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On the Cohomology of Ring Extensions

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INTRODUCTION

In this article we are primarily concerned with the relativized cohomology theory of ring extensions, as initiated by G. Hochschild [14]. Our main objective is to lay the foundation for a general vanishing theory of relativized extension functors. The significance of such a theory rests on the unification it provides for the various classical results from the cohomology theories of groups, associative algebras, and Lie algebras. The approach chosen here heavily emphasizes locally nilpotent operators as an underlying principle for vanishing and isomorphism theorems. Although in algebraic homology theory these concepts have appeared only recently (cf. [5]), they occur implicitly in singular homology theory of topological spaces. The conventional proof of the excision theorem, for instance, resorts, by means of barycentric subdivision, to the construction of a locally nilpotent operator that is chain homotopic to the identity.

In Section 1 we study the question to what an extent properties of operators are inherited by resolutions. It is shown that resolutions defined by injective envelopes and projective covers are suitable tools for ordinary homology. In the framework of relative homological algebra of a given ring extension $R{:}S$ the availability of standard resolutions allows the application of more computational techniques. We show that the commutator algèbra of the centralizer $C_R(S)$ of S in R operates on these resolutions in a natural fashion. This result in conjunction with a certain homotopy is subsequently utilized to show that the associated action on the spaces $\mathrm{Ext}^n_{(R,S)}(M, N)$ is trivial. The succeeding section combines these techniques with basic facts concerning locally finite operators to establish various vanishing theorems for extension functors.

The third section deals with the study of the cohomology theory of Frobenius extensions. We address questions emerging from recent developments in the cohomology theories of finite dimensional modular [9] and finite dimensional restricted Lie algebras [7–9]. Our results are closely

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related to those by Pareigis [22] and Feldvoss [10], who studied the complete cohomology of Frobenius–Hopf algebras. Motivated by well-known facts concerning the cohomology theory of finite groups, we introduce corestriction mappings that naturally extend the usual trace operator of the underlying Frobenius extension. When combined with the change of rings transformation these maps give rise to new vanishing conditions for ordinary extension functors as well as for the complete relative cohomology. Aside from classical applications, such as the cohomology theory of groups and finite dimensional Hopf algebras, our results are also useful in the context of the ordinary cohomology theory of modular Lie algebras.

The concluding section illustrates the utility of our methods for certain module-theoretic questions, thereby complementing the applications touched upon in previous papers [5, 6].

1. Resolutions

In the sequel we study a K-algebra R, defined over a commutative ring K with identity. Given an R-module M, r_M denotes the left multiplication effected by the element $r \in R$. The associated commutator algebra of R will be designated R^-. As usual $\operatorname{ad} u\colon R \to R$; $(\operatorname{ad} u)(r) := [u, r] = ur - ru$ is the adjoint representation of the Lie algebra R^-.

THEOREM 1.1. *Suppose u is an element of R such that $\operatorname{ad} u\colon R \to R$ is locally nilpotent. Let N be an R-module. Then the following statements hold:*

(1) *If u_N is injective and E is an injective envelope of N, then u_E is invertible.*

(2) *If u_N is invertible, then N possesses an injective resolution $(E_i, \partial_i)_{i \geqslant 0}$ such that u_{E_i} is invertible $\forall\ i \geqslant 0$.*

Proof. (1) Owing to the local nilpotence of the adjoint representation of u, the Fitting-0-space $E_0(u_E)$ of the transformation u_E is an R-submodule of E which, by virtue of the injectivity of u_N, intersects N trivially. Since N is an essential submodule of E, it follows that $E_0(u_E) = (0)$. In particular, u_E is injective.

In order to verify the surjectivity of u_E, let m be an element of E, and $r \in R$ such that $ru = 0$. As $\operatorname{ad} u$ is locally nilpotent there is $n \in \mathbb{N}$ such that $0 = (\operatorname{ad} u)^n (r) = u^n r$. Consequently, $u^n r \cdot m = 0$ and $r \cdot m = 0$, by the injectivity of u_E. It follows that the R-linear mapping $f\colon Ru \to E$; $f(ru) = r \cdot m$ is well-defined. As E is injective, f extends to a morphism $g\colon R \to E$ of R-modules. In particular, $m = f(u) = g(u) = u \cdot g(1) = u_E(g(1))$.

(2) We construct E_i inductively, letting E_0 be an injective envelope of

$E_{-1} := N$, $\partial_{-1}: N \to E_0$ the canonical inclusion. Now let $i > 0$, put $E_{-2} := (0)$, and suppose that E_j has already been constructed for $j < i$. Since u operates invertibly on E_{i-1} and E_{i-2}, it readily follows from the exactness of the sequence $E_{i-3} \to E_{i-2} \to E_{i-1}$ that u operates invertibly on $E_{i-1}/\partial_{i-2}(E_{i-2})$. Let E_i be an injective envelope of $E_{i-1}/\partial_{i-2}(E_{i-2})$ and let ∂_{i-1} be the canonical map $E_{i-1} \to E_i$. Part (1) now ensures the invertibility of u_{E_i}, thereby completing the inductive step. ∎

We proceed by considering the dual notion of a projective cover. Our methods partly rest on the following subsidiary result:

For an endomorphism $f: V \to V$ of a K-module V we denote by $V_1(f) := \bigcap_{n \geqslant 1} f^n(V)$ the Fitting-1-component of V relative to f.

LEMMA 1.2. *Suppose g is an endomorphism of V and there is a natural number k such that $(\mathrm{ad} f)^k (g) = 0$. Then $V_1(f)$ is invariant under g.*

Proof. Write W for $V_1(f)$. Note that $f(W) = W$, and that W is contained in $f^n(V)$ for every natural number n. The result evidently holds for $k = 1$, and we proceed by induction on k. The inductive step goes as follows.

Suppose the result has been established for some k, and assume that $(\mathrm{ad} f)^{k+1} (g) = 0$. Applying our inductive hypothesis with $g \circ f - f \circ g$ in the place of g, we conclude that W is stabilized by $f \circ g - g \circ f$. Hence $g(W) = g(f(W))$ is contained in $f(g(W)) + W$. Feeding this relation back into itself $n-1$ times, we deduce that, for every natural number n, $g(W)$ is contained in $f^n(V)$. ∎

DEFINITION. A ring R is said to be *left perfect* if it satisfies the descending chain condition on principal right ideals.

THEOREM 1.3. *Suppose that R is left perfect and let $u \in R$ be an element such that* $\mathrm{ad}\, u$: *$R \to R$ is locally nilpotent. Let M be an R-module such that u_M is surjective. Then the following statements hold:*

(1) *If $P(M)$ is a projective cover of M, then $u_{P(M)}$ is surjective. If u is not a zero divisor, then $u_{P(M)}$ is bijective.*

(2) *Suppose that u is not a zero divisor and that u_M is invertible. Then there exists a projective resolution $(P_i, \partial_i)_{i \geqslant 0}$ such that u_{P_i} is invertible $\forall$ $i \geqslant 0$.*

Proof. (1) Let $\varepsilon: P(M) \twoheadrightarrow M$ be the canonical map. Owing to (1.2), $P(M)_1 (u_{P(M)})$ is an R-submodule of $P(M)$. Consider the descending sequence $(u^n R)_{n \geqslant 1}$ of right ideals of R. Since R is left perfect, there exists a natural number n satisfying $u^n R = u^{n+1} R$. In particular, $u^n = u^{n+1} r$ for a suitably chosen element $r \in R$. It follows that $P(M)_1\,(u_{P(M)}) = u^n_{P(M)}(P(M))$. Now let p be an element of $P(M)$ and write $\varepsilon(p) = u^n_M(x)$. If $q \in P(M)$ satisfies $\varepsilon(q) = x$, then we obtain $\varepsilon(p) = u^n_M(\varepsilon(q)) = \varepsilon(u^n_{P(M)}(q))$. By the above observation there results a representation $P(M) = \ker \varepsilon + P(M)_1\,(u_{P(M)})$. Since $P(M)$ is a projective cover, we conclude that $P(M) = P(M)_1\,(u_{P(M)})$. In particular, $u_{P(M)}$: $P(M) \to P(M)$ is surjective. If u is not a zero divisor, then u acts injectively on any free R-module, hence on the projective module $P(M)$.

(2) We proceed inductively, setting $P_{-2} := (0)$ and $P_{-1} := M$. By virtue of Bass' theorem [1, p. 315], M possesses a projective cover $P(M)$ that is mapped onto M via ε: $P(M) \twoheadrightarrow M$. The first part of our proof ensures the invertibility of $u_{P(M)}$. Now let $i > 0$ and suppose we already have an exact sequence

$$P_{i-1} \to P_{i-2} \to \cdots \to P_{-1} \to (0)$$

of projective R-modules with the desired property. Let $X := \ker \partial_{i-1}$. As u acts invertibily on both P_{i-1} and P_{i-2}, it readily follows that u_X is bijective. We may now complete the argument by letting P_i be a projective cover of X. ∎

Throughout the remainder of this section we consider a pair (R, S) consisting of a K-algebra R and a subalgebra $S \subset R$. Following Hochschild [14], we begin by constructing the so-called standard (R, S)-projective resolution of a given R-module M. For $n \geqslant -1$, let $S_n(R)$ denote the $(n+1)$-fold tensor product of the S-bimodule R over S. For $n \geqslant 0$ we endow the K-module $X_n := S_n(R) \otimes_S M$ with the structure of a left R-module via

$$a \cdot (a_1 \otimes \cdots \otimes a_{n+1} \otimes m) := aa_1 \otimes \cdots \otimes a_{n+1} \otimes m$$

and consider $X := \oplus_{n \geqslant 0} X_n$. Define a K-linear map d: $X \to X$ of degree -1 by means of $d_0 := 0$ and

$$d_n(a_1 \otimes \cdots \otimes a_{n+1} \otimes m) = \sum_{i=1}^{n} (-1)^{i+1}\, a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \otimes m$$
$$+ (-1)^n\, a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} m \qquad \forall n \geqslant 1.$$

This mapping is obviously also R-linear. Let s: $X \to X$ be the S-linear map of degree 1 given by

$$s_n(a_1 \otimes \cdots \otimes a_{n+1} \otimes m) := 1 \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes m.$$

An elementary calculation shows that

$$d_{n+1} \circ s_n + s_{n-1} \circ d_n = \mathrm{id}_{X_n} \qquad \forall n \geqslant 1. \qquad (*)$$

By using $(*)$ in conjunction with the R-linearity of d, one now verifies inductively the identity $d_{n-1} \circ d_n = 0 \ \forall n \geqslant 1$. This in turn implies $\ker d_n = \operatorname{im} d_{n+1} \ \forall n \geqslant 1$. Consider the R-linear mapping $\varepsilon: X_0 \to M$; $\varepsilon(a \otimes m) = am$ as well as the S-linear map $s_{-1}: M \to X_0$; $s_{-1}(m) = 1 \otimes m$. Then we have $d_1 \circ s_0 + s_{-1} \circ \varepsilon = \mathrm{id}_{X_0}$. It now follows from Lemma 2 of [14] that the complex (X, d) together with $\varepsilon: X_0 \to M$ forms an (R, S)-projective resolution of the R-module M. We shall refer to $X(M) := X$ as the (R, S)-*projective standard complex of* M.

