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MODEL-COMPLETIONS AND MODULES

Paul EKLOF and Gabriel SABBAGH
Yale University

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1. Introduction and preliminaries

In this paper we study some first-order properties of the theory of modules over a fixed ring Λ. In particular we investigate first-order definability of injectivity and related notions and, as a consequence, prove that the theory of modules over Λ has a model-completion if and only if Λ is coherent.

Throughout this paper "ring" means associative ring with identity $1 \neq 0$ and "module" means unitary left module. For any ring Λ, let L_Λ be the first-order language whose only non-logical constants are the equality symbol, a constant 0, and the following function symbols: a binary function f, and for each $\lambda \in \Lambda$, a unary function g_λ. (For the definitions of logical terms see e.g. Bell-Slomson [5].) We write $x + y$ for $f(x, y)$, λx for $g_\lambda(x)$, $-x$ for $-1x$. A Λ-module becomes a structure for L_Λ in the obvious way.

We axiomatize the theory of Λ-modules in L_Λ; consider the following sentences:

(1.1) $\forall xyz[(x + y) + z = x + (y + z)]$

(1.2) $\forall x[x + 0 = x]$

(1.3) $\forall x[x + (-x) = 0]$

(1.4) $\forall xy[x+y=y+x]$

(1.5) $\forall x(1x=x]$.

For any $\lambda, \sigma, \tau \in \Lambda$, such that $\tau = \lambda + \sigma$,

$(1.6)_{\lambda,\sigma,\tau}$ $\forall x[\lambda x + \sigma x = \tau x]$.

For any $\lambda \in \Lambda$,

$(1.7)_{\lambda}$ $\forall xy[\lambda(x+y) = \lambda x + \lambda y]$.

For any $\lambda, \sigma, \rho \in \Lambda$ such that $\lambda\sigma = \rho$,

$(1.8)_{\lambda,\sigma,\rho}$ $\forall x[\lambda(\sigma x) = \rho x]$.

M is a model of $(1.1)-(1.8)_{\lambda,\sigma,\rho}$ if and only if M is a Λ-module. Thus, by the Completeness Theorem, if K_{Λ} is the deductive closure of $(1.1)-(1.8)_{\lambda,\sigma,\rho}$, K_{Λ} is the theory of Λ-modules, i.e. the set of sentences of L_{Λ} true in all Λ-modules.

We introduce in § 2 a natural generalization of model-completion, viz. the model-companion of a theory, which is convenient for our purposes. Also in § 2 for motivational reasons we give a proof of the existence of a model-completion of the theory of abelian groups (i.e. **Z**-modules).

The generalization of the result of § 2 to modules over other rings requires a study of injective modules, which is of some independent interest we believe. In particular we prove:

The property of being an injective Λ*-module is first order if and only if* Λ *is noetherian* (Theorem 3.19).

We define some generalizations of the notion of injective, in particular a notion of $\aleph_0$-injective in which "finitely generated ideal" replaces "ideal" in the definition of injective (Definition 3.5).

A ring Λ is called (left) *coherent* if every finitely generated (left) ideal of Λ is finitely presented. (The class of noetherian rings is a proper subclass of the class of coherent rings: see § 3.) The crucial characterization of coherent rings (for our purposes) is given by:

The property of being an $\aleph_0$-injective Λ-module is first-order if and only if Λ is coherent (Theorem 3.16).

Our principal result on model-completions is:

K_Λ has a model-completion if and only if Λ is coherent (Theorems 4.1 and 4.8).

The model-completion is given as the theory of a certain explicitly defined module M_0 (see § 4). In the case that Λ is either commutative noetherian or artinian we give an explicit axiomatization of the model-completion and a structure theorem for the models of the model-completion (§ 5, 6).

In § 7 we introduce for any first-order theory a definition of algebraically closed structures and relate it to the question of the existence of a model-companion. We interpret our results on modules in that setting. We also prove:

The theory of (non-abelian) groups does not have a model-completion (Theorem 7.17).

We would like to thank Eli Bers for his stimulating presence. More precisely, we are grateful to Jon Barwise and Abraham Robinson for many edifying discussions. We are especially indebted to Abraham Robinson for showing us an unpublished manuscript in which we first learned the definition of injective used in § 3. We would also like to thank Ed Fisher for patiently listening to our arguments and perceptively pointing out the holes in many of them. *

Notation. $\alpha, \beta, \gamma, \delta$, and κ denote cardinals; φ and ψ denote formulas of L_Λ, and other lower-case Greek letters denote elements of Λ. Lower-case Latin letters will be used for elements of a Λ-module. If M is a Λ-module $M^{(\kappa)}$ denotes the direct *sum* of κ copies of M.

* Recently the authors have shown *inter alia* that the property of being (left) projective (resp. flat) is first-order if and only if Λ is left perfect and right coherent (resp. right coherent). A detailed development will appear in: Definability problems for modules and rings.

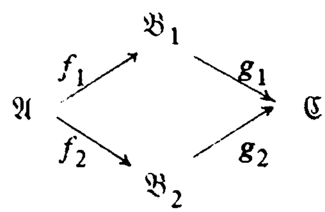
2. The model-completion of abelian groups

We begin by recalling some basic definitions. Let K be a theory (i.e. a deductively-closed consistent set of sentences) in a first-order language L. K is *model-complete* if for any model $\mathfrak{A}$ of K, $K \cup D(\mathfrak{A})$ is complete, where $D(\mathfrak{A})$ is the diagram of $\mathfrak{A}$ (see [32] p. 24). If K and K^* are theories in L, K^* is *model-consistent relative to* K if for any model of K, $K^* \cup D(\mathfrak{A})$ is consistent; K^* is *model-complete relative to* K if for any model $\mathfrak{A}$ of K, $K^* \cup D(\mathfrak{A})$ is complete. K^* is called the *model-completion* of K if $K \subseteq K^*$ and K^* is model-consistent and model-complete relative to K.

A theory K is called *inductive* if it is the deductive closure of a set of $\forall\exists$-sentences. Equivalently K is inductive if the class of models of K is closed under unions of chains ([32] Theorem 3.4.7).

A generalization of the notion of model-completion has been suggested by Eli Bers. We say that K^* is the *model-companion* of K if $K \subseteq K^*$ and K^* is model-consistent relative to K and model-complete. (The definition of model-companion given in [4] is equivalent to ours when K is inductive.) The model-companion of K, if it exists, is unique ([4] Theorem 5.3); the model-completion of K, if it exists, is the model-companion. (As an example, we note that the theory of formally real fields has a model-companion but not a model-completion.)

We say K has the *amalgamation property* if whenever $\mathfrak{A}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ are models of K and f_i: $\mathfrak{A} \to \mathfrak{B}_i$ are embeddings ($i = 1, 2$) then there is a model $\mathfrak{C}$ of K and embeddings $g_i : \mathfrak{B}_i \to \mathfrak{C}$ such that the following diagram is commutative.



The following lemma is due to Eli Bers.

Lemma 2.1. *Let K be a theory which has a model-companion K^*. Then K has the amalgamation property $\Longleftrightarrow$ K has a model-completion.*

Proof. ($\Rightarrow$) We need to show that K^* is model-complete *relative to K*. So let $\mathfrak{A}$ be a model of K and $\mathfrak{B}_1, \mathfrak{B}_2$ models of K^* such that $\mathfrak{A} \subseteq \mathfrak{B}_1$, $\mathfrak{A} \subseteq \mathfrak{B}_2$. Since $K \subseteq K^*$ and since K has the amalgamation property, there is a model $\mathfrak{C}$ of K such that the following diagram is commutative:

$$\begin{array}{ccccc} & & \mathfrak{B}_1 & & \\ & \subseteq & & \subseteq & \\ \mathfrak{A} & & & & \mathfrak{C} \\ & \subseteq & & \subseteq & \\ & & \mathfrak{B}_2 & & \end{array}$$

Since K^* is model-consistent relative to K, we may assume in fact that $\mathfrak{C}$ is a model of K^*. Then, because K^* is model-complete, $\mathfrak{B}_i$ is an elementary substructure of $\mathfrak{C}$ ($i = 1, 2$) and it follows immediately that $(\mathfrak{B}_1, a)_{a \in \mathfrak{A}} \equiv (\mathfrak{B}_2, a)_{a \in \mathfrak{A}}$.

($\Leftarrow$) The proof of this implication is implicit in Robinson [32]. Let $f_i : \mathfrak{A} \to \mathfrak{B}_i$, $i = 1, 2$, be embeddings of models of K. We can in fact assume that $\mathfrak{A} \subseteq \mathfrak{B}_i$ and f_i is the inclusion map. Since K^* is model-consistent relative to K, we can embed $\mathfrak{B}_i$ in a model $\mathfrak{B}_i^*$ of K^*. Let D_0, D_1^*, D_2^* be the respective diagrams of $\mathfrak{A}, \mathfrak{B}_1^*, \mathfrak{B}_2^*$ formulated in terms of the individuals of the corresponding structures. It suffices to show that $K^* \cup D_1^* \cup D_2^*$ is consistent. The proof is then exactly as in [32], Theorem 5.5.13.

Since abelian groups have the amalgamation property (see the proof of Lemma 3.2) we can confine ourselves to looking for the model-*companion* of the theory of abelian groups. Let K be the theory of abelian groups either in the usual language L_{ab} in which we have only two function symbols – for $x + y$ and $-x$ – or in the language $L_{\mathbf{Z}}$ of $\mathbf{Z}$-modules, in which there is a function for every $n \in \mathbf{Z}$: it will be clear that our results apply to both cases.

Consider the following set of sentences (either in L_{ab} or $L_{\mathbf{Z}}$):
For each $0 \neq n \in \mathbf{Z}$,

$$(2.2)_n \qquad \forall x \, \exists y [ny = x] \, .$$

For each prime p and each $m > 0$,

$$(2.3)_{p,m} \quad \exists x_1 \dots x_m \, [\bigwedge_{i \neq j} (x_i \neq x_j) \wedge \bigwedge_{i=1}^{m} (0 \neq x_i \wedge px_i = 0)] \, .$$

The class of models of $(2.2)_n - (2.3)_{p,m}$ is the class $\mathcal{D}$ of all divisible abelian groups A such that A contains, for each p, an infinite number of elements of order p. Thus $A \in \mathcal{D} \Longleftrightarrow A \cong (\bigoplus_p Z(p^\infty)^{(\kappa_p)}) \oplus \mathbf{Q}^{(\kappa)}$ where $\kappa_p \geq \aleph_0$, $\kappa \geq 0$ and $Z(p^\infty)$ is the group of all complex p^n-th roots of unity. (For the structure of divisible abelian groups and other algebraic facts about abelian groups see Kaplansky [23].) Let K^* be the theory of $\mathcal{D}$.

