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Generalized Games and Non-compact Quasi-variational Inequalities

Xian-Zhi Yuan* and Kok-Keong Tan

Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, B3H 3J5, Canada

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In this paper, by developing an approximation approach which is originally due to Tuleca in 1986, we prove the existence of equilibria for generalized games in which constraint mappings (correspondences) are lower (resp., upper) semicontinuous instead of having lower (resp., upper) open sections or open graphs in infinite dimensional topological spaces. Then, existence theorems of solutions for quasi-variational inequalities and non-compact generalized quasi-variational inequalities are also established. Finally, existence theorems of constrained games with non-compact strategy sets are derived. Our results unify and generalize many well known results given in the existing literature. In particular, we answer the question raised by Yannelis and Prabhakar in 1983 in the affirmative under more weaker conditions.

1. INTRODUCTION

In the last three decades, the classical Arrow–Debreu existence theorem of Walrasian equilibria [2] has been generalized in many directions. In the finite dimensional spaces, Gale and Mas-Colell [20] proved the existence of a competitive equilibrium without the assumptions of total or transitive preference mappings (correspondences). Shafer and Sonnenschein [36] gave results in the same direction and they proved the Arrow–Debreu Lemma for abstract economies for the case where preference correspondences may not be total or transitive. For infinite dimensional strategy spaces and finite or infinite many agents, the existence results of equilibria of generalized games were given by Aubin [3], Aubin and Ekeland [4],

*To whom correspondence should be addressed. E-mail address: yuan@cs.dal.ca. and xzy@axiom.math.uq.edu.au. The current corresponding address is: George Xian-Zhi Yuan, Dept. of Mathematics, Dalhousie University, Halifax, Nova Scotia, B3H 3J5, Canada.

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Bewley [7], Border [8], Flam [17], Florenzano [18], Aliprantis and Brown [1], Yannelis and Prabhakar [47], Khan and Vohra [24], Toussaint [43], Tulcea [44, 45], Khan and Papageorgiou [25], Yannelis [46], Kim and Richter [27], Kim *et al*. [28], Tarafdar and Mehta [40], Chang [9], Tian [41], Ding and Tan [13], Ding *et al*. [14], and others. Most existence theorems mentioned above, however, are obtained by assuming that the constraint and preference correspondences have open graphs or have open lower (resp., upper) open sections. Besides this, strategy sets in most of those models are assumed to be compact in topological vector spaces. These are restricted assumptions since it is well known that if a correspondence has an open graph, then it has open upper and lower sections and thus the correspondences with open lower sections are lower semicontinuous. However, a continuous correspondence does not hold open lower (or upper) section properties in general; and we also know that in the infinite settings, the set of feasible allocations generally is not compact in any topology of the commodity spaces. The motivations for economists continually to be interested in setting forth conditions for the existence of equilibria come from the importance of generalized games (also called abstract economies) in the study of markets and other general games and from the restrictions of the existence theorems.

In this paper, by developing an approximate method which was first used by Tulcea [44, 45] and Chang [9], we give existence theorems of equilibria for generalized games in which constraint correspondences are lower (resp., upper) semicontinuous instead of having lower (resp., upper) open sections or open graphs in infinite dimensional locally convex topological vector spaces. Also, in our framework, strategy spaces may be infinitely dimensional and non-compact, the number of agents may be uncountable infinite, and preference correspondences may be nontotal-nontransitive and may not have open lower (resp., upper) section properties. Thus, our results unify and generalize many of the existence theorems, on equilibria of generalized games by relaxing the compactness of strategy spaces and continuity of constraint and preference mappings (correspondences). In particular, we answer the question raised by Yannelis and Prabhakar [47, p. 243] in the affirmative. As a consequence of equilibria of generalized games, the Fan and Glicksberg fixed point theorem is derived. Then, by existence theorems of generalized games with lower (resp., upper) semicontinuous constraint correspondence, existence theorems of solutions for non-compact quasi-variational inequalities and non-compact generalized quasi-variational inequalities are also given in locally convex topological vector space. Finally, with applications of quasi-variational inequalities, existence theorems of non-compact constrained games are established. Our results unify and generalize corresponding results due to Aubin [3], Aubin and Ekeland [4], Chang [9], Borglin and Keiding [6], Ding and Tan [13], Shafer and Sonnenschein [36], Toussaint [43], Tulcea [44, 45], Yannelis [46], Yannelis and Prabhakar [47], Shih and Tan [38], and others.