Let $C_R(S)$ denote the centralizer of S in R. For every element $u \in C_R(S)$ we define, in analogy with [5, pp. 653 f], K-linear maps $\rho_u, \Theta_u: X \to X$ of degrees 1 and 0, respectively, via

$$\begin{aligned}\rho_u^{(n)}(a_1 \otimes \cdots \otimes a_{n+1} \otimes m) &:= \sum_{i=1}^{n+1} (-1)^{i-1} a_1 \otimes \cdots \otimes a_i \otimes u \otimes a_{i+1} \\ &\quad \otimes \cdots \otimes a_{n+1} \otimes m, \\ \Theta_u^{(n)}(a_1 \otimes \cdots \otimes a_{n+1} \otimes m) &:= a_1 u \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes m \\ &\quad + \sum_{i=2}^{n+1} a_1 \otimes \cdots \otimes a_{i-1} \otimes [a_i, u] \otimes a_{i+1} \\ &\quad \otimes \cdots \otimes a_{n+1} \otimes m - a_1 \otimes \cdots \otimes a_{n+1} \otimes um.\end{aligned}$$

Both mappings are homomorphisms of R-modules. We let l_u denote the left multiplication on X effected by $u \in C_R(S)$ and put $\tau_u := l_u \circ s$. An elementary computation shows that

(1) $\rho_u \circ s = s \circ \tau_u - s \circ \rho_u \ \forall u \in C_R(S)$, and

(2) $\Theta_u \circ s = \tau_u - s \circ l_u + s \circ \Theta_u \ \forall u \in C_R(S)$.

PROPOSITION 1.4. $d \circ \rho_u + \rho_u \circ d = \Theta_u \ \forall u \in C_R(S)$.

Proof. We shall establish inductively the validity of

$$d_{n+1} \circ \rho_u^{(n)} + \rho_u^{(n-1)} \circ d_n = \Theta_u^{(n)} \qquad \forall n \geqslant 0.$$

The cases $n = 0, 1$ can be verified directly ($\rho_u^{(-1)} = 0$). Now suppose that $n > 1$. Since all mappings involved are R-linear, it suffices to verify the identity on the generating set $s_{n-1}(X_{n-1})$ of X_n. Using $(*)$ and (1) we obtain

$$\begin{aligned}d_{n+1} \circ \rho_u^{(n)} \circ s_{n-1} &= d_{n+1} \circ s_n \circ \tau_u^{(n-1)} - d_{n+1} \circ s_n \circ \rho_u^{(n-1)} = \tau_u^{(n-1)} - \rho_u^{(n-1)} \\ &\quad + s_{n-1} \circ d_n \circ \rho_u^{(n-1)} - s_{n-1} \circ d_n \circ \tau_u^{(n-1)}\end{aligned}$$

as well as

$$\begin{aligned}\rho_u^{(n-1)}\circ d_n\circ s_{n-1} &= \rho_u^{(n-1)}-\rho_u^{(n-1)}\circ s_{n-2}\circ d_{n-1}\\ &= \rho_u^{(n-1)}-s_{n-1}\circ\tau_u^{(n-2)}\circ d_{n-1}+s_{n-1}\circ\rho_u^{(n-2)}\circ d_{n-1}.\end{aligned}$$

Addition now yields, observing $d\circ l_u=l_u\circ d$,

$$\begin{aligned}&d_{n+1}\circ\rho_u^{(n)}\circ s_{n-1}+\rho_u^{(n-1)}\circ d_n\circ s_{n-1}\\ &\quad=\tau_u^{(n-1)}+s_{n-1}\circ(d_n\circ\rho_u^{(n-1)}+\rho_u^{(n-2)}\circ d_{n-1})-s_{n-1}\circ l_u^{(n-1)}.\end{aligned}$$

By inductive hypothesis the right-hand side coincides with

$$\tau_u^{(n-1)}+s_{n-1}\circ\Theta_u^{(n-1)}-s_{n-1}\circ l_u^{(n-1)}$$

and the assertion now follows from (2). ∎

We let $\mathscr{H}\mathrm{om}_R(X,\cdot)$ and $\mathscr{H}\mathrm{om}_S(X,\cdot)$ denote the graded Hom functor defined on the category of R- and S-complexes, respectively. Let N be another R-module. According to [14] the relative extension functor $\mathrm{Ext}^n_{(R,S)}(M,N)$ is the nth homology group of the positive complex $(\mathscr{H}\mathrm{om}_R(X,N),\delta)$, where $\delta_n:=\mathrm{Hom}_R(d_{n+1},\mathrm{id}_N)$ $\forall n\geqslant 0$. For an element $a\in C_R(S)$ consider the mapping μ_a: $\mathscr{H}\mathrm{om}_S(X,N)\to\mathscr{H}\mathrm{om}_S(X,N)$ defined via

$$\begin{aligned}&\mu_a^{(n)}(f)(a_1\otimes a_2\cdots\otimes a_{n+1}\otimes m)\\ &\quad=af(a_1\otimes a_2\cdots\otimes a_{n+1}\otimes m)\\ &\qquad-\sum_{i=1}^{n+1}f(a_1\otimes\cdots\otimes a_{i-1}\otimes[a,a_i]\otimes a_{i+1}\otimes\cdots\otimes a_{n+1}\otimes m)\\ &\qquad-f(a_1\otimes\cdots\otimes a_{n+1}\otimes am).\end{aligned}$$

If we endow X with the tensor product representation of the Lie algebra $C_R(S)^-$, then $a\mapsto\mu_a$ defines a representation of $C_R(S)^-$ in the graded K-module $\mathscr{H}\mathrm{om}_S(X,N)$.

THEOREM 1.5. *Let N be an R-module. The the following statements hold:*

(1) *$\mathscr{H}\mathrm{om}_R(X,N)$ is a $C_R(S)^-$-submodule of $\mathscr{H}\mathrm{om}_S(X,N)$.*

(2) *The boundary operator δ: $\mathscr{H}\mathrm{om}_R(X,N)\to\mathscr{H}\mathrm{om}_R(X,N)$ is a homomorphism of $C_R(S)^-$-modules.*

(3) *The representation Γ: $C_R(S)^-\to gl(\mathscr{H}\mathrm{om}_R(X,N))$, $\Gamma(u)=\mu_u$, induces the trivial representation on $\mathrm{Ext}^n_{(R,S)}(M,N)$ $\forall n\geqslant 0$.*

Proof. (1) One readily verifies that $\mu_u|_{\mathscr{H}om_R(X,N)} = \mathscr{H}om_R(\Theta_u, \mathrm{id}_N)$ $\forall u \in C_R(S)$.

(2) By employing the R-linearity of ρ_u we define mappings $\gamma_u := \mathscr{H}om_R(\rho_u, \mathrm{id}_N)$ via $\gamma_u^{(n)} := \mathrm{Hom}_R(\rho_u^{(n-1)}, \mathrm{id}_N)$ and obtain, by applying the contravariant and additive functor $\mathscr{H}om_R(\cdot, N)$ to (1.4), the identity

$$\delta_{n-1} \circ \gamma_u^{(n)} + \gamma_u^{(n+1)} \circ \delta_n = \mu_u^{(n)} \qquad \forall n \geqslant 0.$$

We therefore conclude that $\delta \circ \gamma_u + \gamma_u \circ \delta = \mu_u$ $\forall u \in C_R(S)$. This in turn yields $\delta \circ \mu_u = \delta \circ \gamma_u \circ \delta = \mu_u \circ \delta$, thereby qualifying δ as a homomorphism of $C_R(S)^-$-modules.

(3) The subspaces $\mathrm{Hom}_R(X_n, N)$ are $C_R(S)^-$-submodules of $\mathscr{H}om_R(X, N)$. By (2) the same applies to $\ker \delta_n$ and $\mathrm{im}\, \delta_{n-1}$. The above equation then yields $\mu_u^{(n)}(f) = (\delta_{n-1} \circ \gamma_u^{(n)})(f) \in \mathrm{im}\, \delta_{n-1}$ $\forall f \in \ker \delta_n$. Hence $C_R(S)^-$ operates trivially on $\mathrm{Ext}^n_{(R,S)}(M, N) = \ker \delta_n / \mathrm{im}\, \delta_{n-1}$ $\forall n \geqslant 0$ $(\delta_{-1} = 0)$. ∎

We shall briefly illustrate the corresponding facts for the standard (R, S)-injective resolution of a given R-module N. For $n \geqslant 0$ we define

$$\begin{aligned} C^n(N) := \{ f\colon R^{n+1} \to N;\, & f\ K\text{-multilinear}, f(sr_1, r_2, ..., r_{n+1}) \\ &= sf(r_1, r_2, ..., r_{n+1}), f(r_1, ..., r_i s, r_{i+1}, ..., r_{n+1}) \\ &= f(r_1, ..., r_i, sr_{i+1}, ..., r_{n+1})\ \forall s \in S \\ &\ \forall r_1, ..., r_{n+1} \in R,\ 1 \leqslant i \leqslant n \}. \end{aligned}$$

$C^n(N)$ obtains the structure of a left R-module via

$$(r \cdot f)(r_1, ..., r_{n+1}) = f(r_1, ..., r_{n+1} r).$$

Setting $C^{-1}(N) := N$ we see that the map $\psi\colon \mathrm{Hom}_S(R, C^{n-1}(N)) \to C^n(N)$; $\psi(f)(r_1, ..., r_{n+1}) = f(r_{n+1})(r_1, ..., r_n)$ is an isomorphism of R-modules for $n \geqslant 0$. Lemma 1 of [14] now ensures the (R, S)-injectivity of $C^n(N)$ $\forall n \geqslant 0$. Put $C(N) := \bigoplus_{n \geqslant 0} C^n(N)$. The coboundary operator on the complex $C(N)$ is the degree 1 mapping $\partial\colon C(N) \to C(N)$, which is defined via

$$\begin{aligned} \partial_n(f)(r_1, ..., r_{n+2}) &= r_1 f(r_2, ..., r_{n+2}) \\ &\quad + \sum_{i=1}^{n+1} (-1)^i f(r_1, ..., r_i r_{i+1}, ..., r_{n+2}) \qquad \forall f \in C^n(N). \end{aligned}$$

Note that ∂ is a homomorphism of R-modules. In order to show that the sequence

$$(0) \to N \to C^0(N) \to \cdots \to C^n(N) \to C^{n+1}(N) \to \cdots$$

is (R, S)-exact we define an S-linear contracting homotopy t of degree -1 by means of

$$t_n(f)(r_1, ..., r_n) = (-1)^n f(r_1, ..., r_n, 1) \qquad \forall f \in C^n(N), n \geq 0.$$

An elementary computation shows that

$$\partial_{n-1} \circ t_n + t_{n+1} \circ \partial_n = \mathrm{id}_{C^n(N)} \qquad \forall n \geq 1.$$

To verify the identify $\partial_{n+1} \circ \partial_n = 0$ $\forall n \geq 0$ by induction on n, we note that a map $f \in C^n(N)$ is the zero map if and only if $t_n(r \cdot f) = 0$ for every $r \in R$. The inductive step can now be obtained from the following calculation, keeping in mind the R-linearity of ∂:

$$\begin{aligned} t_{n+2} \circ \partial_{n+1} \circ \partial_n &= (\mathrm{id}_{C^{n+1}(N)} - \partial_n \circ t_{n+1}) \circ \partial_n = \partial_n - \partial_n \circ t_{n+1} \circ \partial_n \\ &= \partial_n - \partial_n \circ (\mathrm{id}_{C^n(N)} - \partial_{n-1} \circ t_n) = 0. \end{aligned}$$

If ι denotes the natural map $N \to C^0(N)$, then $\iota \circ t_0 + t_1 \circ \partial_0 = \mathrm{id}_{C^0(N)}$. As a result the above sequence constitutes an (R, S)-injective resolution of N. We shall refer to $C(N)$ as the *standard* (R, S)-*injective complex of* N.

Now let u be an element of $C_R(S)$. Define a map $\rho_u\colon C(N) \to C(N)$ of degree -1 via

$$\rho_u^{(n)}(f)(r_1, ..., r_n) = \sum_{i=0}^{n-1} (-1)^i f(r_1, ..., r_i, u, r_{i+1}, ..., r_n) \qquad \forall f \in C^n(N), n \geq 1$$

and set $\rho_u^{(0)} = 0$. We also consider the degree 0 operator $\Theta_u\colon C(N) \to C(N)$ given by

$$\begin{aligned} \Theta_u^{(n)}(f)(r_1, ..., r_{n+1}) = uf(r_1, ..., r_{n+1}) - \sum_{i=1}^{n+1} f(r_1, ..., (\mathrm{ad}\, u)(r_i), ..., r_{n+1}) \\ - f(r_1, ..., r_{n+1}u) \qquad \forall f \in C^n(N), n \geq 0. \end{aligned}$$

Setting $l_u(f)(r_1, ..., r_n) = f(r_1, ..., r_n u)$, and $\tau_u := t \circ l_u$ we obtain the identities

$$t \circ \rho_u = -\rho_u \circ t - \tau_u \circ t \tag{1}$$

$$t \circ \Theta_u = \Theta_u \circ t - \tau_u + l_u \circ t. \tag{2}$$

PROPOSITION 1.6. $\partial \circ \rho_u + \rho_u \circ \partial = \Theta_u$ $\forall u \in C_R(S)$.