Theorem 2.4. *K^* is the model-companion (and hence the model-completion) of K, the theory of abelian groups.*

Proof. (Since a more general theorem will be proved in § 5, we only sketch a proof here.) Any abelian group can be embedded in a divisible abelian group ([23] Exercise 5, p. 12), and hence can be embedded in a model of K^*. Thus K^* is model-consistent relative to K. A Löwenheim-Skolem argument ([5] p. 80) shows that to prove K^* model-complete it suffices to consider countable models $A \subseteq B$ of K^* and prove that B is an elementary extension of A. Let D be a non-principal ultrafilter on $I = \{n \in \mathbf{Z} : n > 0\}$ and let $A^* = A^I/D$, $B^* = B^I/D$. Then $\mathrm{Card}(A^*) = c = \mathrm{Card}(B^*)$ where $c = 2^{\aleph_0}$ ([5] p. 129). We claim that

$$A^* \cong \bigoplus_p Z(p^\infty)^{(c)} \oplus \mathbf{Q}^{(c)} \cong B^* \, .$$

It suffices to prove that A^* contains c elements of order p for each prime p and a torsion-free subgroup of cardinality c. But if $Z(p)$ is the cyclic group of order p, $A^* \supseteq (Z(p)^{(\aleph_0)})^I/D$ which is a set of elements of order p; and if p_n is the nth prime, $A^* \supseteq \prod_n Z(p_n)^{(\aleph_0)}/D$, which is a torsion-free group. We now have

$$\begin{array}{ccc} A^* & & B^* \\ \uparrow & & \uparrow \\ A & \subseteq & B \end{array}$$

where the vertical maps are elementary embeddings. Since A is divisible, $A^* = A \oplus A_1$, $B^* = A \oplus B_1$ and since A is countable

$$A_1 \cong (\bigoplus_p Z(p^\infty)^{(c)}) \oplus \mathbf{Q}^{(c)} \cong B_1 \;.$$

It follows immediately that A is an elementary substructure of B. This completes the proof. $\cdot$

The divisible abelian groups are precisely the injective **Z**-modules. In order to generalize the above theorem it will be necessary to study the injective Λ-modules; we undertake that task in the next section.

3. α-injective and injective modules

In this section Λ denotes a fixed ring with 1, M a fixed (unitary) left Λ-module and α a fixed cardinal ≥ 2. By ideal, we mean a left ideal of Λ. By module, we mean a left Λ-module. By homomorphism, we mean a Λ-module homomorphism. We will follow in general the terminology of N.Bourbaki.

Definition 3.1. An *α-system* is a set of fewer than α equations in a single variable x, all of the form $\lambda_i x = a_i$ where $\lambda_i \in \Lambda$ and $a_i \in M$.

If $\mathcal{S} = \{\lambda_i x = a_i : i \in \mathcal{I}\}$ is such a system, we denote by $\mathcal{C}(\mathcal{S})$ or $\mathcal{C}$ the set $\{\lambda_i\}$ of coefficients of $\mathcal{S}$ and by $I(\mathcal{S})$ or I the ideal generated by $\mathcal{C}$. With this notation we have

Lemma 3.2. *The following assertions are equivalent:*
(i) *$\mathcal{S}$ has a solution in an extension of M.*
(ii) *A linear relation of the form $\sum \mu_i \lambda_i = 0$ implies $\sum \mu_i a_i = 0$.*

(iii) *There is a homomorphism $g : I \to M$ such that $g(\lambda_i) = a_i$ for all $i \in \mathcal{I}$.*

Proof. The implications (i) ⇒ (ii) and (ii) ⇒ (iii) are obvious.

(iii) ⇒ (i): Let i be the inclusion mapping of I into Λ. Let $M \oplus_I \Lambda$ denoted the amalgamated sum of M and Λ with respect to g and i, that is the module quotient of $M \times \Lambda$ by the submodule $\{(g(\nu), -i(\nu)) : \nu \in I\}$ (cf. [6], p. 258, ex. 5). Let g' (respectively i') denote the canonical homomorphism of Λ (respectively M) into $M \oplus_I \Lambda$. It is easy to verify that i' is one-one and that $g'(1)$ is a solution of $\mathcal{S}$ in $M \oplus_I \Lambda$.

Definition 3.3. An α-system $\mathcal{S}$ is *consistent* if one of the equivalent assertions of Lemma 3.2 is verified.

It is clear that an α-system $\mathcal{S}$ is consistent if and only if every finite subsystem of $\mathcal{S}$ (i.e. $\aleph_0$-subsystem) is consistent.

Lemma 3.4. *The following conditions are equivalent:*

(i) *Every consistent α-system has a solution in M.*

(ii) *For every ideal I having a generating subset of less than α elements, any homomorphism of I into M can be extended to a homomorphism of Λ into M.*

Proof. (i) ⇒ (ii): Let I be an ideal having a generating subset $\mathcal{C}$ of less than α elements. For any homomorphism f of I into M we define an α-system $\mathcal{S}_f = \{\lambda x = f(\lambda) : \lambda \in \mathcal{C}\}$. It is clear that $\mathcal{C}(\mathcal{S}_f) = \mathcal{C}$ and $I(\mathcal{S}_f) = I$. From Lemma 3.2 it then follows that $\mathcal{S}_f$ is consistent. Let s be a solution of $\mathcal{S}_f$ in M. The homomorphism of Λ into M which sends μ into μs extends f.

(ii) ⇒ (i): Let $\mathcal{S}$ be a consistent α-system $= \{\lambda_i x = a_i : i \in \mathcal{I}\}$. From Lemma 3.2 it follows that there exists a homomorphism f of $I(\mathcal{S})$ into M such that $f(\lambda_i) = a_i$. If g is a homomorphism of Λ into M which extends f, $g(1)$ is a solution of $\mathcal{S}$ in M.

Definition 3.5. A module M is *α-injective* if one of the equivalent conditions of Lemma 3.4 is satisfied.

Definition 3.5 1/2. Let $\gamma = \gamma(\Lambda)$ denote the smallest cardinal such that every ideal has a generating subset of less than γ elements. It is clear that a module is β-injective for all cardinals $\beta \geq 2$ if and only if it is γ-injective. Such a module will be called *injective.*

This definition coincides with the usual one: more precisely, one has the following classical result.

Proposition 3.6 (Baer) ([2] or [6], p. 265–266 ex. 11). *The following assertions are equivalent:*
(i) *M is injective.*
(ii) *For any module P and any submodule Q of P, any homomorphism of Q into M can be extended to a homomorphism of P into M.*
(iii) *M is a direct summand of any module which contains it.*

The concept of an injective envelope, due to Eckmann and Schopf [14], will play an important part in the following sections.

Definition 3.7. An injective module E is an *injective envelope* of M if E contains M and if any one-one homomorphism of M into an injective module N can be extended to a one-one homomorphism of E into N.

Proposition 3.8 (cf. [6], p. 268–269, ex. 18). *M can be embedded in an injective envelope of M. Furthermore, if E(M) and E'(M) are two injective envelopes of M, there exists an isomorphism of E(M) onto E'(M) which leaves the elements of M invariant.*

For any module N, $E(N)$ will denote a fixed injective envelope of N. (By *abus de langage*, $E(N)$ will be called *the* injective envelope of N.)

It is well-known that for any modules P and Q one has $E(P \oplus Q) \cong$ $\cong E(P) \oplus E(Q)$ (cf. [6], p. 269, ex. 21).

The two following results are immediate generalizations of well-known ones.

Proposition 3.9. *A direct product* $\prod_{\tau \in \mathcal{T}} P_\tau$ *of modules is α-injective if and only if each of the* P_τ *is α-injective.*

Proof. Easy.

Proposition 3.10. *If $\alpha \leq \aleph_0$, a direct sum of α-injective modules is α-injective.*

Proof. Let $P = \bigoplus_{\tau \in \mathcal{T}} P_\tau$ be the direct sum of a family $\{P_\tau\}_{\tau \in \mathcal{T}}$ of α-injective modules. Let f be a homomorphism of a finitely generated ideal I into P. Since I is finitely generated there exists a finite subset $\mathcal{F}$ of $\mathcal{T}$ such that $\tau \notin \mathcal{F}$ implies $(p_\tau \circ f)(I) = \{0\}$, where p_τ denotes the canonical projection of P onto P_τ. We can then consider f as a homomorphism of I into the module $Q = \bigoplus_{\tau \in \mathcal{F}} P_\tau$. According to the preceding proposition, Q is α-injective. Therefore, if I has a generating subset of less than α elements, f can be extended to a homomorphism g of Λ into Q. If h is the canonical embedding of Q into P, the homomorphism $h \circ g$ of Λ into P extends f.

We will henceforth be concerned with the elementary properties of α-injective modules, particularly of $\aleph_0$-injective and injective modules. One of our main tools will be the notion of ultraproduct, for the definition and properties of which one may consult [5], Chapter 5.

Lemma 3.11. *Every ultraproduct of α-injective modules is α-injective if and only if every ultrapower of an α-injective module is α-injective.*

Proof. We assume that every ultrapower of an α-injective module is α-injective. Let $P = \prod_{\tau \in \mathcal{T}} P_\tau / D$ be an ultraproduct of a family $\{P_\tau\}_{\tau \in \mathcal{T}}$ of α-injective modules. We want to show that P is α-injective.

Let Q denote the direct product $\prod_{\tau \in \mathcal{T}} P_\tau$; by Proposition 3.9, Q is α-injective, and therefore, by hypothesis, $Q^{\mathcal{T}}/D$ is α-injective. We prove that P is a direct summand of $Q^{\mathcal{T}}/D$ by defining homomorphisms $\bar{f} : Q^{\mathcal{T}}/D \to P$, $\bar{g} : P \to Q^{\mathcal{T}}/D$ such that $\bar{f} \circ \bar{g} =$ identity on P. Let f be the map: $Q^{\mathcal{T}} \to Q = \prod_{\tau \in \mathcal{T}} P_\tau$ induced by the family of maps $f_\tau = p_\tau \circ q_\tau$: $Q^{\mathcal{T}} \to P_\tau$ where $q_\tau : Q^{\mathcal{T}} \to Q$, $p_\tau : Q \to P_\tau$ are the canonical projections. Let g be the map: $Q \to Q^{\mathcal{T}}$ induced by the family of maps $g_\tau = j_\tau \circ p_\tau$.

$Q \to Q$ where $j_\tau : P_\tau \to Q$ is the canonical embedding. Then $f \circ g$ = identity on Q, because

$$p_\tau \circ f \circ g = f_\tau \circ g = p_\tau \circ q_\tau \circ g = p_\tau \circ g_\tau = p_\tau \circ j_\tau \circ p_\tau = p_\tau$$

for all $\tau \in \mathcal{T}$. It is then easy to see that f and g induce the maps $\bar{f}$ and $\bar{g}$ on the ultraproducts, as desired. Therefore P is a direct summand of $Q^{\mathcal{T}}/D$ and by Proposition 3.9, P is α-injective.

Remark. What we have really done is to show the following general fact: Every ultraproduct of a family of modules is a direct summand of an ultrapower of the product of this family. The same proof shows that such an ultraproduct is also a direct summand of an ultrapower of the direct sum of this family.

Before stating the main results of this section, let us recall the following well-known definitions: A class $\mathcal{M}$ of structures (of the same type) is said to be *elementary in the wider sense* if $\mathcal{M}$ is the class of models of a first-order theory. $\mathcal{M}$ is said to be *elementarily closed* if any structure elementarily equivalent to an element of $\mathcal{M}$ belongs to $\mathcal{M}$.

Theorem 3.12. *If $\alpha \leq \aleph_0$, the following conditions are equivalent:*
(i) *The α-injective modules constitute an elementary class in the wider sense.*
(ii) *Any ultraproduct of injective modules is α-injective.*
(iii) *For every positive integer n less than α, Λ satisfies the following property* (C_n):
The kernel of every homomorphism of Λ^n into Λ is finitely generated.

Proof. It is enough to show that (ii) implies (iii) and (iii) implies (i).
(ii) ⇒ (iii): Let n be a positive integer less than α and f a homomorphism of Λ^n into Λ. One sees immediately that there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that

$$\forall (\mu_1, \ldots, \mu_n) \in \Lambda^n \; f(\mu_1, \ldots, \mu_n) = \sum_{i=1}^{n} \mu_i \lambda_i .$$

Assuming that the kernel of f, Ker$f$, is not finitely generated, we will exhibit an ultraproduct of injective modules which is not α-injective.

