 is the Green's function for (see [2] for an explicit representation)

$$\begin{aligned} \Delta^n y(k) &= 0, & k &\in I_0 = \{0,1,\dots,T\}, \\ \Delta^i y(0) &= 0, & 0 &\le i \le n-2, \\ \Delta^p y(T+n-p) &= 0, & 1 &\le p \le n-1. \end{aligned}$$

We now use Lemmas 2.1 and 2.2 and Krasnoselskii's fixed-point theorem to establish our main result.

THEOREM 2.3. *Suppose the following conditions are satisfied:*

$$\begin{gathered} f : J_{n-1} \times [0,\infty) \to \mathbf{R} \text{ is continuous and there} \\ \text{exists a constant } M > 0 \text{ with } f(i,u) + M \ge 0 \\ \text{for } (i,u) \in J_{n-1} \times [0,\infty), \end{gathered} \tag{2.2}$$

$$\begin{gathered} f(i,u) + M \le \psi(u) \text{ on } J_{n-1} \times [0,\infty) \text{ with} \\ \psi : [0,\infty) \to [0,\infty) \text{ continuous and nondecreasing} \\ \text{and } \psi(u) > 0, \qquad \text{for } u > 0, \end{gathered} \tag{2.3}$$

$$\exists r \ge \frac{\mu M (T+n)^{(n-1)}}{(n-1)!}\left[\frac{T+1}{n-p} + \frac{n-1}{n}\right], \qquad \text{with } \frac{r}{\psi(r)} \ge \mu \sup_{k \in I_n} \sum_{j=n-1}^{T+n-1} G(k,j), \tag{2.4}$$

$$\begin{gathered} \text{there exists a continuous, nondecreasing function } g : (0,\infty) \to (0,\infty), \\ \text{with } f(i,u) + M \ge g(u), \qquad \text{for } (i,u) \in J_{n-1} \times (0,\infty), \end{gathered} \tag{2.5}$$

and

$$\exists R > r, \qquad \text{with } \frac{R}{g\left(\epsilon R\,(n-1)^{(n-1)}/(T+n)^{(n-1)}\right)} \le \mu \sum_{j=n-1}^{T+n-1} G(\sigma,j); \tag{2.6}$$

here $\epsilon > 0$ is any constant (choose and fix it) so that

$$1 - \frac{\mu M (T+n)^{(n-1)}}{(n-1)!}\,\frac{[T+1]}{R\,(n-p)} \ge \epsilon$$

(note, ϵ exists since $R > r \ge (\mu M (T+n)^{(n-1)}/(n-1)!)([T+1]/(n-p)))$, and $\sigma \in I_n$ is such that

$$\sum_{j=n-1}^{T+n-1} G(\sigma,j) = \max_{i \in I_n} \sum_{j=n-1}^{T+n-1} G(i,j).$$

Then (2.1) has a solution $y \in C(I_n)$ with $y(i) > 0$ for $i \in J_{n-1}$.

PROOF. To show (2.1) has a nonnegative solution, we will look at the boundary value problem

$$\begin{aligned} \Delta^n y(k-n+1) + \mu f^\star(k, y(k) - \phi(k)) &= 0, & k &\in J_{n-1}, \\ \Delta^i y(0) &= 0, & 0 &\le i \le n-2, \\ \Delta^p y(T+n-p) &= 0, & 1 &\le p \le n-1, \end{aligned} \tag{2.7}$$

where $\phi(i) = \mu M w(i)$ (w is as in Lemma 2.2) and

$$f^\star(i,v) = \begin{cases} f(i,v) + M, & v \geq 0, \\ f(i,0) + M, & v \leq 0. \end{cases}$$

We will show, using Theorem 1.1, that there exists a solution y_1 to (2.7) with $y_1(i) \geq \phi(i)$ for $i \in I_n$ (note, $\phi(i) > 0$ for $i \in J_{n-1}$). If this is true, then $u(i) = y_1(i) - \phi(i)$ is a nonnegative solution (positive on J_{n-1}) of (2.1), since, for $k \in J_{n-1}$, we have

$$\begin{aligned} \Delta^n u(k-n+1) &= \Delta^n y_1(k-n+1) + \mu M = -\mu f^\star(k, y(k) - \phi(k)) + \mu M \\ &= -\mu\left[f(k, y(k) - \phi(k)) + M\right] + \mu M = -\mu f(k, u(k)). \end{aligned}$$