Proof. As in the proof of (1.4) one proceeds inductively. Since all mappings involved are R-linear it suffices to verify the identity $t \circ \partial \circ \rho_u + t \circ \rho_u \circ \partial = t \circ \Theta_u$. This can be done by the method employed in the proof of (1.4). ∎

We note that $C^n(N)$ carries the structure of a $C_R(S)^-$-module via

$$(u\cdot f)(r_1, ..., r_{n+1}) := uf(r_1, ..., r_{n+1}) - \sum_{i=1}^{n+1} f(r_1, ..., (\operatorname{ad} u)(r_i), ..., r_{n+1}).$$

If M is an R-module there thus results a canonical $C_R(S)^-$-module structure on the complex $(\mathscr{H}om_S(M, C(N), \delta)$, where $\delta_n := \operatorname{Hom}_R(\operatorname{id}_M, \partial_n)\ \forall n \geqslant 0$. We shall denote the action of u by μ_u. The arguments employed in the proof of (1.5) may now be adapted to yield:

THEOREM 1.7. *Let M be an R-module. Then the following statements hold*

(1) *$\mathscr{H}om_R(M, C(N))$ is a $C_R(S)^-$-submodule of $\mathscr{H}om_S(X, C(N))$.*

(2) *The boundary operator $\delta\colon \mathscr{H}om_R(M, C(N)) \to \mathscr{H}om_R(M, C(N))$ is a homomorphism of $C_R(S)^-$-modules.*

(3) *The representation $\Gamma\colon C_R(S)^- \to gl(\mathscr{H}om_R(M, C(N)); \Gamma(u) = \mu_u$ induces the trivial representation on $\operatorname{Ext}^n_{(R,S)}(M, N)\ \forall n \geqslant 0$.*

DEFINITION. An endomorphism $f\colon V \to V$ of a K-module V is called *locally finite* if for every $v \in V$ there exists a finitely generated submodule *$\mathscr{P}(v)$ such that $\sum_{n \geqslant 0} Kf^n(v) \subset \mathscr{P}(v)$.*

Remark. If K is noetherian, then f is locally finite if and only if $\sum_{n \geqslant 0} Kf^n(v)$ is finitely generated for every $v \in V$.

THEOREM 1.8. *Let u be an element of the center $\mathscr{Z}(S)$ of S such that* $\operatorname{ad} u$*: $R \to R$ is locally nilpotent. Let M be an R-module with (R, S)-projective standard complex $(P_i, \partial_i)_{i \geqslant 0}$. Then the following statements hold*:

(1) *If u_M is invertible, then each u_{P_i} is invertible.*

(2) *If u_M is surjective and* $\ker u_M$ *is an S-direct summand of M, then each u_{P_i} is surjective.*

(3) *If u_M is locally nilpotent, then each u_{P_i} is locally nilpotent.*

(4) *Suppose that K is artinian. If u_M is locally finite and injective, then u_M is invertible and each u_{P_i} is locally finite and invertible.*

Proof. (1), (2), (3) As u is contained in the center of S, it readily follows that $\Theta_u = 0$. Consequently, $u_{\mathscr{P}} = u_{\mathscr{P}} - \Theta_u$ coincides with the tensor product representation of $u \in C_R(S)^-$ on the (R, S)-standard complex $\mathscr{P}$. Owing to our assumption concerning $\operatorname{ad} u$ the mapping $\alpha\colon P_i \to P_i$; $\alpha(r_1 \otimes \cdots \otimes r_{i+1} \otimes m) := \sum_{j=1}^{i+1} r_1 \otimes \cdots \otimes (\operatorname{ad} u)(r_j) \otimes \cdots \otimes r_{i+1} \otimes m$ is locally nilpotent. We have $u_{P_i} = \alpha + \operatorname{id}_{S_i(R)} \otimes u_M$, with the summands commuting. This readily implies (1) and (3). In case (2) there exists an

S-linear map $\lambda\colon M \to M$ such that $u_M \circ \lambda = \mathrm{id}_M$. Let $\beta := \mathrm{id}_{S_i(R)} \otimes u_M$, $\gamma := \mathrm{id}_{S_i(R)} \otimes \lambda$. Then $\beta \circ \gamma = \mathrm{id}_{P_i}$. Since α and γ commute the composite map $\alpha \circ \gamma$ is locally nilpotent and $(\alpha + \beta) \circ \gamma = \alpha \circ \gamma + \mathrm{id}_{P_i}$ is invertible. Consequently, $u_{P_i} = \alpha + \beta$ is surjective.

(4) Now suppose K to be artinian and assume u_M to be injective and locally finite. Let v be an element of M and put $\mathscr{Q}(v) := \sum_{n \geqslant 0} K(u_M)^n(v)$. By assumption there exists a finitely generated K-submodule $\mathscr{P}(v) \subset M$ that contains $\mathscr{Q}(v)$. As K is artinian $\mathscr{P}(v)$ is artinian and the same applies to $\mathscr{Q}(v)$. It follows that the injective K-linear map $u_M\colon \mathscr{Q}(v) \to \mathscr{Q}(v)$ is bijective. Consequently, $v \in \operatorname{im} u_M$, and u_M is an invertible map. In view of part (1) we may conclude the proof by showing that u operates on P_i by a locally finite transformation. As above we write $u_{P_i} = \alpha + \beta$ and note that β is locally finite. Let v be an element of P_i, $\mathscr{P}(v)$ a finitely generated K-submodule of P_i such that $\sum_{n \geqslant 0} K\beta^n(v) \subset \mathscr{P}(v)$. As α is locally nilpotent, there is $k \in \mathbb{N}$ with $\alpha^k(\mathscr{P}(v)) = (0)$. Since α and β commute the binomial formula gives rise to

$$(u_{P_i})^n(v) \in \sum_{i=0}^{k} \alpha^i(\mathscr{P}(v)) \qquad \forall n \geqslant 0.$$

Consequently, u_{P_i} is locally finite. ∎

For the sake of completeness we record the dual result:

THEOREM 1.9. *Let $u \in \mathscr{Z}(S)$ be an element of R such that* ad u: $R \to R$ *is locally nilpotent. Let N be an R-module with (R, S)-injective standard complex $(E_i, \partial_i)_{i \geqslant 0}$. Then the following statements hold:*

(1) *If u_N is invertible, then u_{E_i} is invertible for $i \geqslant 0$.*

(2) *If u_N is injective and* im u_N *is an S-direct summand of N, then u_{E_i} is injective for $i \geqslant 0$.*

(3) *If R is a finitely generated S-module and u_N is locally nilpotent, then u_{E_i} is locally nilpotent for $i \geqslant 0$.*

(4) *Suppose that K is a field and that R is a finitely generated S-module. If u_N is locally finite, then u_{E_i} is locally finite for $i \geqslant 0$.*

Proof. (1) As u belongs to $\mathscr{Z}(S)$ one readily obtains $\Theta_u = 0$. Consequently,

$$u_{E_i}(f) = u_N \circ f - f \circ \alpha,$$

where α is a locally nilpotent map. As u_N and ad u are homomorphisms of S-modules (4.2) of [5] shows that u_{E_i} is bijective.

(2) By assumption there exists an S-linear map $\lambda\colon N \to N$ such that $\lambda \circ u_N = \mathrm{id}_N$. Let $\mathscr{R}\colon E_i \to E_i$ be given by $\mathscr{R}(f) := \lambda \circ f \circ \alpha$. Then $(\mathscr{R}^n(f))_{n \geqslant 0}$

is a summable family of mappings for every $f \in E_i$. It follows that the map $\mathrm{id}_{E_i} - \mathscr{R}$ is invertible. Since $u_{E_i} = (\mathrm{id}_{E_i} - \mathscr{R}) \circ \mathrm{Hom}_S(\mathrm{id}, u_N)$ we obtain the injectivity of u_{E_i}.

(3) Let f be an element of $E_i = C^i(N)$. Since R is a finitely generated S-module $f(R^{i+1}) \subset N$ is a finitely generated S-submodule. Consequently, a suitable power of the S-linear map u_N annihilates $\mathrm{im}\, f$. Furthermore, our present assumption entails the nilpotence of the S-linear map ad u. Hence α is nilpotent as well, and the binomial theorem provides a number k such that $u_{E_i}^k(f) = 0$.

(4) For $f \in E_i$ let $\mathscr{Q}$ be a finite set of generators of $f(R^{i+1})$. As u_N is locally finite, the set $\mathscr{Q}$ is contained in a finite dimensional u_N-invariant K-subspace of N. Hence there exist $t \in \mathbb{N}$ as well as $k_1, ..., k_{t-1} \in K$ such that

$$u_N^t(q) = \sum_{i-1}^{t-1} k_i u_N^i(q) \qquad \forall q \in \mathscr{Q}.$$

The S-linearity of u_N implies that the above equation holds on $f(R^{i+1})$ as well. Consequently, $u_N^t \circ f = \sum_{i=1}^{t-1} k_i u_N^i \circ f$, which proves that the mapping $f \mapsto u_N \circ f$ is locally finite. Since α is nilpotent, an application of the binomial theorem shows that u_{E_i} is locally finite. ∎

2. Vanishing Results

In this section we are mainly concerned with vanishing theorems for extension functors. With the notable exception of (2.1) our results could have equally well been stated for torsion functors. Since the proofs necessitate only minor modifications of those presented here, we shall leave them to the interested reader. We adopt the general conventions of the preceding section.

Theorem 2.1. *Suppose there is $u \in R$ such that* ad u: $R \to R$ *is locally nilpotent. Let M and N be R-modules such that*

(a) *u_M is locally nilpotent*

(b) *u_N is invertible.*

Then the following statements hold:

(1) $\mathrm{Ext}_R^n(M, N) = (0)\ \forall n \geq 0$.

(2) *If $u \in \mathscr{Z}(R)$ and u_M is nilpotent,* $\mathrm{Ext}_R^n(N, M) = (0)\ \forall n \geq 0$.

Proof. Given any two R-modules X, Y such that u_X is locally nilpotent and u_Y is injective, it readily follows that $\mathrm{Hom}_R(X, Y) = (0)$.

(1) According to Theorem 1.1 there exists an injective resolution

$$(0) \to N \to E_0 \to E_1 \to \cdots$$

of N such that u_{E_i} is invertible for every $i \geq 0$. Our above remark now implies the triviality of the complex $\oplus_{n \geq 0} \operatorname{Hom}_R(M, E_n)$. Consequently, $\operatorname{Ext}^n_R(M, N) = (0)\ \forall n \geq 0$.

(2) The central element u induces a mapping $\mu := \operatorname{Ext}^n_R(u_N, \mathrm{id}_M) = \operatorname{Ext}^n_R(\mathrm{id}_N, u_M)$. Our assumption then yields both the invertibility and the nilpotency of μ. Consequently, the zero map is invertible on $\operatorname{Ext}^n_R(N, M)$, whence $\operatorname{Ext}^n_R(N, M) = (0)\ \forall n \geq 0$. ∎

Remark. Suppose that $\operatorname{char}(K) = p$, a prime, and let $\operatorname{ad} u: R \to R$ be nilpotent. Then there is k such that u^{p^k} lies in the center of R and part (2) of Theorem 1.1 continues to hold under the assumption of ad-nilpotence.