Let β denote the smallest cardinal such that there exists a generating subset $\{a_\tau\}_{\tau<\beta}$ of Ker$f$ of cardinal β. β is infinite and if $\langle a_\tau\rangle_{\tau<\nu}$ denotes the submodule of Ker$f$ generated by $\{a_\tau\}_{\tau<\nu}$ one has

(1) $$\forall \nu < \beta\ \exists \nu' < \beta\ a_{\nu'} \notin \langle a_\tau\rangle_{\tau<\nu} .$$

For each $\nu < \beta$ we can embed the quotient module $\Lambda^n/\langle a_\tau\rangle_{\tau<\nu}$ in an injective module E_ν (cf. Proposition 3.8). Let D be a uniform ultrafilter on β. We claim that the ultraproduct $\prod_{\nu<\beta} E_\nu/D$ is not α-injective.

Indeed, for each i between 1 and n let e_i be the element of Λ^n whose ith component is equal to 1 and all other components are equal to 0. For each ν less than β let $e_{i,\nu}$ be the image of e_i under the canonical homomorphism of Λ^n onto $\Lambda^n/\langle a_\tau\rangle_{\tau<\nu}$. Let $\bar{e}_i$ be the equivalence class modulo D of $(e_{i,\nu})_{\nu<\beta}$. It is enough to show that the system of equations $\{\lambda_i x = \bar{e}_i\}_{1\leq i\leq n}$ is consistent but has no solution in $\prod_{\nu<\beta} E_\nu/D$.

This system is consistent: By Lemma 3.2 and Definition 3.3, it is sufficient to show that for each $(\mu_1, ..., \mu_n) \in$ Ker$f$ we have $\sum_{i=1}^{n} u_i\bar{e}_i = 0$. Clearly we can restrict ourselves to the case where $(\mu_1, ..., \mu_n)$ is an element a_ν of the generating subset $\{a_\tau\}_{\tau<\beta}$. But in this case $\sum_{i=1}^{n} \mu_i e_{i,\nu'}$ is the image of a_ν under the canonical homomorphism of Λ^n onto $\Lambda^n/\langle a_\tau\rangle_{\tau<\nu'}$, for each $\nu' < \beta$. It then follows that for each ν' greater than ν and less than β the element $\sum_{i=1}^{n} \mu_i e_{i,\nu'}$ is equal to 0. But then, since $\{\nu' \mid \nu < \nu' < \beta\} \in D$ (because D is uniform), we have $\sum_{i=1}^{n} \mu_i\bar{e}_i = 0$.

This system has no solution in $\prod_{\nu<\beta} E_\nu/D$: let us suppose that there exists an element $(s_\nu)_{\nu<\beta}$ in $\prod_{\nu<\beta} E_\nu$ whose image under the canonical homomorphism of $\prod_{\nu<\beta} E_\nu$ onto $\prod_{\nu<\beta} E_\nu/D$ is a solution of this system. In this case there would exist an index $\nu < \beta$ such that $\lambda_i s_\nu = e_{i,\nu}$ for all

i between 1 and n. It would follow that for each element $a_{\nu'} = (\mu'_1, ..., \mu'_n)$ of the generating subset $\{a_\tau\}_{\tau<\beta}$ we would have

$\sum_{i=1}^{n} \mu'_i e_{i,\nu} = 0$ or $a_{\nu'} \in \langle a_\tau \rangle_{\tau<\nu}$ which would contradict (1).

(iii) ⇒ (i): Let F_α denote the set of (finite) non-empty sequences $\mathcal{C}$ of elements of Λ of length $< \alpha$. We are going to define for every element $\mathcal{C} = \{\lambda_1, ..., \lambda_n\}$ of F_α a first-order sentence $\varphi_{\mathcal{C}}$ such that the module M is α-injective if and only if M is a model of the set $\{\varphi_{\mathcal{C}} | \mathcal{C} \in F_\alpha\}$. Let $\mathcal{S} = \{\lambda_i x = a_i\}_{1 \leq i \leq n}$ be an α-system such that $\mathcal{C}(\mathcal{S}) = \mathcal{C}$. Let $J = J_{\mathcal{C}}$ denote the kernel of the homomorphism f of Λ^n into Λ defined by:

$f(\mu_1, ..., \mu_n) = \sum_{i=1}^{n} \mu_i \lambda_i$. Let $B = B_{\mathcal{C}} = \{b_j\}_{1 \leq j \leq m}$ denote a finite generating subset of J. If $b_j = (\mu_{1,j}, ..., \mu_{n,j})$ the α-system $\mathcal{S}$ is consistent if and only if

$$\bigwedge_{j=1}^{m} \left(\sum_{i=1}^{n} \mu_{i,j} a_i = 0\right).$$

If $\varphi_{\mathcal{C}}$ is the sentence:

$$\forall a_1 \dots a_n \left(\bigwedge_{j=1}^{m} \left(\sum_{i=1}^{n} \mu_{i,j} a_i = 0\right) \rightarrow \exists x \left(\bigwedge_{i=1}^{n} \lambda_i x = a_i\right)\right),$$

it is easy to verify that the module M is α-injective if and only if M is a model of the set $\{\varphi_{\mathcal{C}} | \mathcal{C} \in F_\alpha\}$. The proof is complete.

It is rather remarkable that the condition (C_n) appears in the literature in a slightly different form: We have the following easy

Lemma 3.13. *For every positive integer n, the two following conditions are equivalent:*
(C_n): *The kernel of every momomorphism of* Λ^n *into* Λ *is finitely generated.*

(C'_n): *Every ideal of* Λ *having a generating subset of n elements is finitely presented.*

Proof. (C_n) ⇒ (C'_n): Let I be an ideal of Λ having a generating subset of n elements. There exists a homomorphism f of Λ^n onto I. The kernel of f is finitely generated, which yields a finite presentation of I.

(C'_n) ⇒ (C_n): This implication follows immediately from ([8], p. 37, Lemme 9).

The following definition is due to Bourbaki (cf. [8], p. 62–63, ex. 11 and 12), although the coherent rings were first studied by Chase (cf. [12]).

Definition 3.14. Λ is *coherent* if every finitely generated ideal of Λ is finitely presented, i.e. if Λ satisfies (C_n) for every positive integer n.

Clearly any noetherian ring is coherent. Examples of coherent rings which are not necessarily noetherian are semihereditary rings [12] (which include Prüfer rings * [11] and hence valuation rings [10]) and rings of polynomials (in any number, finite or infinite, of indeterminates) over a commutative noetherian ring.

The following proposition due to Chase subsumes some of the known characterizations of the coherent rings. It will not be used here and is given without proof for the information of the reader.

Proposition 3.15. Λ *is coherent if and only if it satisfies one of the two following equivalent conditions:*

(i) *Any product of flat right modules is flat.*

(ii) *The annihilator of every element of* Λ *is finitely generated and the intersection of any two finitely generated ideals is finitely generated.*

The following theorem is essentially a restatement of our Theorem 3.12 and gives new characterizations of the coherent rings.

Theorem 3.16. *The following conditions are equivalent:*

* For example, Peano rings, i.e. rings elementarily equivalent to the ring Z of integers, are Prüfer rings.

(i) *The* $\aleph_0$*-injective modules constitute an elementary class in the wider sense.*
(ii) *The* $\aleph_0$*-injective modules constitute an elementarily closed class.*
(iii) *Any ultrapower of* $\aleph_0$*-injective modules is* $\aleph_0$*-injective.*
(iv) *Any ultraproduct of* $\aleph_0$*-injective modules is* $\aleph_0$*-injective.*
(v) Λ *is coherent.*

Proof. One shows (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (i). (i) ⇒ (ii) is trivial. (ii) ⇒ (iii) follows from the properties of ultrapowers. (iii) ⇒ (iv) follows from Lemma 3.11. (iv) ⇒ (v) follows from Theorem 3.12. (v) ⇒ (i) is contained in Theorem 3.12.

Lemma 3.17. 1) *Every elementary substructure of an* $\aleph_0$*-injective module is* $\aleph_0$*-injective.*

2) *If* Λ *is coherent, every* $\aleph_0$*-injective module is an elementary substructure of an injective module.*

Proof. 1) We assume P $\aleph_0$-injective and $M \prec P$. Let $\{\lambda_i x = a_i\}_{1 \leq i \leq n} = \mathcal{S}$ be a consistent $\aleph_0$-system where n is an integer and all the a_i belong to M. One can consider $\mathcal{S}$ as a consistent $\aleph_0$-system whose "parameters" a_i belong to P. Since P is $\aleph_0$-injective, $\mathcal{S}$ has a solution in P. Since $M \prec P$, $\mathcal{S}$ has a solution in M.

2) From Theorem 1.7, p. 220 of [5], it follows that every module is an elementary substructure of an α-saturated module for arbitrarily large cardinals α. Let M be $\aleph_0$-injective. We have $M \prec P$ where P is a γ-saturated module. Since Λ is coherent, P is $\aleph_0$-injective. But it is immediate to verify that an $\aleph_0$-injective module which is γ-saturated is γ-injective. Since a γ-injective module is injective, the proof is complete.

Proposition 3.18. *The following assertions are equivalent:*
(i) *Every* $\aleph_0$*-injective module is injective.*
(ii) *For every countable sequence of cyclic modules* $\{M_n\}_{n \in \omega}$, *the direct sum* $\bigoplus_{n \in \omega} E(M_n)$ *is injective.*
(iii) Λ *is noetherian.*

Proof. (i) ⇒ (ii): This is an immediate consequence of Proposition 3.10.

(ii) ⇒ (iii): This implication is proved, although it is not stated, in ([12], p. 471), where it is attributed to Bass.

(iii) ⇒ (i): This implication is obvious.

Theorem 3.19. *The following conditions are equivalent:*
(i) *The injective modules constitute an elementary class in the wider sense.*
(ii) *The injective modules constitute an elementarily closed class.*
(iii) Λ *is noetherian.*

Proof. It is enough to show that (ii) implies (iii) and (iii) implies (i).
(ii) ⇒ (iii): By assumption any ultrapower of injective modules is injective. From Lemma 3.11 it then follows that any ultraproduct of injective modules is injective, therefore is $\aleph_0$-injective. By Theorem 3.12 and Definition 3.14 this implies that Λ is coherent. By Lemma 3.17 every $\aleph_0$-injective module is then elementarily equivalent to an injective module. Therefore every $\aleph_0$-injective module is injective. One applies then the preceding proposition.
(iii) ⇒ (i): Since Λ is noetherian, Λ is coherent and the injective modules are the $\aleph_0$-injective modules. The result then follows from Theorem 3.16. The proof is complete.

Remarks. We will present here some results which are not necessary for understanding the remaining sections: we will therefore be content with giving a sketch of the proofs.

1. Let α be a cardinal (strictly) greater than $\aleph_0$. One may ask when the α-injective modules constitute an elementary class in the wider sense. This is clearly the case if Λ is noetherian. The converse is also true: if the α-injective modules constitute an elementary class in the wider sense, one may show, as in the proof of Theorem 3.19, that every $\aleph_0$-injective module is $\aleph_1$-injective. But it then follows from ([12], p. 471) that Λ is noetherian.

2. The reader will have noticed that the Theorems 3.16 and 3.19 are not exactly parallel. The reason is that, although we are not able to characterize the rings Λ such that any ultraproduct of injective Λ-modules is injective, it is easy to see that such a ring is not necessarily noetherian. It may be of interest to point out first the following:

Proposition 3.20. *The following conditions are equivalent:*
(i) *Any ideal having a generating subset of less than α elements is projective.*
(ii) *Any quotient of an α-injective module is α-injective.*
(iii) *Any quotient of an injective module is α-injective.*

We omit the proof which is very similar to that given in ([11], Theorem 5.4, p. 14).

It follows from the preceding proposition (or from the above reference) that, if Λ is (left) hereditary, i.e. if every (left) ideal of Λ is a projective module, any quotient of an injective module is injective. Since any product of injective modules is injective, we can deduce that, if Λ is hereditary, any ultraproduct of injective modules is injective. But it is well-known that a hereditary ring is not necessarily noetherian.