This paper is organized as follows. Notations and definitions are given in Section 1. In Section 2, one maximal element theorem is proved and the approximation theorem for upper semicontinuous correspondences of Tulcea [45, p. 288] is recalled. They are the auxiliary results which will be needed in Section 3. The main results of this paper are given in Section 3. That is, we describe briefly an approximation approach to prove the existence of equilibria for generalized games in which constraint mappings (correspondences) are lower (resp., upper) semicontinuous instead of having lower (resp., upper) open sections or open graphs in infinite dimensional topological spaces. Also, in our framework, strategy spaces may be infinite dimensional and non-compact, the number of agents may be uncountable infinite, and preference correspondences may be nontotal-nontransitive. With applications of existence theorems of generalized games in Section 4, non-compact quasi-variational inequalities and non-compact generalized quasi-variational inequalities are established in Section 5. Finally, one existence theorem of non-compact constrained games is proved by employing quasi-variational inequalities established in Section 4.

Now we introduce some notations and definitions. Let A be a non-empty subset of a vector space D. We shall denote by co A the convex hull of A. If S is a subset of a topological space X, the closure and interior of S in X are denoted by $\mathrm{cl}_X S$ and by $\mathrm{int}_X S$, respectively. A subset S of X is said to be *compactly open* in X if the set $S \cap X$ is open in C for each non-empty subset C of X. Let X be a non-empty set. We denote by 2^X the family of all subsets of X. Suppose X and Y are two non-empty sets and $F\colon X \to 2^Y$ is a set-valued mapping. Then the graph of F, denoted by Graph F, is the set $\{(x, y) \in X \times Y : y \in F(x)\}$. Let X and Y be two topological spaces and $F\colon X \to 2^Y$ a set-valued mapping. Then we have

(1) F is said to have an open graph if the Graph F of F is open in $X \times Y$;

(2) the mapping $\overline{F}\colon X \to 2^Y$ is defined by $\overline{F}(x) := \{y \in Y : (x, y) \in \mathrm{cl}_{X\times Y}\mathrm{Graph}(F)\}$ for each $x \in X$;

(3) F is said to be lower (resp., upper) semicontinuous if for each closed (resp., open) subset C of Y, the set $\{x \in X\colon F(x) \subset C\}$ is closed (resp., open) in X;

(4) F is said to have the compactly open lower (resp., upper) sections property if the set $F^{-1}(y) := \{x \in X : y \in F(x)\}$ is open in each non-empty compact subset C of X for each $y \in Y$ (resp., $F(x)$ is open in each non-empty compact subset C of Y for each $x \in X$);

(5) F is said to be compact if for each $x \in X$, there exists a neighborhood V_x at x such that the set $F(V_x) := \bigcup_{z \in V_x} F(z)$ is relative compact in Y; and

(6) a subset X of a topological vector space E is said to have the property (K) if for every compact B of X, the convex hull of B is relative compact in E.

Let X and Y be two sets and $P\colon X \to 2^Y$ a set-valued mapping. We also recall that a point $x \in X$ is said to be a maximal element of P if $P(x) = \emptyset$.

Let X be a topological space and Y a non-empty subset of a vector space E. Suppose $\theta\colon X \to E$ is a (single-valued) mapping and $\phi\colon X \to 2^Y$ is a set-valued mapping. Then we have

(1) ϕ is said to be of class $L_{\theta,C}$ if for every $x \in X$, $\operatorname{co}\phi(x) \subset Y$, $\theta(x) \notin \operatorname{co}\phi(x)$ and the set $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is compactly open in X for each $y \in Y$;

(2) a mapping $\phi_x\colon X \to 2^Y$ is said to be an $L_{\theta,C}$-majorant of ϕ at $x \in X$ if there exists an open neighborhood N_x of x in X such that (a) for each $z \in N_x$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin \operatorname{co}\phi_x(z)$; (b) for each $z \in X$, $\operatorname{co}\phi_x(z) \subset Y$; and (c) for each $y \in Y$, $\phi_x^{-1}(y)$ is compactly open in X; and

(3) ϕ is said to be $L_{\theta,C}$-majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an $L_{\theta,C}$—majorant of ϕ at x in X.