An *augmented ring* is a triple (R, Q, τ) consisting of a ring R, a left R-module Q, and a surjective homomorphism $\tau: R \to Q$ of left R-modules. One customarily defines $H^n(R, N) = \operatorname{Ext}^n_R(Q, N)\ \forall n \geq 0$ for every left R-module N.

COROLLARY 2.2. *Let (R, Q, τ) be an augmented ring and let $u \in \ker \tau$ be such that* ad u: *$R \to R$ is locally nilpotent. If N is an R-module on which u operates invertibly, then $H^n(R, N) = (0)\ \forall n \geq 0$.*

Proof. Let q be an element of Q, $r \in R$ a preimage of q under τ. By assumption there exists $n \in \mathbb{N}$ such that $(\operatorname{ad} u)^n\ (r) = 0$. Since u lies in the kernel of τ, we have $\tau \circ \operatorname{ad} u = u_Q \circ \tau$. Consequently, $(u_Q)^n\ (q) = 0$ and u operates locally nilpotently on Q. The assertion now follows from Theorem 2.1. ∎

Before turning to relative extension functors, we briefly digress to illustrate a result pertaining to torsion functors:

PROPOSITION 2.3. *Let u be an element of the center of R. Suppose that M is a right R-module, N a left R-module such that u_M is invertible and u_N is locally nilpotent. Then* $\operatorname{Tor}^R_n(M, N) = (0)\ \forall n \geq 0$.

Proof. We put $N_i := \ker u^i_N$ for every $i > 0$. As u_N is locally nilpotent, we have $N = \varinjlim N_i$. Since u acts nilpotently on N_i, the reasoning of (2.1) yields $\operatorname{Tor}^R_n(M, N_i) = (0)\ \forall n, i \geq 0$. Consequently, $\operatorname{Tor}^R_n(M, N) = \operatorname{Tor}^R_n(M, \varinjlim N_i) = \varinjlim \operatorname{Tor}^R_n(M, N_i) = (0)\ \forall n \geq 0$. ∎

PROPOSITION 2.4. *Let $S \subset R$ be rings and suppose that $u \in \mathscr{Z}(S)$ is an element such that* ad u: *$R \to R$ is locally nilpotent. Let M and N be left R-modules. Then the following statements hold:*

(1) *If* u_M *is locally nilpotent and* u_N *is injective, then* $\mathrm{Ext}^n_{(R,S)}(M, N) = (0)\ \forall n \geqslant 0$.

(2) *If* u_M *is surjective,* ker u_M *is an S-direct summand of M, and* u_N *is nilpotent, then* $\mathrm{Ext}^n_{(R,S)}(M, N) = (0)\ \forall n \geqslant 0$.

Proof. Let X and Y be two R-modules. If u_X is locally nilpotent and u_Y is injective or if u_X is surjective and u_Y is nilpotent, then $\mathrm{Hom}_R(X, Y) = (0)$. It now readily follows from (1.8) that $\mathrm{Hom}_R(P_i, N) = (0)$ for the standard (R, S)-projective resolution $(P_i)_{i \geqslant 0}$ of M. The homology of the corresponding complex is therefore trivial. This implies (1) and (2). ∎

Remark. Note that the conditions of (2.4) actually entail the vanishing of $\mathrm{Ext}^n_{(R,S)}(M, Q)$ for every R-submodule $Q \subset N$. If u lies in the center of R, then the hypotheses of (2) imply that the sequence $(0) \to \ker u_M \to M \to M \to (0)$ is (R, S)-exact with u operating trivially on $\ker u_M$. It now follows from (2) that $\ker u_M$ is an R-direct summand of M.

COROLLARY 2.5. *Let* $u \in R$ *be an element such that* ad u: $R \to R$ *is locally nilpotent and suppose that M and N are R-modules such that* u_M *is invertible and* u_N *is nilpotent. Then* $\mathrm{Ext}^1_R(M, N) = (0)$.

Proof. Let $S := K[u]$ so that $u \in \mathscr{Z}(S)$. Consecutive application of (2.1) and (2.4) yields $\mathrm{Ext}^1_{(R,S)}(M, N) = (0) = \mathrm{Ext}^1_S(M, N)$. To verify the triviality of $\mathrm{Ext}^1_R(M, N)$ we consider an exact sequence $(0) \to N \to X \to M \to (0)$. As $\mathrm{Ext}^1_S(M, N) = (0)$ this sequence splits over S and is thereby (R, S)-exact. According to [14, p. 254] the triviality of $\mathrm{Ext}^1_{(R,S)}(M, N)$ now implies the splitting of the above sequence over R. Consequently, $\mathrm{Ext}^1_R(M, N) = (0)$. ∎

The following example, which also illustrates (2.4), shows that the argument employed in the proof of (2.5) is not generalizable to extension functors of arbitrary order. Let K be a field of characteristic 0, L a finite dimensional semisimple Lie algebra over K. The universal enveloping algebra of L will be denoted $\mathscr{U}(L)$. Owing to [3, Theorem 21.1] the space $\mathrm{Ext}^3_{\mathscr{U}(L)}(K, K)$ does not vanish. We let $\mathscr{U}(L)$ act on itself by left multiplication. The short exact sequence associated to the canonical supplementation ε: $\mathscr{U}(L) \to K$ induces an exact cohomology sequence

$$\mathrm{Ext}^3_{\mathscr{U}(L)}(K, \mathscr{U}(L)) \to \mathrm{Ext}^3_{\mathscr{U}(L)}(K, K) \to \mathrm{Ext}^4_{\mathscr{U}(L)}(K, \mathscr{U}(L)^+),$$

where $\mathscr{U}(L)^+ = \ker \varepsilon$. It follows that one of the extreme terms is not trivial. Let u be a positive root vector of L. Then ad u is nilpotent on L, hence locally nilpotent on $\mathscr{U}(L)$. Since $\mathscr{U}(L)$ is free of zero divisors, u operates on $\mathscr{U}(L)$ and $\mathscr{U}(L)^+$ via an injective transformation. By the Poincaré–Birkhoff–Witt theorem the subalgebra, S say, of $\mathscr{U}(L)$ that is generated by u is a polynomial ring in one variable. Consequently, $\mathrm{Ext}^n_S(\cdot,\cdot)=(0)$ for $n\geqslant 2$. Moreover, Proposition 2.4 implies $\mathrm{Ext}^n_{(\mathscr{U}(L),S)}(K,\mathscr{U}(L))=(0)=\mathrm{Ext}^n_{(\mathscr{U}(L),S)}(K,\mathscr{U}(L)^+)$ $\forall n\geqslant 0$. Hence the vanishing of $\mathrm{Ext}^n_S(M,N)$ and $\mathrm{Ext}^n_{(R,S)}(M,N)$ does not generally ensure the triviality of $\mathrm{Ext}^n_R(M,N)$.

Under additional hypotheses Corollary 2.5 may be strengthened as follows:

PROPOSITION 2.6. *Suppose that $S\subset R$ are rings such that R is flat as a right S-module. Let $u\in\mathscr{Z}(S)$ be an element such that* ad u: $R\to R$ *is locally nilpotent. Suppose that M and N are R-modules. If u_M is invertible and u_N is nilpotent, then* $\mathrm{Ext}^n_R(M,N)=(0)$ $\forall n\geqslant 0$.

Proof. We proceed by induction on n, the case $n=0$ being trivial. If $n>0$ we consider the exact sequence $(0)\to X\to R\otimes_S M\to M\to(0)$ as well as the resulting long exact cohomology sequence

$$\cdots\to\mathrm{Ext}^{n-1}_R(X,N)\to\mathrm{Ext}^n_R(M,N)\to\mathrm{Ext}^n_R(R\otimes_S M,N)\to\cdots.$$

Since R is flat as a right S-module there exists an isomorphism

$$\mathrm{Ext}^n_R(R\otimes_S M,N)\cong\mathrm{Ext}^n_S(M,N).$$

The groups on the right-hand side vanish because of (2.1). As u is contained in the center of S, we have

$$u_{R\otimes_S M}=(\mathrm{ad}\,u)\otimes\mathrm{id}_M+\mathrm{id}_R\otimes u_M.$$

Consequently, $u_{R\otimes_S M}$ and u_X are invertible and the inductive hypothesis now implies $\mathrm{Ext}^n_R(M,N)=(0)$. ∎

THEOREM 2.7. *Let $S\subset R$ be rings and suppose that u is an element of $C_R(S)$ such that* ad u: $R\to R$ *is locally nilpotent. Let M and N be left R-modules. Then the following statements hold:*

(1) *If u_M is locally nilpotent and u_N is invertible, then* $\mathrm{Ext}^n_{(R,S)}(M,N)=(0)$ $\forall n\geqslant 0$.

(2) *If u_M is invertible and u_N is nilpotent, then* $\mathrm{Ext}^n_{(R,S)}(M,N)=(0)$ $\forall n\geqslant 0$.

(3) *Suppose that K is artinian. If u_M is locally finite and injective and u_N is locally nilpotent, then* $\mathrm{Ext}^n_{(R,S)}(M,N)=(0)$ $\forall n\geqslant 0$.

Proof. Throughout, we let $(P_i,\partial_i)_{i\geqslant 0}$ denote the (R,S)-standard complex of the R-module M ($P_{-1}:=M$).

(1) Let $n\geqslant 0$. Then, in the notation of (1.8), we have $\mu_u(f)=u_N\circ f-f\circ(\alpha+\beta)$ $\forall f\in\mathrm{Hom}_S(P_{n-1},N)$, with $\alpha+\beta$ locally nilpotent. Since

u_N and $\alpha+\beta$ are S-linear, [5, (4.2)] ensures the invertibility of μ_u. An elementary computation shows that the map $\mathfrak{X}\colon \operatorname{Hom}_R(P_n, N) \to \operatorname{Hom}_S(P_{n-1}, N)$ given by $\mathfrak{X}(f)(r_1 \otimes \cdots \otimes r_n \otimes m) = f(1 \otimes r_1 \otimes \cdots \otimes r_n \otimes m)$ is an isomorphism of $C_R(S)^-$-modules. Hence u also operates invertibly on $\operatorname{Hom}_R(P_n, N)$. Owing to (1.5), μ_u commutes with the coboundary operator δ, and we now obtain the invertibility of the action of u on $\operatorname{Ext}^n_{(R,S)}(M, N)$. Since the latter is, according to (3) of (1.5), also trivial, $\operatorname{Ext}^n_{(R,S)}(M, N)$ vanishes.

(2), (3) In order to verify (3) we observe that, since ad u is locally nilpotent and u_M is invertible, $\alpha+\beta$ is invertible and locally finite (cf. proof of (1.8(4)). Let v be an element of P_n. Since $\mathcal{Q}(v) := \sum_{i \geqslant 0} K(\alpha+\beta)^i\,(v)$ is artinian, $\alpha+\beta$ operates invertibly on $\mathcal{Q}(v)$. Hence we may write

$$v = \sum_{i=1}^{k} a_i(\alpha+\beta)^i\,(v), \qquad a_i \in K.$$

Now let $\Theta := (\alpha+\beta)^{-1}$ and put $\mathcal{R}_n(v) := \sum_{i=0}^{n} K\Theta^i(v)$ for $n \geqslant 0$. We then obtain, for $n \geqslant k$, $\Theta^n(v) = \sum_{i=1}^{k} a_i \Theta^{n-i}(v)$, proving that $\mathcal{R}_n(v) = \mathcal{R}_{n-1}(v)$. Consequently, $\sum_{n \geqslant 0} K\Theta^n(v) = \mathcal{R}_k(v)$ is finitely generated and Θ is locally finite. We consider the mapping $\Omega\colon \operatorname{Hom}_S(P_n, N) \to \operatorname{Hom}_S(P_n, N)$; $\Omega(f) = u_N \circ f \circ \Theta$. For $v \in P_n$ there exists a finitely generated K-module $\mathcal{P}(v)$ containing $\sum_{n \geqslant 0} K\Theta^n(v)$. Given $f \in \operatorname{Hom}_S(P_n, N)$ we thus have $u^k f(\sum_{n \geqslant 0} K\Theta^n(v)) = (0)$ for a suitably chosen $k \in \mathbb{N}$. Consequently, $\Omega^k(f)(v) = 0$ and the family $(\Omega^j(f))_{j \geqslant 0}$ is summable. Under the conditions of (2), the summability of $(\Omega^j(f))_{j \geqslant 0}$ is a direct consequence of the nilpotence of u_N. In either case it follows that $\Omega - \mathrm{id}_{\operatorname{Hom}_S(P_n, N)}$ is invertible and the same holds for $\mu_u := \operatorname{Hom}_S(\alpha+\beta, \mathrm{id}_N) \circ (\Omega - \mathrm{id}_{\operatorname{Hom}_S(P_n, N)})$. The arguments employed in the first part now yield the assertion. ▌

3. Cohomology Groups of Frobenius Extensions

Let R be a ring. Given a ring homomorphism α of R and a left (right) R-module M we shall denote by ${}_\alpha M$ (M_α) the left (right) R-module with underlying group M and operation $r \cdot m := \alpha(r)\, m$ ($m \cdot r := m\alpha(r)$).