3. Let us recall the following definition, which is equivalent to one given by P.M.Cohn [13].

Definition 3.21. A submodule N of a module M is *pure* in M if any finite set of linear equations (over Λ) with constants in N which has a solution in M has a solution in N.

The following definition has been introduced in ([27], p. 155).

Definition 3.22. A module is *absolutely pure* if it is a pure submodule of any module which contains it.

We would like to compare the notions of injective, absolutely pure and $\aleph_0$-injective modules. One has clearly:

$$\text{Injective} \Rightarrow \text{absolutely pure} \Rightarrow \aleph_0\text{-injective}.$$

Proposition 3.23. *If* Λ *is coherent, every* $\aleph_0$*-injective module is absolutely pure.*

Proof. By the second part of Lemma 3.17, if Λ is coherent, every $\aleph_0$-injective module is an elementary substructure of an injective module.

Since an injective module is absolutely pure, one completes the proof by observing that every elementary substructure of an absolutely pure module is absolutely pure (cf. the proof of the first part of Lemma 3.17).

The preceding proof has a metamathematical flavour. It is possible to replace it by a simple "algebraic" argument which the interested reader may provide. We do not know for what rings the $\aleph_0$-injective modules are absolutely pure.

In the case where Λ is a Prüfer ring, a better result than Proposition 3.23 is available: in this case a submodule N of a module M is pure (in M) if and only if for each $\lambda \in \Lambda$ $\lambda M \cap N = \lambda N$ (cf. [9], p. 132, ex. 6, where Λ is assumed to be a valuation ring, which is enough, by localization, or [36], p. 706, Corollary 5). It is then immediate that, if Λ is a Prüfer ring, every 2-injective module is absolutely pure.

The following proposition, which is essentially a restatement of Proposition 3.18, answers a question left open in [27].

Proposition 3.24. *The following assertions are equivalent:*
(i) *Every $\aleph_0$-injective module is injective.*
(ii) *Every absolutely pure module is injective.*
(iii) *Λ is noetherian.*

4. The 2-injective modules have been studied, under the name of divisible, in a paper by Hattori ([17]). One of the results of this paper is that every module is 2-injective if and only if Λ is a regular ring in the sense of Von Neumann. Such a ring is coherent ([34]) and has the property that every finitely generated ideal is principal. The following proposition is then immediate.

Proposition 3.25. *The following conditions are equivalent:*
(i) *Every module is absolutely pure.*
(ii) *Every module is $\aleph_0$-injective.*
(iii) *Λ is regular in the sense of Von Neumann.*

4. The model-completion of K_Λ

Since K_Λ has the amalgamation property (see the proof of Lemma 3.2), K_Λ has a model-completion if and only if it has a model-companion (Lemma 2.1).

Theorem 4.1. *If* K_Λ *has a model-companion, then* Λ *is coherent.*

We first prove a lemma.

Lemma 4.2. *Let* $M_i \subseteq N_i$ *be modules with* M_i $\aleph_0$*-injective for all* $i \in I$. *Let* D *be an ultrafilter on* I *and* $M^* = \prod_I M_i/D$, $N^* = \prod_I N_i/D$. *If* N^* *is* $\aleph_0$*-injective, then* M^* *is* $\aleph_0$*-injective.*

Proof. Let $\mathcal{S} = \{\lambda_j x = a_j^* : j = 1, ..., n\}$ be a consistent finite system of equations with coefficients $a_j^* \in M^*$. Choose a representative element $(a_j(i))_{i\in I}$ in $\prod_I M_i$ for each a_j^*, so that $a_j^* = (a_j(i))/D$. Since $M^* \subseteq N^*$ and N^* is $\aleph_0$-injective, $\mathcal{S}$ has a solution $b^* = (b(i)/D$ in N^*. Then the system of equations $\mathcal{S}(i) = \{\lambda_j x = a_j(i) : j = 1, ..., n\}$ has a solution $b(i)$ in N_i for i in a set of D. But then since M_i is $\aleph_0$-injective and $M_i \subseteq N_i$, $\mathcal{S}(i)$ has a solution in M_i for i in a set of D. Hence $\mathcal{S}$ has a solution in M^*. Since this is true for any consistent finite system $\mathcal{S}$, M^* is $\aleph_0$-injective.

Proof of 4.1. By Theorem 3.16, it suffices to prove that any ultraproduct $\prod_I M_i/D$ of $\aleph_0$-injectives M_i is $\aleph_0$-injective. If K_Λ has a model-companion – say K^* – then each M_i can be embedded in a model N_i of K^* (K^* is model-consistent relative to K_Λ).

If we show that $N^* = \prod_I N_i/D$ is $\aleph_0$-injective, then by Lemma 4.2 we are done. But if $\mathcal{S} = \{\lambda_i x = a_i\}$ is a consistent finite system of equations with $a_i \in N^*$, then there is an $N_1 \supseteq N^*$ such that $\mathcal{S}$ has a solution in N_1. Since K^* is model-consistent relative to K we may assume that N_1 is a model of K^*. Now N^* is also a model of K^*, so N_1 is an elementary extension of N^* (because K^* is model-complete). Therefore the sentence which asserts the existence of a solution for $\mathcal{S}$ is true in N^* since it is true in N_1. This completes the proof of 4.1.

Let $\mathcal{E} = \{E(\Lambda/I): I$ an ideal in $\Lambda\}$ i.e. $\mathcal{E}$ = the set of all injective envelopes of cyclic modules. Index $\mathcal{E}$ by a set J so that $\mathcal{E} = \{E_j : j \in J\}$. Let $M_0 = \bigoplus_{j\in J} E_j^{(\aleph_0)}$ and let $K_\Lambda^* = \mathrm{Th}(M_0)$ = set of all sentences of L_Λ true in M_0. Note that M_0 is $\aleph_0$-injective (Proposition 3.10). We will prove that if Λ is coherent then K_Λ^* is the model-completion of K_Λ. Our first results, however, do not depend on the fact that Λ is coherent.

We first prove a technical lemma.

Lemma 4.3. *Let A and B be modules. Suppose $\mathcal{F}(B)$ is a family of submodules of A such that for any $B' \in \mathcal{F}(B)$, $B' \cong B$ and for any B', $B'' \in \mathcal{F}(B)$, either $B'' = B'$ or $B'' \cap B' = \{0\}$. Suppose $\mathcal{F}(B)$ has cardinality κ and suppose $A = C_1 \oplus C_2$ where* $\mathrm{Card}(C_1) < \kappa$. *Then C_2 contains a submodule isomorphic to B.*

Proof. We may suppose $A = C_1 \oplus C_2$ is an internal direct sum, so that C_i is a submodule of A $(i = 1, 2)$. Let $p_i : A \to C_i$, $i = 1, 2$, be the canonical projections. Then it suffices to show that $p_2 | B'$ is one-one for some $B' \in \mathcal{F}(B)$. Now if $B', B'' \in \mathcal{F}(B)$, $B' \neq B''$, and $b' \in B'$, $b'' \in B''$ are non-zero elements such that $p_2(b') = 0 = p_2(b'')$, then $b' = p_1(b')$ and $b'' = p_1(b'')$ are distinct elements of C_1 $(B' \cap B'' = \{0\})$. Since $\mathrm{Card}(\mathcal{F}(B)) = \kappa$ and $\mathrm{Card}(C_1) < \kappa$, it follows that $p_2 | B'$ is one-one for some $B' \in \mathcal{F}(B)$. This completes the proof of 4.3:

For the sake of brevity we will write "E is i.e.f.g." if E is the injective envelope of a finitely generated module, that is, if $E = E(\Lambda^{(n)}/R)$ for some $n < \omega$ and some submodule R of $\Lambda^{(n)}$. Let κ be a cardinal such that $\kappa > \{\mathrm{Card}(E) : E$ is i.e.f.g.$\}$. Note that $M_0^{(\kappa)} \cong \bigoplus_{j\in J} E_j^{(\kappa)}$ is a model of K_Λ^* (cf. [16]).

If M and N are modules, $A \subseteq M$, $B \subseteq N$ and $f : A \to B$ is a set bijection we write

$$(M, a)_{a\in A} \equiv (N, f(a))_{a\in A}$$

if for any formula $\varphi(v_1 \ldots v_n)$ in L_Λ and any $a_1, \ldots, a_n \in A$

$$M \models \varphi[a_1 \ldots a_n] \Longleftrightarrow N \models \varphi[f(a_1) \ldots f(a_n)] \ .$$

Lemma 4.4. *Let M, N be injective modules which contain submodules isomorphic to $M_0^{(\kappa)}$. Let $f : A \to B$ be an isomorphism of modules where $A \subseteq M$ and $B \subseteq N$ are finitely generated. Then*

$$(M, a)_{a \in A} \equiv (N, f(a))_{a \in A} \ .$$

Proof. In fact we prove that

$$(M, a)_{a \in A} \equiv_{\infty\omega} (N, f(a))_{a \in A} \ .$$

By replacing A and B by their injective envelopes, we can assume that A and B are i.e.f.g.

Let $\mathcal{G}$ = the set of all isomorphisms $g : S_1 \to T_1$ such that g is an extension of f and $S_1 \subseteq M$, $T_1 \subseteq N$ are i.e.f.g. We will prove that for any $c \in M$ (resp. $d \in N$) and any $g \in \mathcal{G}$ there exists $g' \in \mathcal{G}$ such that g' extends g and $c \in$ domain of g' (resp. $d \in$ range of g'). If we prove this, then the conclusion of the lemma follows easily by an induction on formulas (cf. [24]). So let $g : S_1 \to T_1$ and let $c \in M$ (the proof that we can extend the range of g to $d \in N$ is identical). Since S_1 and T_1 are injective, we can write $M = S_1 \oplus S_2$, $N = T_1 \oplus T_2$. We can assume $c \in S_2$ (since if $c = c_1 + c_2$ where $c_i \in S_i$ ($i = 1, 2$) it suffices to extend g to $c_2 \in S_2$). Then $E(\Lambda c)$ is a direct summand of S_2; we have:

$$M = S_1 \oplus E(\Lambda c) \oplus S_2'$$

and $S_1 \oplus E(\Lambda c)$ is i.e.f.g. (see the remark following Proposition 3.8). Now since $N \supseteq M_0^{(\kappa)}$, there is a family $\mathcal{F}(E(\Lambda c))$ of submodules of N satisfying the hypothesis of Lemma 4.3 such that $\mathcal{F}(E(\Lambda c))$ is of cardinality κ. Since T_1 is of cardinality $< \kappa$ (by choice of κ : T_1 is i.e.f.g.) Lemma 4.3 implies T_2 contains a submodule – say T_2' – isomorphic to $E(\Lambda c)$. Clearly we can extend g to an isomorphism $g' : S_1 \oplus E(\Lambda c) \to$ $\to T_1 \oplus T_2'$.

Remark. If we take $A = B = \{0\}$ in Lemma 4.4 we obtain: if M, N are injective modules which contain submodules isomorphic to $M_0^{(\kappa)}$, then $M \equiv N$.

A natural question to ask is whether two injectives which contain M_0 are elementarily equivalent. We shall give an affirmative answer to this question in the case that Λ is coherent. First we need a general model-theoretic fact.