As a result, we will concentrate our study on (2.7). Let $E = (C(I_n), |\,.\,|_0)$ and

$$K = \left\{ u \in C(I_n) : u(k) \geq \frac{k^{(n-1)}}{(T+n)^{(n-1)}} |u|_0, \text{ for } k \in I_n \right\}.$$

Clearly, K is a cone of E. Let

$$\Omega_1 = \{u \in C(I_n) : |u|_0 < r\} \qquad \text{and} \qquad \Omega_2 = \{u \in C(I_n) : |u|_0 < R\}.$$

Next let $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to C(I_n)$ be defined by

$$A\,y(k) = \mu \sum_{j=n-1}^{T+n-1} G(k,j)\, f^\star(j, y(j) - \phi(j)).$$

First, we show $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$. If $u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, then (2.2) and the known sign of G (see [1,2]) guarantees that

$$\begin{aligned} \Delta^n A\,u(k-n+1) &\leq 0, & k &\in J_{n-1}, \\ \Delta^i A\,u(0) &= 0, & 0 &\leq i \leq n-2, \\ \Delta^p A\,u(T+n-p) &= 0, & 1 &\leq p \leq n-1, \end{aligned}$$

and so Lemma 2.1 implies $A\,u(k) \geq (k^{(n-1)}/(T+n)^{(n-1)})\,|A\,u|_0$ for $k \in I_n$. Consequently, $A\,u \in K$ so $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$. It is well known [2] that $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ is continuous and compact.

We now show

$$|A\,y|_0 \leq |y|_0, \qquad \text{for } y \in K \cap \partial\Omega_1. \tag{2.8}$$

To see this let $y \in K \cap \partial\Omega_1$, so $|y|_0 = r$ and $y(k) \geq (k^{(n-1)}/(T+n)^{(n-1)})\,r$ for $k \in I_n$. Now, for $k \in I_n$, we have

$$\begin{aligned} A\,y(k) &= \mu \sum_{j=n-1}^{T+n-1} G(k,j)\, f^\star(j, y(j) - \phi(j)) \\ &\leq \mu \sum_{j=n-1}^{T+n-1} G(k,j)\, \psi(y(j)) \\ &\leq \mu\, \psi(|y|_0) \sum_{j=n-1}^{T+n-1} G(k,j) \\ &\leq \mu\, \psi(r) \sup_{k \in I_n} \sum_{j=n-1}^{T+n-1} G(k,j), \end{aligned}$$

since for $j \in J_{n-1}$ we have (note, $y(k) \geq 0$ for $k \in I_n$)

$$f^\star(j, y(j) - \phi(j)) = \begin{cases} f(j, y(j) - \phi(j)) + M \leq \psi(y(j) - \phi(j)) \leq \psi(y(j)), & \text{if } y(j) - \phi(j) \geq 0, \\ f(j, 0) + M \leq \psi(0) \leq \psi(y(j)), & \text{if } y(j) - \phi(j) < 0; \end{cases}$$

in fact, one can show $y(j) - \phi(j) \geq 0$ for $j \in J_{n-1}$ (see the argument below). This together with (2.4) yields

$$|A\,y|_0 \leq \mu\,\psi(r) \sup_{k \in I_n} \sum_{j=n-1}^{T+n-1} G(k,j) \leq r = |y|_0,$$

so (2.8) holds.

Next, we show

$$|A\,y|_0 \geq |y|_0, \qquad \text{for } y \in K \cap \partial\Omega_2. \tag{2.9}$$

To see this, let $y \in K \cap \partial\Omega_2$ so $|y|_0 = R$ and $y(k) \geq (k^{(n-1)}/(T+n)^{(n-1)})\,R$ for $k \in I_n$. Let ϵ be as in the statement of Theorem 2.3. For $j \in J_{n-1}$, we have from Lemma 2.2 that

$$\begin{aligned} y(j) - \phi(j) &= y(j) - \mu\,M\,w(j) \geq y(j) - \mu\,M\,\frac{(T+1)\,j^{(n-1)}}{(n-1)!\,(n-p)} \\ &\geq y(j)\left[1 - \frac{\mu\,M\,(T+1)}{(n-1)!\,(n-p)}\,\frac{(T+n)^{(n-1)}}{R}\right] \\ &\geq \epsilon\,y(j) \geq \epsilon\,\frac{j^{(n-1)}}{(T+n)^{(n-1)}}\,R \geq \epsilon\,\frac{(n-1)^{(n-1)}}{(T+n)^{(n-1)}}\,R. \end{aligned}$$