In the sequel let $R : S$ be an α-Frobenius extension in the sense of [18]. We shall study the ordinary cohomology of such an extension. By definition there exists an isomorphism $\varphi\colon {}_R R_S \to \operatorname{Hom}_S(R, {}_\alpha S)$ of R-S-bimodules. Since R is finitely generated and projective as a right S-module, there is, for an arbitrary S-module V, a natural isomorphism $\tau\colon \operatorname{Hom}_S(R, {}_\alpha S) \otimes_S V \to \operatorname{Hom}_S(R, {}_\alpha V)$ of R-modules such that $\tau(f \otimes v)(r) = f(r)\, v$. Combining τ with $\varphi \otimes \mathrm{id}_V$ we obtain a natural equivalence

$$\Gamma_V\colon R \otimes_S V \to \operatorname{Hom}_S(R, {}_\alpha V); \qquad \Gamma_V(r \otimes v)(u) = \varphi(r)(u)\, v$$

of left R-modules. If V is an R-module, there thus results an R-linear natural transformation $\mathrm{Hom}_S(R, {}_\alpha V) \to V$. As R is also a projective left S-module, the latter induces natural corestriction maps

$$\mathrm{Cor}_n\colon \mathrm{Ext}_S^n(M, {}_\alpha V) \to \mathrm{Ext}_R^n(M, V)$$

for every R-module M and $n \geq 0$.

To obtain control of these mappings, we recall the notion of a trace map (cf. [20, pp. 9, 12)]. Since R, and thereby R_α, is a finitely generated projective S-module, the mapping Ω: $R_\alpha \otimes_S R \to \mathrm{Hom}_S(\mathrm{Hom}_S(R, {}_\alpha S), R)$; $\Omega(x \otimes y)(f) = xf(y)$ is an isomorphism of additive groups. Consequently, there are $x_1, ..., x_n, y_1, ..., y_n \in R$ such that $\Omega(\sum_{i=1}^n x_i \otimes y_i) = \varphi^{-1}$. Now let M, N be two R-modules. The mapping

$$\mathrm{Tr}\colon \mathrm{Hom}_S(M, {}_\alpha N) \to \mathrm{Hom}_R(M, N); \qquad \mathrm{Tr}(f)(m) = \sum_i x_i f(y_i m)$$

is customarily referred to as the *trace map*.

Given two R-modules M and N, we let Res_n: $\mathrm{Ext}_R^n(M, N) \to \mathrm{Ext}_S^n(M, N)$ denote the canonical change of rings map. Note that Res_n is a natural transformation of functors.

THEOREM 3.1. *Let $R:S$ be an α-Frobenius extension, M and N two R-modules. Then the following statements hold*:

(1) $\mathrm{Cor}_0 = \mathrm{Tr}$.

(2) *Suppose that α can be extended to a ring homomorphism α: $R \to R$. Then*

(a) $\chi_N\colon {}_\alpha N \to N$; $\chi_N(n) = \sum_i x_i \alpha(y_i) n$ *is an R-linear map.*

(b) $\mathrm{Cor}_n \circ \mathrm{Res}_n = \mathrm{Ext}_R^n(\mathrm{id}_M, \chi_N)$ $\forall n \geq 0$.

Proof. (1) Let f be an element of $\mathrm{Hom}_S(M, {}_\alpha N)$. The corestriction map first sends f onto $g \in \mathrm{Hom}_R(M, \mathrm{Hom}_S(R, {}_\alpha N))$, where $g(m)(r) = f(rm)$. Next, we apply $\mathrm{Hom}_R(\mathrm{id}_M, \tau^{-1})$ and obtain h. Note that if $h(m) = \sum_j \gamma_j \otimes n_j$, then

$$f(rm) = g(m)(r) = \sum_j \gamma_j(r)\, n_j. \qquad (*)$$

The image of f under the corestriction map is now obtained by applying $\mathrm{Hom}_R(\mathrm{id}_M, \varphi^{-1} \otimes \mathrm{id}_N)$ to h and then applying the canonical map $R \otimes_S N \to N$ to the result. Consequently, observing $(*)$ we conclude

$$\mathrm{Cor}_0(f)(m) = \sum_{i,j} \Omega(x_i \otimes y_i)(\gamma_j)\, n_j = \sum_{i,j} x_i \gamma_j(y_i)\, n_j = \sum_i x_i f(y_i m) = \mathrm{Tr}(f)(m).$$

(2) Note that $\mathrm{id}_N \in \mathrm{Hom}_S({}_\alpha N, {}_\alpha N)$. Thus $\mathrm{Tr}(\mathrm{id}_N) \in \mathrm{Hom}_R({}_\alpha N, N)$ and $\mathrm{Tr}(\mathrm{id}_N)(n) = \sum_i x_i \, \mathrm{id}_N(y_i \cdot n) = \sum_i x_i \alpha(y_i) n = \chi_N(n)$. This verifies assertion (a). Part (b) will be established by induction on n. Suppose that $n = 0$. Then $\mathrm{Cor}_0 \circ \mathrm{Res}_0(f)(m) = \sum_i x_i f(y_i m) = \sum_i x_i \alpha(y_i) f(m) = \mathrm{Hom}_R(\mathrm{id}_M, \chi_N)(f)(m)$ for every $f \in \mathrm{Hom}_R(M, {}_\alpha N)$. If n is positive, we consider a projective presentation $(0) \to X \to P \to M \to (0)$ of M. Since P is also S-projective, the long exact cohomology sequences provide a commutative diagram

$$\begin{array}{ccccc} \mathrm{Ext}_R^{n-1}(X, {}_\alpha N) & \longrightarrow & \mathrm{Ext}_R^n(M, {}_\alpha N) & \longrightarrow & (0) \\ \downarrow & & \downarrow & & \\ \mathrm{Ext}_R^{n-1}(X, N) & \longrightarrow & \mathrm{Ext}_R^n(M, N) & \longrightarrow & (0) \end{array}$$

with exact rows. By inductive hypothesis the asserted identity holds on the left-hand side. Since the corestriction maps are induced by natural transformations, they commute with connecting homomorphisms. Consequently, $\mathrm{Cor}_n \circ \mathrm{Res}_n = \mathrm{Ext}_R^n(\mathrm{id}_M, \chi_N)$. ∎

EXAMPLE. Let G be a group, H a subgroup of finite index $k := (G : H)$. Suppose that $Q := \{x_1, ..., x_k\}$, $x_1 = e$, is a set of representatives of the set of right cosets of H. Then $K[G] : K[H]$ is a Frobenius extension of first kind with Frobenius homomorphism $\lambda: K[G] \to K[H]$ induced by the projection of the decomposition $K[G] = \oplus_{i=1}^k K[H]\, x_i$ onto the first summand. If N is a G-module, then $\chi_N = k\, \mathrm{id}_N$ and we thus retrieve [11, (16.4), p. 226].

Remark. Let ψ be the *Nakayama automorphism* of the α-Frobenius extension $R : S$, i.e., the automorhism ψ of $C_R(S)$ satisfying

$$\varphi(1)(ax) = \varphi(1)(x\psi(a)) \qquad \forall a \in C_R(S)\ \forall x \in R.$$

Then ψ coincides with α on the center of S. Hence α can be extended to an automorphism of R whenever $S \subset \mathscr{Z}(R)$.

COROLLARY 3.2. *Let $R : S$ be an α-Frobenius extension, M and N two R-modules. Suppose that α can be extended to an automorphism $\alpha: R \to R$. Then* $\mathrm{Cor}_n \circ \mathrm{Res}_n = \mathrm{Ext}_R^n(\chi_{{}_\beta M}, \mathrm{id}_N)$, *where $\beta := \alpha^{-1}$.*

Proof. Let $(P_n, \partial_n)_{n \geq 0}$ be a projective resolution of the R-module M. Then $({}_\beta P_n, \partial_n)_{n \geq 0}$ is readily seen to be a projective resolution of ${}_\beta M$. Since the groups $\mathrm{Hom}_R({}_\beta P_n, N)$ and $\mathrm{Hom}_R(P_n, {}_\alpha N)$ coincide, it follows that

$$\mathrm{Ext}_R^n({}_\beta M, N) = \mathrm{Ext}_R^n(M, {}_\alpha N) \qquad \forall n \geq 0.$$

Evidently, $\mathrm{Hom}_R(\chi_{\beta M}, \mathrm{id}_N) = \mathrm{Hom}_R(\mathrm{id}_M, \chi_N)$. We proceed by verifying the identity $\mathrm{Ext}^n_R(\chi_{\beta M}, \mathrm{id}_N) = \mathrm{Ext}^n_R(\mathrm{id}_M, \chi_N)$ by induction on n. Let $n \geqslant 1$ and consider an injective presentation $(0) \to N \to E \to Q \to (0)$. There results a commutative diagram

$$\begin{array}{ccccc} \mathrm{Ext}^{n-1}_R({}_\beta M, Q) & \longrightarrow & \mathrm{Ext}^n_R({}_\beta M, N) & \longrightarrow & (0) \\ \downarrow & & \downarrow & & \\ \mathrm{Ext}^{n-1}_R(M, Q) & \longrightarrow & \mathrm{Ext}^n_R(M, N) & \longrightarrow & (0) \end{array}$$

with exact rows. A standard argument now yields $\mathrm{Ext}^n_R(\chi_{\beta M}, \mathrm{id}_N) = \mathrm{Ext}^n_R(\mathrm{id}_M, \chi_N)$. The proof may now be concluded by applying (3.1). ∎

DEFINITION. Let $R : S$ be an α-Frobenius extension such that α is extendable to a homomorphism $\alpha: R \to R$. Then $c_0 := \sum_i x_i \alpha(y_i)$ will be called the *Casimir element* of the extension $R : S$.

COROLLARY 3.3. *Let $R : S$ be an α-Frobenius extension defined by an endomorphism α of R. Let M and N be R-modules and assume that the left global dimension $lD(S)$ of S is bounded by k.*

(1) *If c_0 operates invertibly on N, then* $\mathrm{Ext}^n_R(M, {}_\alpha N) = (0) = \mathrm{Ext}^n_R(M, N)$ $\forall n > k$.

(2) *If α is an automorphism of R and c_0 operates invertibly on M or N, then* $\mathrm{Ext}^n_R(M, N) = (0)$ $\forall n > k$.

Proof. (1) As S has left global dimension $\leqslant k$, we have $\mathrm{Ext}^n_S(M, N) = (0)$ $\forall n > k$. Our present assumption in conjunction with (3.1) now shows that $\mathrm{Ext}^n_R(\mathrm{id}_M, \chi_N)$ is bijective and identically zero for $n > k$. This proves our assertion. Part (2) can be verified analogously, employing (3.2) in place of (3.1). ∎

Remark. Suppose that, under the general hypotheses of (3.3), N is an irreducible R-module. Then ${}_\alpha N$ inherits this property and the R-module homomorphism χ_N is either trivial or bijective. Consequently, c_0 operates either trivially or invertibly on N.