Lemma 4.5. *Let $\mathfrak{A}, \mathfrak{B}$ be structures for a language* L *such that $\mathfrak{B} \subseteq \mathfrak{A}$ (resp. $\mathfrak{B} \prec \mathfrak{A}$). If $\mathfrak{B}'$ is elementarily equivalent to $\mathfrak{B}$, then there is an embedding (resp. elementary embedding) of $\mathfrak{B}'$ into an elementary extension $\mathfrak{A}'$ of $\mathfrak{A}$.*

Proof. Since $\mathfrak{B} \equiv \mathfrak{B}'$, by Frayne's Lemma there is an elementary embedding f of $\mathfrak{B}'$ into an ultrapower $\mathfrak{B}^I/D$ of $\mathfrak{B}$. But then the composition of f with the canonical embedding of $\mathfrak{B}^I/D$ into $\mathfrak{A}^I/D = \mathfrak{A}'$ is an embedding of $\mathfrak{B}'$ into $\mathfrak{A}'$, which is elementary if $\mathfrak{B} \prec \mathfrak{A}$.

Corollary 4.6. *Let* L *be a language with a distinguished constant* 0 *(so that the direct sum of structures for* L *is definable). Let $\mathfrak{A}, \mathfrak{B}$ be structures for* L *and let α be a cardinal $\geq \aleph_0$.*
(a) $\mathfrak{B}^{(\aleph_0)} \equiv \mathfrak{B}^{(\alpha)}$ *(where the notation denotes direct sum)* [16].
(b) *If $\mathfrak{A}$ contains a substructure elementarily equivalent to $\mathfrak{B}^{(\aleph_0)}$, then there is an elementary extension of $\mathfrak{A}$ which contains a substructure isomorphic to $\mathfrak{B}^{(\alpha)}$.*

Proof. (a) In fact $\mathfrak{B}^{(\aleph_0)} \equiv_{\infty\omega} \mathfrak{B}^{(\alpha)}$ (see, for example, [3] Lemma 1.8); (b) follows immediately from (a) and the preceding lemma.

Corollary 4.7. *Assume Λ is coherent.*
(1) *Let M, N be $\aleph_0$-injective modules which contain submodules elementarily equivalent to M_0. Let $f: A \to B$ be an isomorphism of modules where $A \subseteq M$ and $B \subseteq N$ are finitely generated. Then*

(2) $E(M_0) \equiv M_0$. $\qquad (M, a)_{a \in A} \equiv (N, f(a))_{a \in A}$.

Proof. (1) Since $M_0 \cong M_0^{(\aleph_0)}$, it follows from Corollary 4.6 that $M_0^{(\kappa)}$ can be embedded in an elementary extension M' (resp. N') of M (resp. N). Since Λ is coherent, M' and N' are $\aleph_0$-injective (Theorem 3.16). Therefore by Lemma 3.17, there is an elementary extension M'' (resp. N'') of M' (resp. N') sich that M'' and N'' are injective. The conclusion follows from Lemma 4.4 applied to M'' and N''.
(2) By Lemma 3.17, there is an elementary extension M_1 of M_0 which is injective. By (1), $E(M_0) \equiv M_1$.

Remark. We do not know if the corollary remains true if we drop the assumption that Λ is coherent.

Theorem 4.8. *If* Λ *is coherent, then* K_Λ^* *is the model-companion* (*and hence the model-completion*) *of* K_Λ.

Proof. We prove first that K_Λ^* is model-consistent relative to K_Λ. Let M be any module; then $M \subseteq E(M) \oplus M_0$, which is a model of K_Λ^* by Corollary 4.7.

To show that K_Λ^* is model-complete, consider $M_1 \subseteq M_2$ where M_1 and M_2 are models of K_Λ^*. Since $M_i \equiv M_0$, $(i = 1, 2)$, there are elementary embeddings $h_i : M_i \to N_i$ of M_i into an extension N_i of M_0. (For example, use Frayne's Lemma). By Theorem 3.16 N_i is $\aleph_0$-injective. Let $\varphi(v_1 \ldots v_n)$ be a formula of L_Λ and let $a_1, \ldots, a_n \in M_1$ such that $M_1 \models \varphi[a_1 \ldots a_n]$. Since h_1 is an elementary embedding, $N_1 \models \varphi[h_1(a_1) \ldots h_n(a_n)]$. Let $A_i = h_i(\Lambda a_1 + \ldots + \Lambda a_n) \subseteq N_i$. There is an isomorphism $f : A_1 \to A_2$ such that

$$\begin{array}{ccc} A_1 & \xrightarrow{\ f\ } & A_2 \\ {\scriptstyle h_1}\nwarrow & & \nearrow{\scriptstyle h_2} \\ & \Lambda a_1 + \ldots + \Lambda a_n & \end{array}$$

commutes. Since A_1 is finitely generated

$$(N_1, a)_{a \in A_1} \equiv (N_2, f(a))_{a \in A_1}$$

by Corollary 4.7. Therefore $N_1 \models \varphi[h_1(a_1) \dots h_1(a_n)]$ implies $N_2 \models \varphi[fh_1(a_1) \dots fh_1(a_n)]$ which, by the commutativity of the diagram, is the same as $N_2 \models \varphi[h_2(a_1) \dots h_2(a_n)]$. Since h_2 is an elementary embedding we conclude that $M_2 \models \varphi[a_1 \dots a_n]$. Thus $M_1 \prec M_2$ and the proof is complete.

Remark. It is an open question whether for any coherent ring Λ and any model N of K_Λ, $K_\Lambda^* \cup D(N)$ has a prime model. However we can give an affirmative answer to this question when Λ is commutative noetherian or artinian.

Proposition 4.9. *All models of K_Λ^* are injective if and only if Λ is noetherian.*

Proof. It follows immediately from Proposition 3.18 ((ii) and (iii)) that if M_0 is injective, Λ is noetherian. (By definition of M_0, $\bigoplus_{n \in \omega} E(M_n)$ is a direct summand of M_0 for any countable sequence $\{M_n = \Lambda/I_n\}$ of cyclic modules.) Therefore if all models of K_Λ^* are injective, Λ is noetherian. Conversely, if Λ is noetherian, by Proposition 3.18 ((i) and (iii)), M_0 is injective and, since the injective modules constitute an elementarily closed class (Theorem 3.19), all models of K_Λ^* are injective.

5. Commutative noetherian rings

Throughout this section Λ is a commutative noetherian ring. In this case we are able to give additional information about the model-completion K_Λ^* of K_Λ. Our principal tools are the Krull Intersection Theorem and results of Matlis [28] on injectives over noetherian rings.

Let $\mathcal{P}$ = the set of proper prime ideals in Λ, $\mathcal{M}$ = the set of maximal ideals. Also let

$$\mathcal{M}_f = \{Q \in \mathcal{M} : \Lambda/Q \text{ is finite}\},$$

$$\mathcal{M}_i = \{Q \in \mathcal{M} : \Lambda/Q \text{ is infinite}\}.$$

Then any injective E can be written in the form

$$(5.1) \qquad E \cong \bigoplus_{P \in \mathcal{P}} E(\Lambda/P)^{(\alpha_P)},$$

where $\alpha_P \geq 0$ ([28], Theorems 2.5, 3.1). Moreover, the α_P are uniquely determined by E ([1], Theorem 1).

By *abus de langage* we will refer to the latter fact as the "uniqueness of the decomposition (5.1)" (although the isomorphism in (5.1) is not unique). In another instance of *abus de langage* we will freely confuse the notions of inclusion and embedding. For example we will say "E contains $E(\Lambda/P)^{(\alpha_P)}$" when we should strictly say "E contains a submodule isomorphic to $E(\Lambda/P)^{(\alpha_P)}$".

We will prove:

Theorem 5.2. *Let E be a Λ-module. The following assertions are equivalent:*

(I) *E is injective and if E is decomposed as in* (5.1):

a) $\alpha_P > 0$ *for all* $P \in \mathcal{M}$;

b) *If* $P \in \mathcal{M}_f$, $\alpha_P \geq \aleph_0$.

(II) *E is a model of* K^*_Λ.

The fact that any model of K^*_Λ is injective follows from Proposition 4.9. The rest of the proof of 5.2 will be given in the form of two propositions (5.4 and 5.6). First we prove a lemma that gives a criterion for $\alpha_P > 0$. (If $a \in M$, $\mathrm{Ann}(a)$ denotes the *annihilator* of a, i.e. the set $\{\lambda \in \Lambda : \lambda a = 0\}$).

Lemma 5.3. *Let* $E = \bigoplus_{P \in \mathcal{P}} E(\Lambda/P)^{(\alpha_P)}$. *Then* $\alpha_P > 0 \iff$ *there exists* $a \in E$ *such that* $\mathrm{Ann}(a) = P$.

Proof. ($\Rightarrow$) For any $0 \neq a \in \Lambda/P \subseteq E(\Lambda/P)$, $\mathrm{Ann}(a) = P$, because P is prime.

($\Leftarrow$) If $a \in E$, $a \neq 0$, we can write a uniquely in the form $a = a_1 + \ldots + a_n$ where each a_i is non-zero and for some $P_i \in \mathcal{P}$, is an element of a summand $E(\Lambda/P_i)$ of $\bigoplus_{P \in \mathcal{P}} E(\Lambda/P)^{(\alpha_P)}$. Then

$$\mathrm{Ann}(a) = \bigcap_{i=1}^{n} \mathrm{Ann}(a_i)\,.$$

Hence if $\mathrm{Ann}(a) = P$, then $\mathrm{Ann}(a_i) = P$ for some i (because

$\prod_{i=1}^{n} \mathrm{Ann}(a_i) \subseteq P$). But $\mathrm{Ann}(a_i)$ is a P_i-primary ideal ([28], Lemma 3.2). Therefore if $\mathrm{Ann}(a_i) = P$, then $P = P_i$.

Remark. If $a = a_1 + \ldots + a_n$ as above and $\mathrm{Ann}(a) = P$ where P is *maximal*, then $\mathrm{Ann}(a_i) = P$ for *all* i, because $P = \mathrm{Ann}(a) \subseteq \mathrm{Ann}(a_i) \neq \Lambda$.

If $Q \in \mathfrak{M}$ choose a finite basis $\mu_1, \ldots, \mu_r$ for Q and let $\psi_Q(x)$ be the formula

$$x \neq 0 \wedge \bigwedge_{i=1}^{r} (\mu_i x = 0)\,.$$

Thus $M \models \psi_Q[a]$ if and only if $\mathrm{Ann}(a) = Q$. (Since Q is maximal, $\mathrm{Ann}(a) = Q \Longleftrightarrow a \neq 0$ and $\mathrm{Ann}(a) \supseteq Q$.) Let θ_Q^n be the sentence

$$(\exists x_1) \ldots (\exists x_n) [\bigwedge_{i=1}^{n} \psi_Q(x_i) \wedge \bigwedge_{i \neq j} (x_i \neq x_j)]\,.$$

Let T_0 be a set of axioms for the theory of injective modules (Theorem 3.19) and let

$$T = T_0 \cup \{\theta_Q^1 : Q \in \mathfrak{M}_i\} \cup \{\theta_Q^n : n > 0;\ Q \in \mathfrak{M}_f\}\,.$$

Proposition 5.4. *E is a model of $T \Longleftrightarrow E$ satisfies* (I) (a) *and* (b) *of* Theorem 5.2.

Proof. If we write an injective E as in (5.1) then by Lemma 5.3, $\alpha_Q > 0 \Longleftrightarrow E \models \theta_Q^1$, for each $Q \in \mathfrak{M}$. If $Q \in \mathfrak{M}_f$, $\{a \in E(\Lambda/Q) : \mathrm{Ann}(a) \supseteq Q\} \cong \Lambda/Q$, which is finite ([28] Theorem 3.4 (4)). Therefore it follows from Lemma 5.3 and the remark following it that $\alpha_Q \geq \aleph_0 \Longleftrightarrow \{a \in E : \mathrm{Ann}(a) = Q\}$ is infinite $\Longleftrightarrow E \models \theta_Q^n$ for all $n > 0$. The proof is complete.