In this paper, we shall only deal with either the case (I) $X = Y$ and is a non-empty convex subset of a topological vector space and $\theta := I_X$, the identity mapping on X; or the case (II) $X = \Pi_{i \in I} X_i$ and $\theta = \pi_j\colon X \to X_j$ is the projection of X onto X_j and $X_j := Y$ is a non-empty convex subset of a topological vector space. In both cases (I) and (II), we shall write L_C in place of $L_{\theta,C}$.

Let X and Y be topological spaces. We also recall that a mapping $T\colon X \to 2^Y$ is said to be

(1) quasi-regular if (i) it has open lower sections; (ii) $T(x)$ is non-empty and convex for each $x \in X$, and (iii) $\overline{T}(x) = \operatorname{cl}_X T(x)$ for each $x \in X$; and

(2) regular if it is quasi-regular and has an open graph.

Let I be a finite or infinite set of players (resp., agents). A generalized game (resp., an abstract economy) is a family $\Gamma = (X_i; A_i, B_i; P_i)_{i \in I}$ of quadruples $(X_i; A_i, B_i; P_i)$ where for each $i \in I$, X_i is a topological space, $A_i, B_i\colon X := \Pi_{j \in I} X_j \to 2^{X_i}$ are constraint mappings, and $P_i\colon X \to 2^{X_i}$ is a preference mapping. An equilibrium point for Γ is a point $x^* \in X$ such that for each $i \in I$, we have

$$x_i^* = \pi_i(x^*) \in \overline{B}_i(x^*) \qquad \text{and} \qquad A_i(x^*) \cap P_i(x^*) = \emptyset,$$

where π_i: $X \to X_i$ is the ith projection. We recall that a qualitative game is a family $\Gamma = (X_i; P_i)_{i \in I}$ of ordered pairs (X_i, P_i), where for each $i \in I$, X_i is a topological space and P_i: $X = \Pi_{j \in I} X_j \to 2^{X_i}$ is an irreflexive preference mapping (i.e., $x_i \notin P_i(x)$ for all $x \in X$). A point $x^* \in X$ is said to be an equilibrium point of the qualitative game Γ if $P_i(x^*) = \emptyset$ for all $i \in I$.

Throughout this paper, all topological vector spaces are assumed to be Hausdorff.

2. MAXIMAL ELEMENT THEOREMS

By the same argument used in the proof of Lemma 1 of Ding *et al*. [14, p. 510] (see also Lemma 2 of Ding and Tan [13] or Proposition 1 of Tulcea [44, p. 3]), we can have the following result and thus its proof is omitted here to save space.

LEMMA 2.1. *Let X be a regular topological space and Y a non-empty subset of a vector space E. Let θ: $X \to E$ be a single-valued mapping and P: $X \to 2^Y$ $L_{\theta,C}$-majorized. If every open subset of X containing the subset $\{x \in X : P(x) \neq \emptyset\}$ is paracompact, then there exists a mapping ϕ: $X \to 2^Y$ which is of class $L_{\theta,C}$ such that $P(x) \subset \phi(x)$ for each $x \in X$.*

By Lemma 2.1, we have the following theorem concerning the existence of a maximal element in topological vector spaces.