The second part of (3.1) may of course be combined with the results of Section 2 to yield various vanishing results such as:

COROLLARY 3.4. *Let $R : S$ be as in* (3.1) *and suppose there is $u \in S$ such that* ad u: $S \to S$ *is locally nilpotent. If u_M is locally nilpotent and u_N and χ_N are invertible, then* $\mathrm{Ext}^n_R(M, N) = (0)$ $\forall n \geqslant 0$. ∎

The preceding results indicate the significance of α-Frobenius extensions for which α can be extended to an automorphism of R. This class comprises, in accordance with earlier observations,

(a) central Frobenius extensions, i.e., $S \subset \mathscr{Z}(R)$;

(b) Frobenius extensions of first kind, i.e., extensions satisfying $\alpha = \mathrm{id}_S$.

Note that in the latter case (2a) of (3.1) shows that the Casimir element $c_0 = \sum_i x_i y_i$ lies in the center of R. Given any two R-modules M and N, χ_M and χ_N thus induce the same mappings on $\mathrm{Ext}^n_R(M, N)$.

In the sequel, we let $l\dim_R(M)$ denote the left projective dimension of M relative to R.

COROLLARY 3.5. *Let $R:S$ be an α-Frobenius extension given by an automorphism α of R. Suppose that c_0 operates invertibly on the R-module M. Then $l\dim_R(M) = l\dim_S(M)$. In particular, M is R-projective if and only if it is projective as an S-module.*

Proof. Since R is S-projective we have $l\dim_S(M) \leqslant l\dim_R(M)$ Let $k := l\dim_S(M)$. Given any R-module V, $\mathrm{Ext}^n_S(M, V) = (0)$ $\forall n > k$. The arguments employed in the proof of (3.3) therefore yield the triviality of $\mathrm{Ext}^n_R(M, V)$ for $n > k$. Consequently, $l\dim_R(M) \leqslant k$. ▮

COROLLARY 3.6. *Let $R:S$ be as in* (3.5) *and suppose that $\varepsilon: R \to K$ is a supplementation. If $\varepsilon(c_0)$ is invertible, then $H^n(R, M) = (0)$ for every R-module M and every $n > l\dim_S(K)$.*

Remark. The preceding result is closely related to Corollary 6 of [22, p. 172], where the left norm of a Frobenius extension is considered. Evidently, $q := \sum_i x_i \varepsilon(y_i)$ is a left norm such that $\varepsilon(q) = \varepsilon(c_0)$.

COROLLARY 3.7. *Suppose that H is a finite dimensional Hopf algebra over a field K. If $\varepsilon(c_0) \neq 0$, then H is semisimple.*

Proof. Owing to the main theorem of [17], H is a Frobenius algebra over K, and thereby a Frobenius extension of K of first kind. Let $(P_i)_{i \geqslant 0}$ denote a projective resolution of the H-module K. The canonical equivalence $\mathrm{Hom}_H(P_n \otimes_K M, N) \cong \mathrm{Hom}_H(P_n, \mathrm{Hom}_K(M, N))$ (cf. [22, p. 166]) shows that $(P_i \otimes_K M)_{i \geqslant 0}$ is a projective resolution of M. Moreover, we obtain isomorphisms $\mathrm{Ext}^n_H(M, N) \cong H^n(H, \mathrm{Hom}_K(M, N))$ $\forall n \geqslant 0$. An application of (3.6) now implies the triviality of Ext^n_H for $n > 0$. Consequently, H is semisimple. ▮

The above result may be employed to retrieve a well-known fact from the structure theory of Hopf algebras. Let $\int$ denote the one-dimensional space of left integrals of the finite dimensional Hopf algebra H. Suppose that $\varepsilon(\int) \neq (0)$. Let Λ be a left integral satisfying $\varepsilon(\Lambda) \neq 0$. Then Λ is easily seen to be a right integral (cf. [17, Proposition 4]). Since H is a Frobenius algebra, there exists a linear form ζ such that $\varphi: H \to \mathrm{Hom}_K(H, K)$; $\varphi(x)(y) = \zeta(yx)$ is an isomorphism. If ζ annihilates Λ, then $\varphi(\Lambda)(y) = \zeta(y\Lambda) = \varepsilon(y)\zeta(\Lambda) = 0$, whence $\Lambda \in \ker \varphi$, a contradiction. We may therefore assume that $\zeta(\Lambda) = 1$. Choose a basis $\{x_1, ..., x_n\}$ of H over K such that $x_1 = \Lambda$. If $\{y_1, ..., y_n\}$ denotes the dual basis, then we have $\varepsilon(y_1) = 1$ as well as $y_i \in \ker \varepsilon$ for $i \geqslant 2$. It follows that $\varepsilon(c_0) = \varepsilon(\sum_i x_i y_i) = \varepsilon(\Lambda)\varepsilon(y_1) = \varepsilon(\Lambda) \neq (0)$, thereby implying the semisimplicity of H.

In the sequel let $R:S$ be an α-Frobenius extension. We shall apply the methods and results of Section 2 to study the complete relative cohomology of such an extension. Given an R-module M, a *complete (R, S)-projective resolution of M* is an (R, S)-exact sequence

$$\mathscr{P}: \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

consisting of (R, S)-projective modules such that $\mathrm{im}(P_0 \to P_{-1}) \cong M \cong \ker(P_{-1} \to P_{-2})$. Let N be another R-module. For $k \in \mathbb{Z}$ the kth homology group of the complex $\mathrm{Hom}_R(\mathscr{P}, N)$ is called *the kth complete cohomology group of (R, S) with coefficients in M and N*. We shall denote this group by $\hat{H}^k_{(R,S)}(M, N)$.

Let u be an element of $C_R(S)$. Then ad u: $R \to R$ is a homomorphism of S-modules. Since R is a finitely generated left S-module ad u is nilpotent whenever it is locally nilpotent.

THEOREM 3.8. *Let $R:S$ be an α-Frobenius extension with Nakayama automorphism ψ, $u \in C_R(S)$ such that the map* ad u: *$R \to R$ is nilpotent. Let M and N be R-modules. Then the following statements hold*:

(1) *Suppose that u_M is locally nilpotent. If u operates invertibly on N, then $\hat{H}^k_{(R,S)}(M, N) = (0)$ $\forall k \geqslant 0$. If $\psi(u)_N$ is invertible, then $\hat{H}^k_{(R,S)}(M, N) = (0)$ $\forall k < 0$.*

(2) *Suppose that u_M is invertible. If u_N is nilpotent, then $\hat{H}^k_{(R,S)}(M, N) = (0)$ $\forall k \geqslant 0$. If $\psi(u)_N$ is nilpotent, then $\hat{H}^k_{(R,S)}(M, N) = (0)$ $\forall k < 0$.*

(3) *Assume K to be a field. Suppose that u_M is locally finite and injective. If u_N is locally nilpotent, then $\hat{H}^k_{(R,S)}(M, N) = (0)$ $\forall k \geqslant 0$. If $\psi(u)_N$ is locally nilpotent, then $\hat{H}^k_{(R,S)}(M, N) = (0)$ $\forall k < 0$.*

Proof: (1) Let $(X_k(M))_{k \geqslant 0}$ and $(C^k(M))_{k \geqslant 0}$ denote the standard (R, S)-projective and (R, S)-injective resolutions of M, respectively. According to Satz 11 of [20], $C^k(M)$ is (R, S)-projective for every $k \geqslant 0$. We put

$P_i := X_i(M)$ for $i \geqslant 0$ and $P_i := C^{-(i+1)}(M)$ for $i < 0$. Splicing of the exact sequences at M yields a complete (R, S)-projective resolution

$$\mathscr{P}: \cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

$$\begin{array}{ccccc} P_0 & & \longrightarrow & & P_{-1} \\ & \searrow & & \nearrow & \\ & & M & & \\ & \nearrow & & \searrow & \\ (0) & & & & (0) \end{array}$$

We consider the complex $\mathscr{H}om_R(\mathscr{P}, N)$. If u_N is invertible it readily follows from (1) of (2.7) that $H^k_{(R,S)}(M, N) = \operatorname{Ext}^k_{(R,S)}(M, N) = (0)$ $\forall k \geqslant 0$. Now assume $\psi(u)_N$ to be invertible. It follows from the definition of the Nakayama automorphism that

$$\varphi(a)(ur) = \varphi(a\psi(u))(r) \qquad \forall a, r \in R. \qquad (*)$$

Owing to our introductory remarks, the R-linear map $\mathfrak{X}$: $R \otimes_{S\ \alpha^{-1}} C^{n-1}(M) \to C^n(M)$; $\mathfrak{X}(a \otimes f)(r_1, ..., r_{n+1}) = f(r_1, ..., r_n \alpha^{-1}(\varphi(a)(r_{n+1})))$ is an isomorphism for $n \geqslant 0$. An elementary computation, observing $(*)$, shows that

$$\mathfrak{X}(a \otimes \Theta_u(f)) = \Theta_u(\mathfrak{X}(a \otimes f)) + \mathfrak{X}(a\psi(u) \otimes f) - \mathfrak{X}(a \otimes u \cdot f). \qquad (**)$$

We next consider the K-linear isomorphism $\mathfrak{X}^*$: $\operatorname{Hom}_R(C^n(M), N) \to \operatorname{Hom}_S(C^{n-1}(M), {}_\alpha N)$; $\mathfrak{X}^*(\lambda)(f) = \lambda(\mathfrak{X}(1 \otimes f))$ $\forall f \in C^{n-1}(M)$, $\forall \lambda \in \operatorname{Hom}_R(C^n(M), N)$. For $f \in C^{n-1}(M)$ we define

$$\Omega_u(f)(r_1, ..., r_n) = uf(r_1, ..., r_n) - \sum_{i=1}^{n} f(r_1, ..., (\operatorname{ad} u)(r_i), ..., r_n).$$

With this convention, we obtain from $(**)$

$$\begin{aligned} \mathfrak{X}^*(\lambda \circ \Theta_u)(f) &= \lambda \circ \Theta_u(\mathfrak{X}(1 \otimes f)) = \lambda \circ \mathfrak{X}(1 \otimes \Theta_u(f)) \\ &\qquad - \lambda \circ \mathfrak{X}(\psi(u) \otimes f) + \lambda \circ \mathfrak{X}(1 \otimes u \cdot f) \\ &= \mathfrak{X}^*(\lambda)(\Theta_u(f)) - \psi(u)_N \circ \mathfrak{X}^*(\lambda)(f) + \mathfrak{X}^*(\lambda)(u \cdot f) \\ &= \mathfrak{X}^*(\lambda)(\Omega_u(f)) - \psi(u)_N \circ \mathfrak{X}^*(\lambda)(f). \end{aligned}$$

Consequently,

$$\mathfrak{X}^*(\lambda \circ \Theta_u) = \mathfrak{X}^*(\lambda) \circ \Omega_u - \psi(u)_N \circ \mathfrak{X}^*(\lambda) \qquad \forall \lambda \in \operatorname{Hom}_R(C^n(M), N). \qquad (***)$$

We consider the mapping $\mathscr{E}_u$: $\operatorname{Hom}_S(C^{n-1}(M), {}_\alpha N) \to \operatorname{Hom}_S(C^{n-1}(M), {}_\alpha N)$; $\mathscr{E}_u(\gamma) = \psi(u)_N^{-1} \circ \gamma \circ \Omega_u$. Since we have $\Omega_u(f) = \Phi \circ f - f \circ \Psi$, with locally nilpotent S-linear maps Φ and Ψ, it follows from the binomial theorem that the family $(\Omega_u^k(f))_{k\geq 0}$ is summable for every $f \in C^{n-1}(M)$. For an element $\gamma \in \mathrm{Hom}_S(C^{n-1}(M), {}_\alpha N)$ we may therefore define $\mathscr{R}_u(\gamma)(f) = \sum_{k \geq 0} \psi(u)_N^{-k} \circ \gamma \circ \Omega_u^k(f)$. The resulting K-linear map $\mathscr{R}_u: \mathrm{Hom}_S(C^{n-1}(M), {}_\alpha N) \to \mathrm{Hom}_S(C^{n-1}(M), {}_\alpha N)$ is easily seen to be the inverse of id $-\mathscr{E}_u$. It now follows that the K-linear map Ξ_u of $\mathrm{Hom}_S(C^{n-1}(M), {}_\alpha N)$ that sends γ onto $\gamma \circ \Omega_u - \psi(u)_N \circ \gamma$ is invertible. According to identity $(***)$ the map $\mathrm{Hom}_R(\Theta_u, \mathrm{id}_N) = (\mathfrak{X}^*)^{-1} \circ \Xi_u \circ \mathfrak{X}^*$ has the same property. Owing to (1.6) the former map is a chain map which is homotopic to the zero map. Hence the induced maps on the relevant homology groups are both invertible and trivial. Consequently, $H^n_{(R,S)}(M, N) = (0)\ \forall n < 0$.