Now with σ as in the statement of Theorem 2.3, we have

$$\begin{aligned} A\,y(\sigma) &= \mu \sum_{j=n-1}^{T+n-1} G(\sigma,j)\,f^\star(j, y(j) - \phi(j)) \\ &= \mu \sum_{j=n-1}^{T+n-1} G(\sigma,j)\,[f(j, y(j) - \phi(j)) + M] \\ &\geq \mu\,g\left(\epsilon\,R\,\frac{(n-1)^{(n-1)}}{(T+n)^{(n-1)}}\right) \sum_{j=n-1}^{T+n-1} G(\sigma,j), \end{aligned}$$

since for $j \in J_{n-1}$ we have from (2.5) that

$$f(j, y(j) - \phi(j)) + M \geq g(y(j) - \phi(j)) \geq g\left(\epsilon\,R\,\frac{(n-1)^{(n-1)}}{(T+n)^{(n-1)}}\right).$$

This together with (2.6) yields

$$A\,y(\sigma) \geq \mu\,g\left(\epsilon\,R\,\frac{(n-1)^{(n-1)}}{(T+n)^{(n-1)}}\right) \sum_{j=n-1}^{T+n-1} G(\sigma,j) \geq R = |y|_0,$$

so (2.9) holds.

Now Theorem 1.1 implies A has a fixed point $y_1 \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, i.e., $r \leq |y_1|_0 \leq R$ and $y_1(k) \geq (k^{(n-1)}/(T+n)^{(n-1)})\,r$ for $k \in I_n$. To finish the proof, we need to show $y_1(k) \geq \phi(k)$ for $k \in I_n$. This is immediate since Lemma 2.2 with the fact that $r \geq (\mu\,M\,(T+n)^{(n-1)}/(n-1)!)\,[(T+1)/(n-p) + (n-1)/n]$ implies for $k \in I_n$ that

$$y_1(k) \geq \frac{k^{(n-1)}}{(T+n)^{(n-1)}}\,r \geq \frac{k^{(n-1)}\,\mu\,M}{(n-1)!}\,\left[\frac{T+1}{n-p} + \frac{n-1}{n}\right] \geq \mu\,M\,w(k) = \phi(k). \qquad \blacksquare$$

REMARK 2.1. In (2.4), it is possible to replace

$$r \geq \frac{\mu M (T+n)^{(n-1)}}{(n-1)!} \left[\frac{T+1}{n-p} + \frac{n-1}{n}\right]$$

with

$$r \geq \frac{\mu M (T+n)^{(n-1)} (T+1)}{(n-1)! \, (n-p)};$$

see the ideas in the last few lines in the proof of Theorem 2.6.

EXAMPLE. Consider (2.1) with

$$f(k,u) = u^m - 1, \qquad m > 1, \qquad \text{and} \qquad \mu \in \left(0, \frac{(n-1)!}{(T+n)^{(n-1)}} \frac{1}{[(T+1)/(n-p) + (n-1)/n]}\right].$$

Then (2.1) has a solution y with $y(i) > 0$ for $i \in J_{n-1}$.

To see this, we will apply Theorem 2.3 with (here $R > 1$ will be chosen later; in fact we will choose R so that $\epsilon = 1/2$ works, i.e., we choose R so that $1 - \mu (T+n)^{(n-1)}(T+1)/ (n-1)! \, R \, (n-p) \geq 1/2$)

$$M = 1, \qquad \psi(u) = g(u) = u^m, \qquad \text{and} \qquad \epsilon = \frac{1}{2}.$$