Since M_0 obviously satisfies (I)(a) and (b) it follows that $T \subseteq K_\Lambda^*$ and hence any model of K_Λ^* satisfies (I)(a) and (b). This proves the implication (II) ⇒ (I) of Theorem 5.2. We begin the proof of the opposite implication with a lemma that establishes a criterion for E to be a model of K_Λ^*:

Lemma 5.5. *If E is an injective which contains $E(\Lambda/P)^{(\aleph_0)}$ for each $P \in \mathcal{P}$, then $E \equiv M_0$.*

Proof. By definition $M_0 \supseteq E(\Lambda/P)^{(\aleph_0)}$ for each $P \in \mathcal{P}$. Therefore $M_0 \cong \bigoplus_{P \in \mathcal{P}} E(\Lambda/P)^{\beta_P}$ where $\beta_P \geq \aleph_0$ for all $P \in \mathcal{P}$ (by the uniqueness of the decomposition (5.1)). If $M' = \bigoplus_{P \in \mathcal{P}} E(\Lambda/P)^{(\aleph_0)}$ then $M' \equiv M_0$ (by Corollary 4.6(a) and [16]). By hypothesis, if we write E as in (5.1) then $\alpha_P \geq \aleph_0$ for all $P \in \mathcal{P}$. Thus $M' \subseteq E$ and it follows from Corollary 4.7 that $E \equiv M_0$.

Now define

$$M_1 = \bigoplus_{Q \in \mathcal{M}_f} E(\Lambda/Q)^{(\aleph_0)} + \bigoplus_{Q \in \mathcal{M}_i} E(\Lambda/Q) .$$

By the uniqueness of the decomposition (5.1), E contains (a submodule isomorphic to) M_1 if and only if E satisfies (I)(a) and (b) of Theorem 5.2. Therefore we will have proved the implication (I) ⇒ (II) of 5.2 if we prove:

Proposition 5.6. *If E is an injective which contains M_1, then E is a model of K_Λ^*.*

Proof. First of all we observe that there is an elementary extension E of E which contains $E(\Lambda/Q)^{(\aleph_0)}$ for each $Q \in \mathcal{M}_i$ (i.e. $E_1 \supseteq M_1^{(\aleph_0)}$). In fact let $\delta > \max \{\text{Card}\, E(\Lambda/Q) : Q \in \mathcal{M}_i\}$; by taking a suitable ultrapower E^I/D of E we obtain an elementary extension of E which for each $Q \in \mathcal{M}_i$ contains δ elements whose annihilator is Q (viz. the nonzero elements of $(\Lambda/Q)^I/D$). Now by the remark following Lemma 5.3 and since $\delta > \text{Card}\, E(\Lambda/Q)$, we see that $E^I/D \supseteq E(\Lambda/Q)^{(\delta)}$.

We claim that to prove the proposition it suffices to prove that for any module N containing $M_1^{(\aleph_0)}$, there is an elementary extension N_1 of N such that for all $P \in \mathcal{P} - \mathcal{M}$, $N_1 \supseteq N \oplus E(\Lambda/P)$. If this is the case, then we obtain by induction on n an increasing chain of modules N_n such that $N_0 = E_1$, $N_{n-1} \prec N_n$, and for each $P \in \mathcal{P} - \mathcal{M}$, $N_n \supseteq N_{n-1} \oplus E(\Lambda/P)$. Taking the union of the chain we obtain an elementary extension N_ω of N such that for each $P \in \mathcal{P}$, $N_\omega \supseteq E(\Lambda/P)^{(\aleph_0)}$. N_ω is injective by Theorem 3.19 because $N_\omega \equiv E$. Thus by Lemma 5.5, $N_\omega \equiv M_0$ and hence $E \equiv M_0$ i.e. $E \models K_\Lambda^*$. Thus to complete the proof of 5.6 it suffices to prove:

Lemma 5.7. *Let D be a non-principal ultrafilter on $I = \{n \in \mathbf{Z} : n > 0\}$. Let N be a module containing $M_1^{(\aleph_0)}$ and let $N_1 = N^I/D$. Then for any $P \in \mathcal{P} - \mathcal{M}$, $N_1 \supseteq N \oplus E(\Lambda/P)$.*

Proof. Let $P \in \mathcal{P} - \mathcal{M}$ and $Q \in \mathcal{M}$ such that $P \subset Q$. We will prove first that $(E(\Lambda/Q)^{(\aleph_0)})^I/D$ contains a submodule isomorphic to $E(\Lambda/P)$. By the Krull Intersection Theorem, $P = \bigcap_{n>0} (Q^n + P)$ ([36] Theorem 12′, p. 217). Now $Q^n + P$ is a Q-primary ideal ([36], Corollary 1, p. 153). Write $Q^n + P = J_{n,1} \cap \ldots \cap J_{n,r_n}$ as an intersection of irreducible ideals; then each $J_{n,k}$ is an irreducible Q-primary ideal, so by ([28], Lemma 3.2) there is an element $x_{n,k} \in E(\Lambda/Q)$ such that $\mathrm{Ann}(x_{n,k}) = J_{n,k}$. Moreover we can choose the $x_{n,k}$ to lie in different copies of $E(\Lambda/Q)$. Then if $y_n = x_{n,1} + \ldots + x_{n,r_n}$, $\mathrm{Ann}(y_n) = \bigcap_{k=1}^{r_n} \mathrm{Ann}(x_{n,k}) = \bigcap_k J_{n,k} = Q^n + P$. If we let y^* be the element of $(E(\Lambda/Q)^{(\aleph_0)})^I/D$ represented by $(y_n)_n$, then for any $\lambda \in \Lambda$, $\lambda y^* = 0$ implies $\lambda y_n = 0$ for arbitrarily large n (D is non-principal); so $\lambda \in \bigcap_n (Q^n + P) = P$. Thus $\mathrm{Ann}(y^*) \subseteq P$; the opposite inclusion is clear, and hence $\mathrm{Ann}(y^*) = P$. It follows from Lemma 5.3 that $(E(\Lambda/Q)^{(\aleph_0)})^I/D$ contains a submodule A isomorphic to $E(\Lambda/P)$.

Now $E(\Lambda/Q)^{(\aleph_0)})^I/D$ – and hence A – may be regarded as a submodule of N_1. We claim that if $d : N \to N_1$ is the diagonal embedding, $A \cap d(N) = \{0\}$. Indeed, $d(N) \cap (E(\Lambda/Q)^{(\aleph_0)})^I/D = d(E(\Lambda/Q)^{(\aleph_0)})$. Thus if $0 \neq a \in A \cap d(N)$, $a \in d(E(\Lambda/Q)^{(\aleph_0)})$ and so $E(\Lambda/Q)^{(\aleph_0)}$ contains a submodule isomorphic to $E(\Lambda a)$. But also $E(\Lambda a) \cong A \cong E(\Lambda/P)$

because A is indecomposable ([28] Prop. 2.2), which contradicts the uniqueness of the decomposition (5.1). Hence we have proved $A \cap d(N) = \{0\}$ and consequently $N_1 \supseteq d(N) \oplus A$. This completes the proof of Lemma 5.7, Proposition 5.6, and Theorem 5.2.

Corollary 5.8. *For any model N of K_Λ, $K_\Lambda^* \cup D(N)$ has a prime model which is unique up to isomorphism over N.*

Proof. If N is a module, write

$$E(N) = \bigoplus_{P \in \mathcal{P}} E(\Lambda/P)^{(\alpha_P)} .$$

For $P \in \mathfrak{M}_f$, define $\beta_P = \aleph_0$ if $\alpha_P < \aleph_0$; $\beta_P = 0$ otherwise. For $P \in \mathfrak{M}_i$, define $\beta_P = 1$ if $\alpha_P = 0$; $\beta_P = 0$ otherwise. Let

$$E_0 = E(N) \oplus \bigoplus_{P \in \mathfrak{M}} E(\Lambda/P)^{(\beta_P)} .$$

Then $N \subseteq E(N) \subseteq E_0$ and E_0 is a prime model of $K_\Lambda^* \cup D(N)$. Clearly any prime model of $K_\Lambda^* \cup D(N)$ is isomorphic to E_0 by an isomorphism which fixes N.

6. Artinian rings

When Λ is artinian, we can also give a structure theorem for the models of K_Λ^* and a set of axioms for K_Λ^*.

We first handle the case where Λ is semi-simple artinian. We begin by indicating three mathematical facts about Λ and Λ-modules which will be used in the proof.

(6.1). (Artin-Wedderburn Theorem) $\Lambda = \bigoplus_{i=1}^{n} \Lambda e_i$ *where each* Λe_i *is a simple left ideal and* $\{e_i\}_{1 \le i \le n}$ *is a finite set of orthogonal idempotents.*

Clearly we may assume that there is an integer $m \le n$ such that

(6.1a) *if* $1 \leq i$, $j \leq m$ *and* $i \neq j$ *the left* Λ*-modules* Λe_i *and* Λe_j *are not isomorphic.*

(6.1b) *if* $m < i \leq n$ *there is a* $j \leq m$ *such that the left* Λ*-modules* Λe_i *and* Λe_j *are isomorphic.*

(6.2). *Any module M is semi-simple and therefore can be written as a direct sum of simple modules:*

$$M = \bigoplus_{i=1}^{m} L_i^{(\alpha_i)}$$

where $L_i \cong \Lambda e_i$.

The cardinals α_i are uniquely determined by M, which allows us, as in § 5, to refer to the uniqueness of the decomposition without further comment.

(6.3). *A simple module L is isomorphic to* Λe_i *if and only if there is an element* $a \in L$ *such that* $e_i a \neq 0$.

For 6.1 and 6.2 one may see, e.g., [7] or [20]. 6.3 may be easily checked by the reader. The main interest of (6.3) is that it gives a *first-order sentence* which a simple module L satisfies if and only if L is isomorphic to Λe_i.

Remark. An immediate consequence of 6.2 is the fact that any module is injective. It is well known (cf. [20]), but will not be needed here, that this gives a characterization of the artinian semi-simple rings.

We consider now the module M_0 defined in § 4. Clearly we have:

$M_0 \cong \bigoplus_{i=1}^{m} (\Lambda e_i)^{(\alpha_i)}$ where each α_i is $\geq \aleph_0$. By Corollary 4.6(a) and [16] it then follows that M_0 is elementarily equivalent to $\bigoplus_{i=1}^{m} (\Lambda e_i)^{(\aleph_0)}$.

For any $i = 1, ..., m$ and for any integer $k > 0$ let φ_i^k be the sentence

$$(\exists x_1) \dots (\exists x_k) (\bigwedge_{j=1}^{k} e_i x_j \neq 0) \wedge (\bigwedge_{j \neq j'} e_i x_j \neq e_i x_{j'})$$

Let $\mathcal{F}$ be the set $\{i \mid 1 \leq i \leq m$ and Λe_i is finite$\}$. Let T be the set $\{\varphi_i^k : k > 0,\ i \in \mathcal{F}\} \cup \{\varphi_i^1 : i \notin \mathcal{F}\}$.

Theorem 6.4. *Let Λ be semi-simple artinian and M a Λ-module. The following assertions are equivalent:*

(i) *M is a model of T.*

(ii) *The decomposition given in* (6.2) *satisfies:*

a) *$\alpha_i > 0$ for all i;*

b) *If $i \in \mathcal{F}$ then $\alpha_i \geq \aleph_0$.*

(iii) *M is a model of K_Λ^*.*

Proof. (i) $\Rightarrow$ (ii): This is an easy consequence of (6.3). (ii) $\Rightarrow$ (iii): We suppose that M satisfies (ii) and we have to show $M \equiv M_0$ or

$M \equiv \bigoplus_{i=1}^{m} (\Lambda e_i)^{(\aleph_0)}$. By Corollary 4.6(a) and [16], we are done if we show that

(6.5) $i \notin \mathcal{F}$ implies $\Lambda e_i \equiv (\Lambda e_i)^{(\aleph_0)}$.