(2) The first assertion is a direct consequence of (2.7(2)). If $\psi(u)_N$ is nilpotent, then Lemma 4.2 of [5] successively ensures the invertibility of Ω_u and Ξ_u. The arguments employed in the proof of (1) now yield the assertion.

(3) As before, the first assertion was established earlier. Since u_M is bijective, we obtain the invertibility of Ω_u. Recall that $\Omega_u(f) = \Phi \circ f - f \circ \Psi$, where Φ is locally finite. We may now adopt the arguments employed in the proof of (1.9(4)) to see that Ω_u is locally finite. As in the proof of (2.7) we conclude that $\Delta := \Omega_u^{-1}$ inherits this property from Ω_u. Define a mapping $\mathscr{F}_u: \mathrm{Hom}_S(C^{n-1}(M), {}_\alpha N) \to \mathrm{Hom}_S(C^{n-1}(M), {}_\alpha N)$ via $\mathscr{F}_u(\gamma) = \psi(u)_N \circ \gamma \circ \Delta$. As Δ is locally finite, the family $(\mathscr{F}_u^k(\gamma))_{k \geq 0}$ is summable and we conclude that id $- \mathscr{F}_u$ is invertible. Owing to the bijectivity of Ω_u, the same applies to Ξ_u. The proof can now be completed as in (1). ▌

Remark: By definition of the Nakayama automorphism we have $\psi(u) = u$ for every central element $u \in \mathscr{Z}(R)$. Suppose that char$(K) = p$, a prime number. In that case the nilpotence of ad u is equivalent to the existence of $k \in \mathbb{N}$ such that $u^{p^k} \in \mathscr{Z}(R)$. Consequently, $(\psi(u)_N)^{p^k} = (u_N)^{p^k}$ and $\psi(u)_N$ inherits the relevant properties from u_N. There thus result vanishing conditions for $H^n_{(R,S)}(M, N)$ that depend only on the action of u on M and N.

PROPOSITION 3.9. *Let $R:S$ be a Frobenius extension of first kind with Casimir element c_0. Let M and N be R-modules such that $(c_0)_M$ or $(c_0)_N$ is invertible. Then $H^n_{(R,S)}(M, N) = (0)\ \forall n \in \mathbb{Z}$.*

Proof. Let $\mathscr{P} = (P_i, \delta_i)_{i \in \mathbb{Z}}$ be the complete (R, S)-projective resolution constructed in the proof of (3.8). Using the notation of Section 1, we have $\delta_i = d_i$ for $i \geq 1$, $\delta_0 = \iota \circ \varepsilon$, and $\delta_i = \partial_{-(i+1)}$ for $i < 0$. Define a degree 1 S-linear map $\sigma: \mathscr{P} \to \mathscr{P}$ via $\sigma_i := s_i$ for $i \geq 0$, $\sigma_{-1} := s_{-1} \circ t_0$ as well as $\sigma_i := t_{-(i+1)}$ for $i \leq -2$. A direct calculation then shows that σ is a contracting homotopy, i.e.,

$$\delta_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \delta_n = \mathrm{id}_{P_n} \qquad \forall n \in \mathbb{Z}.$$

As δ is R-linear, we have $\mathrm{Tr}(\delta\circ\sigma)=\delta\circ\mathrm{Tr}(\sigma)$ and $\mathrm{Tr}(\sigma\circ\delta)=\mathrm{Tr}(\sigma)\circ\delta$. Consequently,

$$\delta_{n+1}\circ\mathrm{Tr}(\sigma_n)+\mathrm{Tr}(\sigma_{n-1})\circ\delta_n=c_0\,\mathrm{id}_{P_n}\qquad \forall n\in\mathbb{Z}.$$

Application of the Hom functor now shows that $\mathrm{Hom}_R(c_0\mathrm{id}_{P_n},\mathrm{id}_N)=\mathrm{Hom}_R(\mathrm{id}_{P_n},c_0\,\mathrm{id}_N)$ is nullhomotopic. If c_0 acts invertibly on N, then the assertion readily follows. Alternatively, the comparison theorem [23, p. 179] provides a chain map $f\colon\mathscr{P}\to\mathscr{P}$ such that $f\circ(c_0)_{\mathscr{P}}$ is homotopic to the identity map. This again forces the vanishing of $H^n_{(R,S)}(M,N)$ $\forall n\in\mathbb{Z}$. ∎

4. Cohomology of Modular Lie Algebras

In this section we illustrate a non-classical application by considering the ordinary cohomology theory of finite dimensional modular Lie algebras. Accordingly, K is assumed to be a field of positive characteristic p. Let L be a finite dimensional Lie algebra over K, $L_0\subset L$ a subalgebra of codimension k. The universal enveloping algebra $\mathscr{U}(L)$ of L has a canonical filtration $(\mathscr{U}(L)_{(i)})_{i\geqslant 0}$. According to Jacobson's refinement of the Poincaré–Birkhoff–Witt theorem (cf. [24, pp. 58 f.]) there exist, for a cobasis $\{e_1,...,e_k\}$ of L_0 in L natural numbers $m_1,...,m_k$ as well as elements $v_i\in\mathscr{U}(L)_{(p^{m_i}-1)}$ such that

(a) $z_i:=e_i^{p^{m_i}}+v_i$ belongs to the center $\mathscr{Z}(\mathscr{U}(L))$, $1\leqslant i\leqslant k$;

(b) $\mathscr{U}(L)$ is a free left and right module over the algebra $\mathscr{O}(L,L_0)$, generated by $L_0\cup\{z_1,...,z_k\}$;

(c) $\mathscr{O}(L,L_0)\cong K[z_1,...,z_k]\otimes_K\mathscr{U}(L_0)$, where $K[z_1,...,z_k]$ is a polynomial ring in k indeterminates;

(d) if $\varepsilon\colon\mathscr{U}(L)\to K$ denotes the canonical supplementation of $\mathscr{U}(L)$, then $z_i\in\ker\varepsilon$, $1\leqslant i\leqslant k$.

In adopting the conventional multi-index notation (cf. [24, p. 51]) we write

$$\mathscr{U}(L)=\bigoplus_{0\leqslant a\leqslant\tau}\mathscr{O}(L,L_0)\,e^a,$$

where $\tau:=(p^{m_1}-1,...,p^{m_k}-1)$. Consider the projection $\lambda\colon\mathscr{U}(L)\to\mathscr{O}(L,L_0)$ defined by the summand $\mathscr{O}(L,L_0)\,e^\tau$ as well as the automorphism α of $\mathscr{O}(L,L_0)$ that satisfies $\alpha(z_i)=z_i$, $1\leqslant i\leqslant k$, and $\alpha(x)=x-\mathrm{tr}(\mathrm{ad}_{L/L_0}(x))\,1$ $\forall x\in L_0$. Owing to [9, (1.3)], $\mathscr{U}(L)\colon\mathscr{O}(L,L_0)$ is an α-Frobenius extension with Frobenius homomorphism λ. In particular, if $\mathrm{tr}(\mathrm{ad}_{L/L_0}(x))=0$ for every $x\in L_0$, then $\mathscr{U}(L):\mathscr{O}(L,L_0)$ is a Frobenius extension of first kind.

Suppose this to be the case and let $\{u_a; 0 \leqslant a \leqslant \tau\}$ be the dual basis of $\{e^a; 0 \leqslant a \leqslant \tau\}$ (cf. [9, (1.3)], i.e.,

$$\lambda(e^a u_b) = \delta(a, b), \qquad 0 \leqslant a, b \leqslant \tau.$$

An elementary computation shows that $c_0 = \sum_{0 \leqslant a \leqslant \tau} e^a u_a$. Consequently, $\varepsilon(c_0) = \varepsilon(u_0)$.

THEOREM 4.1. *Suppose that* $\varepsilon(u_0) \neq 0$ *and* $\operatorname{tr}(\operatorname{ad}_{L/L_0}(x)) = 0$ *for every* $x \in L_0$. *Let* M *be an* L*-module. If* $z_i \cdot M = (0)$, $1 \leqslant i \leqslant k$, *or if* M *is irreducible, then* $\dim_K H^n(L, M) \leqslant \sum_{i=0}^{t} \binom{k}{i} \dim_K H^{n-i}(L_0, M)$ $\forall n \geqslant 0$, *where* $t := \min(k, n)$.

Proof. If M is irreducible, then z_i operates either trivially or invertibly on M. In the latter case (2.1) ensures the vanishing of $H^n(L, M) \cong \operatorname{Ext}^n_{\mathscr{U}(L)}(K, M)$. For ease of notation, we put $\mathscr{R} := K[z_1, ..., z_k]$ so that $\mathcal{O}(L, L_0) \cong \mathscr{R} \otimes_K \mathscr{U}(L_0)$. Since the z_i act trivially on K and M, we obtain isomorphisms

$$K \otimes_K Q \cong Q, \qquad Q \in \{K, M\},$$

with $\mathcal{O}(L, L_0)$ operating on $K \otimes_K Q$ via

$$(r \otimes u) \cdot (\alpha \otimes q) = \varepsilon(r)\, \alpha \otimes uq.$$

Since $\varepsilon(u_0) \neq 0$ an application of (3.1) ensures the injectivity of the change of rings map Res_n: $H^n(L, M) \to \operatorname{Ext}^n_{\mathcal{O}(L, L_0)}(K, M)$. The above isomorphisms allow the computation of the latter spaces via the $\bigvee$-product (cf. [2, p. 205]). By applying [2, (3.1), p. 209] we obtain an identification

$$\operatorname{Ext}^n_{\mathcal{O}(L, L_0)}(K, M) \cong \bigoplus_{i+j=n} \Lambda^i(L/L_0) \otimes_K H^j(L_0, M),$$

with Λ^i denoting the i-fold wedge product. Now combine the above mappings to arrive at the asserted inequality. ∎

Remarks. (1) If M is finite dimensional, then one can always find $z_1, ..., z_k \in \mathscr{U}(L)^+ := \ker \varepsilon$ that annihilate M (cf. [8]).

(2) Suppose that $(L, [p])$ is a restricted Lie algebra and $L = L_0 \oplus \bigoplus_{i=1}^{k} Ke_i$. Then one usually defines $z_i := e_i^p - e_i^{[p]}$, $1 \leqslant i \leqslant k$, and every restricted L-module satisfies the condition of (4.1).