Clearly (2.2), (2.3), and (2.5) hold. In addition, we know [1, p. 773] that

$$\begin{aligned} \mu \sup_{k \in I_n} \sum_{j=n-1}^{T+n-1} G(k,j) &= \mu \sup_{k \in I_n} \frac{k^{(n-1)}}{(n-1)!} \left[\frac{(T+1)}{(n-p)} - \frac{k-n+1}{n}\right] \\ &\leq \mu \frac{(T+n)^{(n-1)}}{(n-1)!} \left[\frac{(T+1)}{(n-p)} + \frac{n-1}{n}\right], \end{aligned}$$

so (2.4) is true with $r = 1$ since

$$\frac{\mu M (T+n)^{(n-1)}}{(n-1)!} \left[\frac{(T+1)}{(n-p)} + \frac{n-1}{n}\right] \leq 1 = r$$

and

$$\mu \sup_{k \in I_n} \sum_{j=n-1}^{T+n-1} G(k,j) \leq \mu \frac{(T+n)^{(n-1)}}{(n-1)!} \left[\frac{(T+1)}{(n-p)} + \frac{n-1}{n}\right] \leq 1 = \frac{r}{\psi(r)}.$$

Finally, notice (2.6) is satisfied for R large since

$$\frac{R}{g\left(\epsilon R (n-1)^{(n-1)}/(T+n)^{(n-1)}\right)} = \frac{1}{\left(\epsilon (n-1)^{(n-1)}/(T+n)^{(n-1)}\right)^m R^{m-1}} \to 0, \qquad \text{as } R \to \infty.$$

Thus, all the conditions of Theorem 2.3 are satisfied so existence is guaranteed.

Next, we discuss the discrete conjugate boundary value problem

$$\begin{aligned} (-1)^{n-p} \Delta^n y(k-p) &= \mu f(k, y(k)), && k \in J_p, \\ \Delta^i y(0) &= 0, && 0 \leq i \leq p-1 \quad (\text{i.e., } y(0) = \cdots = y(p-1) = 0), \\ \Delta^i y(T+n-i) &= 0, && 0 \leq i \leq n-p-1 \\ & && (\text{i.e., } y(T+p+1) = \cdots = y(T+n) = 0), \end{aligned} \tag{2.10}$$

where $\mu > 0$, $T \in \{1, 2, \dots\}$, $J_p = \{p, p+1, \dots, T+p\}$, $1 \leq p \leq n-1$, and $y : I_n \to \mathbf{R}$.

First, we recall two known results from the literature. The first lemma can be found in [3] and the second in [1, p. 773].

LEMMA 2.4. *Suppose $y : I_n \to \mathbf{R}$ satisfies*

$$\begin{aligned} (-1)^{n-p}\,\Delta^n y(k-p) &\geq 0, & k &\in J_p, \\ \Delta^i y(0) &= 0, & 0 &\leq i \leq p-1, \\ \Delta^i y(T+n-i) &= 0, & 0 &\leq i \leq n-p-1. \end{aligned}$$

Then

$$y(k) \geq \theta\, |y|_0, \qquad \text{for } k \in J_p;$$

here $0 < \theta < 1$ is such that $\theta = \min\{b(p), b(p+1)\}$ with

$$b(x) = \frac{\min\{g(x,p),\, g(x,T+p)\}}{\min\{g(x,[\theta(x)]),\, g(x,[\theta(x)+1]),\, g(x,p),\, g(x,T+p)\}}$$

and

$$g(x,k) = k^{(x-1)}\,(T+n-k)^{(n-x)}, \qquad \theta(x) = \frac{(x-1)\,T + (x-2)\,n + x}{n-1}$$

and $[\,.\,]$ is the greatest integer function.

LEMMA 2.5. *The boundary value problem*

$$\begin{aligned} (-1)^{n-p}\,\Delta^n y(k-p) &= 1, & k &\in J_p, \\ \Delta^i y(0) &= 0, & 0 &\leq i \leq p-1, \\ \Delta^i y(T+n-i) &= 0, & 0 &\leq i \leq n-p-1, \end{aligned}$$

has a solution w with

$$w(k) \leq \frac{1}{n!}\,(T+p)^{(p)}\,(T+n-p)^{(n-p)}, \qquad \text{for } k \in J_p,$$

and

$$w(k) \leq \frac{1}{n!}\,(T+n)^{(p)}\,(T+n)^{(n-p)}, \qquad \text{for } k \in I_n;$$