It follows easily from (6.3) that for any module N elementarily equivalent to Λe_i, any simple submodule of N is isomorphic to Λe_i. Let us suppose now that Λe_i is infinite; there exists (by the Löwenheim-Skolem theorem) a module N elementarily equivalent to Λe_i and of cardinal > cardinal (Λe_i). Since any simple submodule of N is isomorphic to Λe_i, by (6.2) (and the uniqueness of the decomposition) one has

$$N \cong (\Lambda e_i)^{(\alpha)} \quad \text{with} \quad \alpha > \aleph_0 .$$

By Corollary 4.6(a), one then has $N \equiv (\Lambda e_i)^{(\aleph_0)}$ and therefore $\Lambda e_i \equiv (\Lambda e_i)^{(\aleph_0)}$.

(iii) $\Rightarrow$ (i): Immediate, since $M_0 \models T$. This completes the proof.

Now let us suppose only that Λ is artinian (not necessarily semi-simple). Since no new idea is involved, we will be content with briefly indicating how this case may be reduced to the case where Λ is semi-simple artinian.

Let J be the Jacobson radical of Λ. It is well known that Λ/J is semi-simple artinian. We can then apply (6.1) to Λ/J and obtain a decomposition of Λ/J as a direct sum having the properties mentioned in (6.1):

$$\Lambda/J = \bigoplus_{i=1}^{n} (\Lambda/J)\bar{e}_i$$

(where $\bar{e}_i \in \Lambda/J$ denotes the class of an element $e_i \in \Lambda$). We introduce also for Λ/J an integer m having the properties (6.1a) and (6.1b). For any Λ-module M we define the "semi-simple part of M" $S(M) = \{x \in M : Jx = 0\}$. $S(M)$ can be endowed in a canonical way with the structure of a Λ/J module.

The following results are due to Morita, Kawada and Tachikawa [29] and may be considered as a counterpart of the results of Matlis which were used in § 5.

Theorem 6.6. *Let Λ be artinian and let E, E' be injective Λ-modules. Then E and E' are isomorphic if and only if $S(E)$ and $S(E')$ are isomorphic.*

Theorem 6.6½. *Let Λ be artinian and E be an injective Λ-module.*
a) *E is indecomposable if and only if $S(E)$ is simple.*

b) $E \cong \bigoplus_{i=1}^{n} E_i^{(\alpha_i)}$ *where E_i is a (injective) Λ-module such that* $S(E_i) \cong (\Lambda/J)\bar{e}_i$.

Remark. A consequence of Theorem 6.6½ is that M_0 is elementarily equivalent to $\bigoplus_{i=1}^{m} E_i^{(\aleph_0)}$.

We may then state the main result of this section.

Theorem 6.7. *Let Λ be artinian and M be a Λ-module. The following assertions are equivalent:*
(i) *M is injective and if $M \cong \bigoplus_{i=1}^{m} E_i(\alpha_i)$ is the decomposition given by* (6.6½b), *then*

a) $\alpha_i > 0$ *for all* i.
b) *If* $(\Lambda/J)\bar{e}_i$ *is finite*, $\alpha_i \geq \aleph_0$.
(ii) M *is injective and* $S(M)$ *is a model of* $K^*_{\Lambda/J}$.
(iii) M *is a model of* K^*_Λ.

The proof is left to the reader. We also leave to the reader the task of writing down an explicit set of axioms for K^*_Λ using 6.7(ii). This requires defining a formula $\theta(x)$ which asserts $Jx = 0$ (such a formula may be written in our language since J is finitely generated) and "relativizing the sentences of T (see Theorem 6.4) to θ".

As in §5, we obtain, as a consequence of Theorem 6.7, the following Corollary whose proof we also leave to the reader.

Corollary 6.8. *Let* Λ *be artinian. Every model* N *of* K_Λ *can be embedded in a prime model of* $K^*_\Lambda \cup D(N)$ *which is unique up to isomorphism over* N.

Remark. For the case when Λ is a field, the fact that the theory K^*_Λ is model-complete appears as Theorem 3.6.9 of [31] (with a different terminology). The proof in [31] seems also to cover the case when Λ is a division ring.

7. Algebraically closed structures

In this section $\mathfrak{M}$ denotes a fixed class of structures of the same type. All the structures considered are assumed to belong to $\mathfrak{M}$. As the reader will observe, much of this section is implicit in the work of A. Robinson on model-completeness (see e.g. [32] Chapter 4).

Definition 7.1. A substructure N of a structure M is *algebraically closed in* M if any finite set of equations with constants in N which has a solution in M has a solution in N.

Definition 7.2. A substructure N of a structure M is *existentially closed in* M (one could also say strongly algebraically closed in M) if any finite set of equations and inequations with constants in N which has a solution in M has a solution in N.

In other words, a substructure N of a structure M is existentially closed in M if any primitive sentence (cf. [32], p. 92) which is defined in N holds in M only if it holds in N. It is easy to see that a substructure N of a structure M is existentially closed in M if and only if any existential sentence which is defined in N holds in M only if it holds in N.

The two following results are immediate.

Proposition 7.3. *Let M, N, P be structures such that $P \subseteq N \subseteq M$ and P is algebraically (resp. existentially) closed in N and N is algebraically (resp. existentially) closed in M. Then P is algebraically (resp. existentially) closed in M.*

Proposition 7.4. *If M and N are structures such that $N \prec M$, then N is existentially closed in M.*

Definition 7.5. A structure is *algebraically* (resp. *existentially*) *closed* if it is algebraically (resp. existentially) closed in every extension.

Remarks. 1. The preceding definitions have been introduced for the class of groups, with a slightly different terminology, by W.R.Scott ([33]).

2. For each cardinal α, notions of α-algebraically closed and α-existentially closed structures might be defined, which for $\alpha = \aleph_0$ would coincide with those introduced above.

From now on, we will assume that *$\mathfrak{M}$ is closed under ultrapowers.*

Proposition 7.6. *Let N, M_1, M_2 be structures such that $N \subseteq M_1$, $N \subseteq M_2$. If N is existentially closed in M_1, there exists a structure M which is an elementary extension of M_2 and which contains M_1 such that the following diagram is commutative.*

$$\begin{array}{ccc} N & \subseteq & M_1 \\ \cap\!\mid & & \cap\!\mid \\ M_2 & \prec & M \end{array}$$

Proof. Since N is existentially closed in M_1, there exists (cf. [5], Lemma 3.9, p. 187) an embedding of M_1 in an ultrapower N^I/D of N such that the following diagram is commutative:

$$\begin{array}{ccc} N & \subseteq & M_1 \\ d \big\downarrow & \swarrow & \\ N^I/D & & \end{array}$$

(d denotes the canonical embedding of N into N^I/D).

The following diagram is clearly commutative and all its maps are embeddings:

$$\begin{array}{ccc} N & \subseteq & M_2 \\ \downarrow & & \downarrow \\ N^I/D & \subseteq & M_2^I/D \end{array}$$

One may then take for M the ultrapower M_2^I/D.

Corollary 7.7. *Let N be a substructure of a structure M. If N is existentially closed in M and if M is algebraically (resp. existentially) closed, then N is an algebraically (resp. existentially) closed structure.*

Proof. Let us suppose that M is existentially closed (the case where M is algebraically closed is similar). Let φ be a primitive sentence defined in N which holds in an extension M_1 of N. By the preceding proposition there exists an extension M' of M_1 and of M such that the following diagram is commutative:

$$\begin{array}{ccc} N & \subseteq & M \\ \cap| & & \cap| \\ M_1 & \subseteq & M' \end{array}$$

Since φ holds in M_1, φ holds in M' and therefore, since M is existentially closed, φ holds in M. Since N is existentially closed in M, it follows that φ holds in N. Therefore N is existentially closed and the proof is complete.

Corollary 7.8. *Any elementary substructure of an algebraically (resp. existentially) closed structure is an algebraically (resp. existentially) closed structure.*

Corollary 7.9. *If for every structure $M \in \mathfrak{M}$ every substructure of M is existentially closed in M, then $\mathfrak{M}$ has the amalgamation property.*

Remarks. 1. An immediate consequence of Corollary 7.9 is the fact, already proved in § 2, that a model-complete theory has the amalgamation property.

2. It may be shown (see Appendix) that an algebraically closed substructure in an algebraically closed structure need not be algebraically closed, although this is the case if $\mathfrak{M}$ has the amalgamation property (proof as in Corollary 7.7).

In the remainder of this section, we will assume that $\mathfrak{M}$ is *the* [illegible] *of models of a first-order theory K.* δ will denote the cardinal of [illegible] guage of K.

Proposition 7.10. *If K has a model-companion K^*, the models of K^* are exactly the existentially closed models of K.*

Proof. Let M be a model of K^*. Let P be an extension of M and let φ be an existential sentence defined in M which holds in P. Since K^* is model-consistent relative to K, we may assume that P is a model of K^*. But, in this case, since K^* is model-complete, M is an elementary substructure of P and φ holds in M. It follows that M is existentially closed.

We suppose now that M is an existentially closed model of K. Since K^* is inductive (cf. [26]), it is enough to show that M is a model of the $\forall\exists$ sentences of K^*. Let

$$\varphi = (\forall x_1) \dots (\forall x_m)(\exists y_1) \dots (\exists y_n)\psi(x_1, \dots, x_m, y_1, \dots, y_n)$$

(where ψ is quantifier-free) be an $\forall\exists$ sentence of K^*. Since K^* is model-consistent relative to K, there exists a model Q of K^* containing M. For any elements $a_1, \dots, a_m$ of M the existential sentence $(\exists y_1) \dots (\exists y_n)\psi(a_1, \dots, a_m, y_1, \dots, y_n)$ is defined in M and holds in Q; therefore it must hold in M. It follows that M is a model of φ and the proof is complete.

Proposition 7.11. *The class of algebraically (resp. existentially) closed structures is elementary in the wider sense if (and only if) it is closed under ultraproducts.*

Proof. Let $\mathscr{A}$ denote the class of algebraically (resp. existentially) closed structures. It is enough to show that $\mathscr{A}$ is elementarily closed if $\mathscr{A}$ is closed under ultraproducts (cf. [5], p. 151). Since $\mathcal{M}$ is elementarily closed, it is enough to show that, if $M \in \mathcal{M}$ and $A \in \mathscr{A}$, then $M \equiv A$ implies $M \in \mathscr{A}$. But by Frayne's Lemma $M \equiv A$ implies that M is elementary embeddable in an ultrapower of A. The result then follows from Corollary 7.8.

Before continuing, let us give a simple example. We take for K the theory T of semigroups with an unspecified identity. The non-logical symbols of T are the equality $=$ and a function f of two variables ($f(x, y)$ will be denoted by xy) and the axioms of T are:

$$(\forall x)(\forall y)(\forall z) \ ((xy)z = x(yz))$$

$$(\exists x)(\forall y) \ (xy = yx = y)$$

Since every semigroup M can be embedded in a semigroup M' whose identity is different from the identity of M, it is easy to see that no model of T is existentially closed.

That situation does not arive if we assume that K *is an inductive theory*. We have indeed the following result which was rediscovered by Eli Bers.

Theorem 7.12. *If K is inductive, any model of K can be embedded in an existentially closed structure.*

Proof. With minor modifications, the proof given in [33] works.

Remark. Any model M of cardinal $\beta \geq \delta$ can be embedded in an existentially closed structure of cardinal β. Indeed, if M' is an existentially closed structure containing M, there exists, by the Löwenheim-Skolem theorem, a structure M'' such that: $M \subseteq M''$, $M'' \prec M'$, $\mathrm{Card}(M'') = \beta$. By Corollary 7.8, M'' is existentially closed.