PROPOSITION 4.2. *Let L be a finite dimensional modular Lie algebra over the algebraically closed field K. Suppose that*

(a) $L = L_0 \oplus L_1$, *the vector space direct sum of two subalgebras,*

(b) $\operatorname{tr}(\operatorname{ad}_{L/L_0}(x)) = 0$ $\forall x \in L_0$,

(c) L_1 *is restricted and L is a restricted L_1-module.*

We let $\mathcal{O}(L, L_0)$ denote the subalgebra of $\mathcal{U}(L)$ that is generated by $L_0 \cup \{y^p - y^{[p]};\ y \in L_1\}$. Then the following statements are equivalent:

$$(1)\ \varepsilon(u_0) \neq 0, \qquad (2)\ L_1 \text{ is a torus and } \varepsilon(u_0) = (-1)^{\dim_K L_1}.$$

Proof. It obviously suffices to verify (1)$\Rightarrow$(2). Let $\{e_1, ..., e_k\} \subset L_1$ be a basis of L_1 and define $z_i := e_i^p - e_i^{[p]}$, $1 \leqslant i \leqslant k$. Condition (c) ensures that the z_i lie centrally in $\mathcal{U}(L)$. Owing to condition (b), $\mathcal{U}(L) : \mathcal{O}(L, L_0)$ is a Frobenius extension of first kind. As before we let $\lambda: \mathcal{U}(L) \to \mathcal{O}(L, L_0)$ denote the corresponding Frobenius homomorphism. Consider the canonical projection $\pi: \mathcal{U}(L_1) \to u(L_1)$ from $\mathcal{U}(L_1)$ onto the restricted universal enveloping algebra $u(L_1)$. It is well known that $u(L_1)$ is a free Frobenius algebra with K-basis $\{\pi(e^a);\ 0 \leqslant a \leqslant \tau\}$, where $\tau = (p-1, ..., p-1)$. The corresponding Frobenius homomorphism is given by $\mu(\sum_{0 \leqslant a \leqslant \tau} \alpha_a \pi(e^a)) = \alpha_\tau$. By applying Jacobson's refinement of the Poincaré–Birkhoff–Witt theorem we obtain $\lambda(\mathcal{U}(L_1)) \subset \mathcal{U}(L_1)$ as well as

$$\pi \circ \lambda|_{\mathcal{U}(L_1)} = \mu \circ \pi. \qquad (*)$$

The Poincaré–Birkhoff–Witt theorem in conjunction with condition (a) implies that the multiplication map induces a linear isomorphism $\mathcal{U}(L) \cong \mathcal{U}(L_1) \otimes_K \mathcal{U}(L_0)$. We thus write $u_0 = \sum_i y_i x_i$, with $x_i \in \mathcal{U}(L_0)$, $y_i \in \mathcal{U}(L_1)$, and $x_1 = 1$. As $e^a y_i \in \mathcal{U}(L_1)$ we obtain, observing (b),

$$\delta(a, 0) = \lambda(e^a u_0) = \sum_i \lambda(e^a y_i x_i) = \sum_i \lambda(e^a y_i)\, x_i, \qquad 0 \leqslant a \leqslant \tau.$$

It follows that $\lambda(e^a y_i) = 0$ for $i \neq 1$ and all a, whence $y_i = 0$ for $i \neq 1$. Consequently, $u_0 = y_1 \in \mathcal{U}(L_1)$. Now consider $v_0 := \pi(u_0) \in u(L_1)$. Because of $(*)$ we obtain

$$\mu(\pi(e^a)\, v_0) = \mu(\pi(e^a u_0)) = \pi(\lambda(e^a u_0)) = \delta(a, 0), \qquad 0 \leqslant a \leqslant \tau.$$

The assumption $\varepsilon(u_0) \neq 0$ now implies that the supplementation map γ of the Hopf algebra $u(L_1)$ does not annihilate v_0. If c_0 denotes the Casimir element of $u(L_1)$, then, in analogy with our remarks preceding (4.1), we have $\gamma(c_0) = \gamma(u_0) \neq 0$. According to (3.6) this entails the semisimplicity of $u(L_1)$ and a classical result by Hochschild [13] now ensures that L_1 is a torus.

By a theorem of Jacobson (cf. [24, (3.6) p. 82]) the basis $\{e_1, ..., e_k\}$ may now be chosen to consist of toral elements. Writing e_i instead of $\pi(e_i)$ we thus have $e_i^p = e_i$ in $u(L_1)$. We propose to show that $v_0 = \prod_{i=1}^k (e_i^{p-1} - 1)$. To that end, let $a \in \{0, ..., \tau\}$ be given and put $J_0 := \{i;\ a_i = 0\}$, $J_1 := \{i; a_i \neq 0\}$. Then we obtain

$$e^a \prod_{i=1}^{k} (e_i^{p-1} - 1) = \prod_{i \in J_0} (e_i^{p-1} - 1) \prod_{i \in J_1} (e_i^p e_i^{a_i - 1} - e_i^{a_i})$$
$$= \prod_{i \in J_0} (e_i^{p-1} - 1) \prod_{i \in J_1} (e_i^{a_i} - e_i^{a_i}).$$

Hence $e^a \prod_{i=1}^k (e_i^{p-1} - 1) = 0$ unless $a = 0$. Finally, we note that $\prod_{i=1}^k (e_i^{p-1} - 1) = e^\tau + x$, where $x \in \bigoplus_{o \leqslant a < \tau} Ke^a$. Consequently, $\mu(\prod_{i=1}^k (e_i^{p-1} - 1)) = 1$, as desired. The assertion now follows from the identity $\varepsilon(u_0) = \gamma(v_0) = (-1)^k$. ∎

Note that (4.2) applies in the following cases:

(1) $L = \mathscr{B}$, a Borel subalgebra of a classical semisimple Lie algebra, and $L_0 = \mathscr{N}$, the p-nilpotent radical of $\mathscr{B}$. In that case k coincides with the dimension of the standard torus.

(2) $L = W(1; n)_{(0)}$, the zero component of the natural filtration of the Zassenhaus algebra $W(1; n)$, $L_0 = W(1; n)_{(1)}$.

5. Rings of Bounded Global Dimension

In this section we present some ring-theoretic applications of the vanishing results obtained in Section 2. Throughout $R : S$ is assumed to be an extension of K-algebras.

Theorem 5.1. *Suppose that $lD(R) \leqslant n$ and let $u \in C_R(S)$ be an element of R such that* ad u: $R \to R$ *is locally nilpotent. Let M and N be R-modules such that u_M is injective and u_N is nilpotent. Then the following statements hold*:

(1) *If S is semisimple or if $n = 1$, then* $\mathrm{Ext}^n_{(R, S)}(M, N) = (0)$.

(2) *If $u \in \mathscr{Z}(R)$, then* $\mathrm{Ext}^n_R(M, N) = (0)$.

Proof. Let $E(M)$ be an injective envelope of M. According to (1) of Theorem 1.1 the operator $u_{E(M)}$ is bijective. Part (2) of (2.7) now ensures the vanishing of $\mathrm{Ext}^n_{(R, S)}(E(M), N)$. If S is semisimple, this group coincides with $\mathrm{Ext}^n_R(E(M), N)$. Alternatively, we invoke (2.5) to arrive at the same result in the case of $n = 1$. If u lies centrally in R, then (2.1) implies the triviality of $\mathrm{Ext}^n_R(E(M), N)$. The long exact cohomology sequence therefore gives rise to

$$(0) \to \mathrm{Ext}^n_R(M, N) \to \mathrm{Ext}^{n+1}_R(E(M)/M, N).$$

Since R has global dimension $\leqslant n$, the right-hand term vanishes, and we obtain the asserted result. ∎

Given an R-module M, we let $M_0(u) \subset M$ denote the Fitting-0-space of the operator u_M.

COROLLARY 5.2. *Let R be hereditary, $u \in R$, such that* ad $u: R \to R$ *is locally nilpotent. If M is a noetherian R-module, then there exists an R-submodule $V(u)$ such that $M = M_0(u) \oplus V(u)$.*

Proof. Since M is noetherian, $M_0(u)$ is a finitely generated submodule. Consequently, u operates nilpotently on $M_0(u)$. Since u also operates injectively on $M/M_0(u)$ and $lD(R) \leqslant 1$, it follows from (1) of Theorem 5.1 that $\mathrm{Ext}^1_R(M/M_0(u), M_0(u)) = (0)$. As a result, the exact sequence $(0) \to M_0(u) \to M \to M/M_0(u) \to (0)$ splits. ∎

Remark. The preceding result applies in particular to rings with left global dimension 0, i.e., semisimple rings. Next, suppose that M is a finitely generated $R[X]$-module, and let $f: M \to M$ be an R-module homomorphism. Corollary 5.2 then ensures the existence of a "Fitting-decomposition" for f.

COROLLARY 5.3. *Let R be a Dedekind ring, M a finitely generated R-module with torsion submodule $t(M)$. Then the following statements hold:*

(1) *For every $u \in R$ there exists a submodule $V(u)$ of M such that $M = M_0(u) \oplus V(u)$.*

(2) *There exists a projective submodule $V \subset M$ such that $M = t(M) \oplus V$.*

Proof. Since every Dedekind ring is noetherian and hereditary the decomposition asserted in (1) is an immediate consequence of Corollary 5.2. The fact that M is finitely generated readily yields the existence of $u \in R - \{0\}$ satisfying $u.t(M) = (0)$. As $M/t(M)$ is torsion free, u operates injectively on $M/t(M)$. It follows that $\mathrm{Ext}^1_R(M/t(M), t(M)) = (0)$, proving the existence of a direct complement V of $t(M)$. Since V is finitely generated and torsion free it is projective. ∎

THEOREM 5.4. *Let R be a left perfect ring of left global dimension $lD(R) \leqslant n$. Suppose that $u \in R$ is not a zero divisor and such that* ad $u: R \to R$ *is locally nilpotent. Let M and N be modules such that u_M is locally nilpotent and u_N is surjective; then* $\mathrm{Ext}^n_R(M, N) = (0)$.

Proof. According to Bass' theorem [1, p. 315] there exists an exact sequence $(0) \to Q \to P(N) \to N \to (0)$ with $P(N)$ being a projective cover of N. As $lD(R) \leqslant n$ there results an exact sequence

$$\cdots \to \mathrm{Ext}^n_R(M, P(N)) \to \mathrm{Ext}^n_R(M, N) \to (0).$$

A consecutive application of (1.3(1)) and (2.1) shows the vanishing of the left-hand side. Consequently, $\mathrm{Ext}^n_R(M, N) = (0)$. ∎

COROLLARY 5.5. *Let R be a left perfect, hereditary ring. Suppose that $u \in R$ is not a zero divisor and assume that* ad u: $R \to R$ *is locally nilpotent. If M is an R-module then there exists $V \subset M_0(u)$ such that $M = V \oplus M_1(u)$.*

Proof. It is clear from the definitions that u operates surjectively on the submodule $M_1(u)$. Since the descending sequence $(u^nR)_{n \geqslant 1}$ of right ideals is stationary, there exist $n \in \mathbb{N}$ and a sequence $(r_k)_{k \in \mathbb{N}}$ such that $u^n = u^{n+k} r_k \; \forall k \in \mathbb{N}$. Let m be an element of M. Then $u^n m = u^{n+k} r_k m = u^k(u^n r_k m)$, proving that u operates nilpotently on $M/M_1(u)$. Theorem 5.4 now implies the vanishing of $\mathrm{Ext}^1_R(M/M_1(u), M_1(u))$. Hence there is a submodule $V \subset M$ such that $M = M_1(u) \oplus V$. By the above observation V is contained in $M_0(u)$. ∎

PROPOSITION 5.6. *Suppose that K is semisimple and let $u \in R$ be an element such that* ad u: $R \to R$ *is locally nilpotent. Let M be an R-module such that u_M is locally finite. Then $M = M_0(u) \oplus M_1(u)$.*

Proof. As K is semisimple $X := M/M_0(u)$ is a K-projective R-module on which u operates via an injective and locally finite transformation. Since K is artinian and $u_{M_0(u)}$ is locally nilpotent, (3) of (2.7) applies and we conclude that $\mathrm{Ext}^1_{(R,K)}(X, M_0(u)) = (0)$. Since this group coincides, because of the semisimplicity of K, with $\mathrm{Ext}^1_R(X, M_0(u))$, it follows that there exists an R-submodule V with $M = M_0(u) \oplus V$. As u operates invertibly on X, V is contained in $M_1(u)$.

Since K is noetherian the surjective map $u_{M_1(u)}$ is invertible. For $m \in M_1(u)$ write $m = m_0 + v$, where $m_0 \in M_0(u)$ and $v \in V$. Then $m_0 \in M_0(u) \cap M_1(u) = (0)$. Consequently $V = M_1(u)$. ∎

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