here $w(0) = \cdots = w(p-1) = w(T+p+1) = \cdots w(T+n) = 0$ with

$$w(k) = \sum_{j=p}^{T+p} (-1)^{n-p}\,K(k,j) = \sum_{j=0}^{T} (-1)^{n-p}\,K_1(k,j) = \frac{1}{n!}\,k^{(p)}\,(T+n-k)^{(n-p)},$$

for $k \in I_n$, where $K(k,j) = K_1(k,j-p)$ and K_1 is the Green's function for (see [3] for an explicit representation)

$$\begin{aligned} \Delta^n y(k) &= 0, & k &\in I_0 = \{0,1,\dots,T\}, \\ \Delta^i y(0) &= 0, & 0 &\leq i \leq p-1, \\ \Delta^i y(T+n-i) &= 0, & 0 &\leq i \leq n-p-1. \end{aligned}$$

We are now in a position to prove our main result for (2.10).

THEOREM 2.6. *Suppose the following conditions are satisfied:*

$$\begin{gathered} f : J_p \times [0,\infty) \to \mathbf{R} \text{ is continuous and there} \\ \text{exists a constant } M > 0, \qquad \text{with } f(i,u) + M \geq 0, \quad \text{for } (i,u) \in J_p \times [0,\infty), \end{gathered} \tag{2.11}$$

$$\begin{gathered} f(i,u)+M \le \psi(u) \;\; on \;\; J_p \times [0,\infty) \;\; with \\ \psi:[0,\infty)\to[0,\infty) \;\; continuous \; and \; nondecreasing \\ and \;\; \psi(u)>0, \quad for \; u>0, \end{gathered} \tag{2.12}$$

$$\exists\, r \ge \frac{\mu M\,(T+p)^{(p)}\,(T+n-p)^{(n-p)}}{n!\,\theta}, \qquad with \; \frac{r}{\psi(r)} \ge \mu \sup_{k\in I_n} \sum_{j=p}^{T+p} (-1)^{n-p} K(k,j), \tag{2.13}$$

$$\begin{gathered} there \; exists \; a \; continuous, \; nondecreasing \; function \; g:(0,\infty)\to(0,\infty), \\ with \; f(i,u)+M \ge g(u), \quad for \; (i,u)\in J_p\times(0,\infty), \end{gathered} \tag{2.14}$$

and

$$\exists\, R>r, \qquad with \; \frac{R}{g(\epsilon R\theta)} \le \mu \sum_{j=p}^{T+p} (-1)^{n-p} K(\sigma,j); \tag{2.15}$$

here $\epsilon>0$ is any constant (choose and fix it) so that

$$1-\frac{\mu M\,(T+p)^{(p)}\,(T+n-p)^{(n-p)}}{n!\,R\theta} \ge \epsilon$$

and $\sigma \in I_n$ is such that

$$\sum_{j=p}^{T+p} (-1)^{n-p} K(\sigma,j) = \max_{i\in I_n} \sum_{j=p}^{T+p} (-1)^{n-p} K(i,j).$$

Then (2.10) has a solution $y\in C(I_n)$ with $y(i)>0$ for $i\in J_p$.

PROOF. To show (2.1) has a nonnegative solution, we will look at the boundary value problem

$$\begin{aligned} (-1)^{n-p}\,\Delta^n y(k-p) &= \mu f^\star(k, y(k)-\phi(k)), && k\in J_p, \\ \Delta^i y(0) &= 0, && 0\le i\le p-1, \\ \Delta^i y(T+n-i) &= 0, && 0\le i\le n-p-1, \end{aligned} \tag{2.16}$$

where $\phi(i)=\mu M\,w(i)$ (w is as in Lemma 2.5) and $f^\star$ is as in Theorem 2.3. It is enough to show that there exists a solution y_1 to (2.16) with $y_1(i)\ge\phi(i)$ for $i\in I_n$ (note, $\phi(i)>0$ for $i\in J_p$). Let $E=(C(I_n),\,|\,.\,|_0)$ and

$$K=\{u\in C(I_n): u(i)\ge 0, \text{ for } i\in I_n \text{ and } u(k)\ge\theta\,|u|_0, \text{ for } k\in J_p\}.$$