Corollary 7.13. *Let K be inductive. K has a model-companion if and only if the existentially closed models of K constitute an elementary class in the wider sense.*

Proof. If K has a model-companion, it follows from Proposition 7.10 that the existentially closed models of K constitute an elementary class in the wider sense.

To show the converse, let us denote by K^* the first-order theory of the existentially closed models of K. One has clearly $K \subseteq K^*$ and it follows from Theorem 7.12 that K^* is model-consistent relative to K. By applying the model-completeness test ([32], p. 92), one sees immediately that K^* is model-complete. Therefore, K^* is the model-companion of K and the proof is complete.

Corollary 7.14. *If K has a model-completion, then the algebraically closed models of K constitute an elementary class in the wider sense.*

Proof. By Proposition 7.11 it is enough to show that the class of algebraically closed structures is closed under ultraproducts. Let $M^* = \prod_{i \in I} M_i/D$ be an ultraproduct of a family of algebraically closed structures. For every $i \in I$ there exists an existentially closed structure N_i which contains M_i (cf. Proposition 7.10). Let us denote by N^* the ultraproduct $\prod_{i \in I} N_i/D$. For every extension Q of M^*, there exists an extension P of N^* and Q such that the following diagram is commutative (since K has the amalgamation property, cf. Lemma 2.1):

$$\begin{array}{ccc} M^* & \subseteq & N^* \\ \cap\!\mid & & \cap\!\mid \\ Q & \subseteq & P \end{array}$$

It follows that every finite set of equations φ defined in M^* which holds in Q holds in P and therefore holds in N^* since N^* is existentially closed. One then finishes the proof as in Lemma 4.2.

The following short proof of a Theorem of P.Lindström [26] resulted from a discussion with E.Fisher.

Corollary 7.15. *If K satisfies the three following conditions:*
(i) *K is categorical in a cardinal $\alpha \geq \delta$;*
(ii) *K is inductive;*
(iii) *all models of K are infinite;*
then K is model-complete.

Proof. By the model-completeness test ([32], p. 92), it is clearly enough to show that every model of K is existentially closed. We will show that, under the assumptions (i) and (ii), every infinite model M of K is existentially closed. By Corollary 7.8, since every infinite model of K is an elementary substructure of a model of cardinal $\geq \alpha$, we may restrict ourselves to the case where the cardinal of M is $\geq \alpha$. If the cardinal of M is equal to α, the result is an immediate consequence of Theorem 7.12 and of the remark following it. In the general case, if the cardinal of M is $\geq \alpha$, for any primitive sentence φ defined in M, there exists a substructure M' of M of cardinal α such that φ is defined in M'. If φ holds in an extension of M, φ holds in M' since M' is existentially closed, and therefore φ holds in M. The proof is complete.

Our next result deals with the inductive hull of K (cf. [21]). It has been shown by K.Kaiser that if K is inductive there exists a (unique) theory $\widetilde{K}$, called the inductive hull of K, which has the following properties:

(i) $K \subseteq \widetilde{K}$
(ii) $\widetilde{K}$ is model-consistent relative to K.
(iii) $\widetilde{K}$ is inductive.
(iv) Any theory satisfying (i), (ii) and (iii) is contained in $\widetilde{K}$.

Corollary 7.16. *If K is inductive, $\widetilde{K}$ is the deductive closure of the set of $\forall\exists$ sentences which hold in all existentially closed models of K.*

Proof. Let us denote by K' the deductive closure of the set of $\forall\exists$ sentences which hold in all existentially closed models of K. It is obvious that K' satisfies (i) and (iii) and it follows from Theorem 7.12 that K' satisfies (ii). We have then $K' \subseteq \widetilde{K}$.

To show the inclusion $\widetilde{K} \subseteq K'$, it is enough to prove that, for any $\forall\exists$ sentence φ of $\widetilde{K}$ and any existentially closed model M of K, φ holds in M. This is done by embedding M in a model of φ, which is possible since $\widetilde{K}$ is model-consistent relative to K.

We are now going to restate some of our results about modules in the terminology of this section. The algebraically closed modules were called absolutely pure in the remarks at the end of § 3. It was found that the existentially closed modules constitute an elementary class in the wider sense if and only if the algebraically closed modules constitute an elementary class in the wider sense (compare with Corollary 7.14 of this section) if and only if Λ is coherent. It was proved that each existentially closed module is injective if and only if Λ is noetherian.

It may be shown that for any ring Λ the inductive hull of the theory K_Λ of Λ-modules is the set of consequences of the set of $\forall\exists$ sentences which hold in an explicitly given module, namely the module M_0 which was defined in § 4.

It may be of interest to point out that a Z-module, namely an abelian group, is algebraically closed if and only if it is divisible, i.e. injective, and that a divisible abelian group, for example the additive group of rationals Q, is not always existentially closed. It then seems natural to look at the theory of groups, since it is well-known that a group is injective (in the categorical sense of the word) if and only if it is trivial (see e.g. [15]) and since it has been shown by B.H.Neumann that a non-trivial group is existentially closed if and only if it is algebraically closed ([30]). In this connection we have

Theorem 7.17. *The theory T_1 of groups has no model-companion.*

Proof. We find it convenient (but this is inessential) to axiomatize T_1 with a function symbol of two variables (for the multiplication), a function symbol of one variable (for the inverse) and a constant symbol e (for the identity). There is no need to write the axioms.

We prove the theorem by showing that the class of existentially closed groups is not closed under ultraproducts and by applying Corollary 7.13. More precisely, we show that, for any ω-incomplete ultrafilter D over any set I and any existentially closed group G, the ultrapower G^I/D is not existentially closed.

For every positive integer m the sentence

$$(\varphi_m)\ (\exists x)(x^m = e \wedge \bigwedge_{j<m} x^j \neq e)$$

holds in an extension H of G (one may take $H = G \times Z(m)$). It follows that for every positive integer m, G has an element of order m.

Now let $(u_n)_{n\in\omega}$, $(v_n)_{n\in\omega}$ be two stricly increasing sequences of positive integers such that for any $n \in \omega$ $u_n \neq v_n$. For each $n \in \omega$ let a_n (resp. b_n) be an element of G of order u_n (resp. v_n). Let $(I_n)_{n\in\omega}$ be a sequence of subsets of I such that:

$$\forall m \neq n \in \omega (I_m \cap I_n = \phi); \qquad \forall_m \in \omega (I_m \notin D);$$

$$I = \bigcup_{n\in\omega} I_n .$$

Let a (resp. b) be the element of G^I whose ith component a_i (resp. b_i) is equal to a_n (resp. b_n) if $i \in I_n$. Let $\bar{a}$ (resp. $\bar{b}$) be the image of a (resp. b) under the canonical homomorphism of G^I onto G^I/D. It is easy to verify that $\bar{a}$ and $\bar{b}$ are not of finite order. It then follows from a result due to G.Higman, B.H.Neumann and H.Neumann ([18], Corollary of Theorem I′, p. 249) that the equation $x^{-1}\bar{a}x = \bar{b}$ has a solution in an extension of G^I/D. The proof will be complete if we show that the equation $x^{-1}\bar{a}x = \bar{b}$ has no solution in G^I/D.

Let us assume that the equation $x^{-1}\bar{a}x = \bar{b}$ has a solution in G^I/D. In this case the equation $x^{-1}a_ix = b_i$ would have a solution in G for some $i \in I$, which is clearly impossible since a_i and b_i do not have the same order.

Remarks. 1. This result, namely the fact that the existentially closed groups do not constitute an elementary class in the wider sense, answers a question implicit in ([22], p. 129).

2. From Theorem 7.17, it follows that every theory T' such that
(i) $T_1 \subseteq T'$,
(ii) T' is model-consistent relative to T_1
has a model which is not an existentially closed group (if not, T' would be the model-companion of T_1). Therefore if one takes for T' the inductive hull of T_1 or the forcing-companion of T_1 ([4]), T' has a (infinite) model which is not existentially closed.

3. The following question has been raised by Eli Bers and seems open: are two existentially closed groups elementarily equivalent?

4. Let us denote by T_2 the theory of monoids (i.e. the theory of semi-groups with a specified identity e) whose axioms are:

$$(\forall x)(\forall y)(\forall z)\,((xy)z = x(yz))$$

$$(\forall x)(xe = ex = x)$$

By appropriately modifying the proof of Theorem 7.17 and by using, instead of the Higman-Neumann-Neumann result, a generalization of it (Theorem 1 of [19]), one may show that T_2 has no model-companion.

Appendix

We given an example of a theory T with models $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, $\mathfrak{A}$ is algebraically closed in $\mathfrak{B}$, $\mathfrak{B}$ is algebraically closed, but $\mathfrak{A}$ is not algebraically closed.

Let L be a language with equality and two binary relations P and Q. Let T be the deductive closure of

(1) $\forall x\, Q(x, x)$

(2) $\forall x[(\exists y(P(x, y) \vee P(y, x)) \Rightarrow \forall z((Q(x, z) \vee Q(z, x)) \Rightarrow x = z)]$

Let $B = \{b_1, b_2, b_3\}$ and let $\mathfrak{B}$ = the model of T with universe B on which the relations $Q(b_1, b_2)$, $Q(b_2, b_1)$, $Q(b_1, b_1)$, $Q(b_2, b_2)$, $Q(b_3, b_3)$, and $P(b_3, b_3)$ are imposed. We claim $\mathfrak{B}$ is algebraically closed. Indeed, suppose $\mathfrak{C} = \langle C, ...\rangle$ is an extension of $\mathfrak{B}$ such that in $\mathfrak{C}$ there is a solution to a system $\mathcal{S}$ of equations of the form

$$Q(\alpha, \beta) \quad \text{or} \quad P(\alpha, \beta)$$

where the α, β are either variables x_i or elements b_j of B. We have to show $\mathcal{S}$ has a solution in $\mathfrak{B}$. We can assume no variable x_i occurs in both a relation of the form $P(\alpha, \beta)$ and in one of the form $Q(\alpha, \beta)$. In fact, if we have $P(x_i, \beta)$ and $Q(x_i, \beta')$ (the other possibilities are handled similarly), then by (2), $x_i = \beta'$; thus we can (by (1)) remove $Q(x_i, \beta')$ from $\mathcal{S}$ and eliminate x_i from $\mathcal{S}$ by replacing x_i by β' everywhere. If b_3 occurs in a relation of the form $Q(x_i, b_3)$ or $Q(b_3, x_i)$ then by (2) $x_i = b_3$. Thus we can assume that b_3 does not occur in any relation $Q(\alpha, \beta)$ of $\mathcal{S}$. Note also that b_1 and b_2 do not occur in any relation $P(\alpha, \beta)$ of $\mathcal{S}$. (This follows from (2) because $Q(b_1, b_2)$). Hence we get a solution to $\mathcal{S}$ in $\mathfrak{B}$ by setting $x_i = b_1$ for any variable x_i occurring in a relation of the form $Q(x_i, \beta)$ or $Q(\alpha, x_i)$ and by setting $x_i = b_3$ for any x_i occurring in a relation of the form $P(x_i, \beta)$ or $P(\alpha, x_i)$.

Now let $\mathfrak{A}$ be the substructure of $\mathfrak{B}$ with universe $A = \{b_1, b_3\}$. Then $\mathfrak{A}$ is a model of T (T is universal) and an argument similar to that above shows that $\mathfrak{A}$ is algebraically closed in $\mathfrak{B}$. On the other hand, $\mathfrak{A}$ is not algebraically closed; in fact, there is a model $\mathcal{D} = \langle D, ... \rangle$ of T which is an extension of $\mathfrak{A}$ and contains an element $c \in D - A$ such that $\mathcal{D} \models P[c, b_1]$. Thus $P(x, b_1)$ has a solution in $\mathcal{D}$ but not in $\mathfrak{A}$.

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