Also, let

$$\Omega_1=\{u\in C(I_n): |u|_0<r\} \qquad \text{and} \qquad \Omega_2=\{u\in C(I_n): |u|_0<R\}$$

and let $A: K\cap(\bar\Omega_2\setminus\Omega_1)\to C(I_n)$ be defined by

$$A\,y(k)=\mu\sum_{j=p}^{T+p}(-1)^{n-p}K(k,j)\,f^\star(j,y(j)-\phi(j)).$$

Essentially the same reasoning as in Theorem 2.3 guarantees that $A: K\cap(\bar\Omega_2\setminus\Omega_1)\to K$ is continuous and compact with

$$|A\,y|_0 \le |y|_0, \qquad \text{for } y\in K\cap\partial\Omega_1. \tag{2.17}$$

Next, we show

$$|A\,y|_0 \ge |y|_0, \qquad \text{for } y\in K\cap\partial\Omega_2. \tag{2.18}$$

To see this, let $y \in K \cap \partial\Omega_2$ so $|y|_0 = R$ and $y(k) \geq \theta R$ for $k \in J_p$ and $y(k) \geq 0$ for $k \in I_n$. Let ϵ be as in the statement of Theorem 2.6. For $j \in J_p$ we have from Lemma 2.5 that

$$\begin{aligned} y(j) - \phi(j) = y(j) - \mu M w(j) &\geq y(j) - \frac{\mu M (T+p)^{(p)} (T+n-p)^{(n-p)}}{n!} \\ &\geq y(j) \left[1 - \frac{\mu M (T+p)^{(p)} (T+n-p)^{(n-p)}}{n!\, \theta R}\right] \\ &\geq \epsilon y(j) \geq \epsilon \theta R. \end{aligned}$$

Now with σ as in the statement of Theorem 2.6, we have

$$\begin{aligned} A y(\sigma) &= \mu \sum_{j=p}^{T+p} (-1)^{n-p} K(\sigma, j) [f(j, y(j) - \phi(j)) + M] \\ &\geq \mu g(\epsilon \theta R) \sum_{j=p}^{T+p} (-1)^{n-p} K(\sigma, j) \\ &\geq R = |y|_0, \end{aligned}$$

from (2.15). Thus, (2.18) holds.

Now Theorem 1.1 implies A has a fixed point $y_1 \in K \cap (\bar{\Omega}_2 \backslash \Omega_1)$, i.e., $r \leq |y_1|_0 \leq R$ and $y_1(k) \geq 0$ for $k \in I_n$ and $y_1(k) \geq \theta r$ for $k \in J_p$. To finish the proof, we need to show $y_1(k) \geq \phi(k)$ for $k \in I_n$. First, if $k \in I_n \setminus J_p$, then since $y_1(j) \geq 0$ for $j \in I_n$ and $\phi(j) = 0$ for $j \in I_n \setminus J_p$, we have $y_1(k) \geq 0 = \phi(k)$ for $k \in I_n \setminus J_p$. It remains to consider the case $k \in J_p$. If $k \in J_p$, then $r \geq \mu M (T+p)^{(p)} (T+n-p)^{(n-p)}/n!\, \theta$ and Lemma 2.5 implies

$$y_1(k) \geq \theta r \geq \frac{\mu M (T+p)^{(p)} (T+n-p)^{(n-p)}}{n!} \geq \mu M w(k) = \phi(k). \qquad \blacksquare$$

REFERENCES

1. R.P. Agarwal, *Difference Equations and Inequalities*, Second Edition, Marcel Dekker, New York, (2000).
2. R.P. Agarwal and D. O'Regan, Singular discrete (n,p) boundary value problems, *Appl. Math. Lett.* **12** (8), 113–119, (1999).
3. R.P. Agarwal and D. O'Regan, Discrete conjugate boundary value problems, *Appl. Math. Lett.* **13** (2), 97–104, (2000).
4. P.W. Eloe, A generalization of concavity for finite differences, *Computers Math. Applic.* **36** (10–12), 109–113, (1998).
5. P. Hartman, Difference equations: Disconjugacy, principal solutions, Green's functions and monotonicity, *Trans. Amer. Math. Soc.* **246**, 1–30, (1978).
6. A. Lasota, A discrete boundary value problem, *Ann. Polon. Math.* **20**, 183–190, (1968).
7. R.P. Agarwal and D. O'Regan, Nonpositone discrete boundary value problems, *Nonlinear Analysis* **39**, 207–215, (2000).