THEOREM 2.2. *Let X be a non-empty paracompact convex subset of a topological vector space and P: $X \to 2^X$ L_C-majorized* (i.e., $L_{I_X,C}$-majorized). *Suppose that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in \operatorname{co}(X_0 \cup \{y\})$ such that $x \in \operatorname{co} P(y)$. Then there exists an $\hat{x} \in K$ such that $P(\hat{x}) = \emptyset$.*

Proof. Suppose that for each $x \in X$, $P(x) \neq \emptyset$. Then P: $X \to 2^X$ is a mapping such that $P(x) \neq \emptyset$ for each $x \in X$. By Lemma 2.1, there exists a mapping ϕ: $X \to 2^X$ which is of class L such that $P(x) \subset \phi(x)$ for all $x \in X$. Moreover, for each $y \in X \setminus K$, there is $x \in \operatorname{co}(X_0 \cup \{y\})$ such that $x \in \operatorname{co} P(y)$, but then $x \in \operatorname{co} \phi(y)$. By Theorem 3″ of Ding and Tan [12], there exists $\hat{x} \in X$ such that $\hat{x} \in \operatorname{co} \phi(\hat{x})$, which contradicts our assumption that ϕ is of class L. Thus there must exist some $\hat{x} \in X$ such that $P(\hat{x}) = \emptyset$. By our hypotheses, $\hat{x}$ must be in K and thus we complete the proof. ∎

Remark 2.3. Theorem 2.2 indeed generalizes Theorem 2.1 of Toussaint [43, p. 101], Theorem 1 of Yannelis [46], and Corollary 1 of Borglin and Keiding [6, p. 314] in several aspects.

Let X be a non-empty set, Y a non-empty subset of the topological vector space E and $F\colon X \to 2^Y$. A family $(f_j)_{j \in J}$ of correspondences between X and Y, indexed by a non-empty filtering set J (we denote by $\leq$ the order relation in J) is said to be an upper approximating family for F if

(A_I) $F(x) \subset f_j(x)$ for all $x \in X$ and all $j \in J$;

(A_{II}) for each $j \in J$, there is $j^* \in J$ such that for each $h \geq j^*$ and $h \in J$, $f_h(x) \subset f_j(x)$ for each $x \in X$; and

(A_{III}) for each $x \in X$ and $V \in \mathcal{B}$, where $\mathcal{B}$ is the fundamental system of zero of topological vector space E, there is $j_{x,V} \in J$ such that $f_h(x) \subset F(x) + V$ if $h \in J$ and $j_{x,V} \leq h$.

From (A_I)–(A_{III}), it is easy to deduce that:

(A_{IV}) for each $x \in X$ and $k \in J$

$$F(x) \subset \bigcap_{j \in J} f_j(x) = \bigcap_{k \leq j,\, k \in J} f_j(x) \subset \overline{F(x)} \subset \bar{F}(x).$$

We now state the following Lemma 2.4 which is Theorem 3 of Tulcea [45, p. 280].

LEMMA 2.4. *Let X be a paracompact space and Y a non-empty convex subset of locally convex topological vector space E such that Y has the property* (K). *Suppose $F\colon X \to 2^Y$ is such that*

(1) *F is compact and upper semi-continuous*; *and*

(2) *$F(x)$ is non-empty compact and convex for each $x \in X$.*

Then there is a family $(f_j)_{j \in J}$ of correspondences between X and Y, indexed by a directed set J, such that

(a) *for every $j \in J$, the correspondence f_j is regular*;

(b) *$(f_j)_{j \in J}$ and $(\bar{f}_j)_{j \in J}$ are upper approximating families for F*;

(c) *for every $j \in J$ the correspondence $\bar{f}_j$ is continuous if Y is compact.*

3. EQUILIBRIA

In this section, in order to employ the approximation method to study existence of equilibrium points for generalized games in which constraint mappings are upper semicontinuous, we shall first need the following result which is Lemma 5.3 of Tan and Yuan [39]:

LEMMA 3.1. *Let X be a topological space, Y a non-empty subset of a topological vector space E, $\mathcal{B}$ a base for the zero neighborhoods in E, and B: $X \to 2^Y$. For each $V \in \mathcal{B}$, let B_V: $X \to 2^Y$ be defined by*

$$B_V(x) := (B(x) + V) \cap Y$$

for each $x \in X$. Suppose $\hat{x} \in X$ and $\hat{y} \in Y$ are such that $\hat{y} \in \bigcap_{V \in \mathcal{B}} \overline{B_V}(\hat{x})$. Then we have $\hat{y} \in \overline{B}(\hat{x})$.

As an application of Theorem 2.2, we first have the following existence theorem of equilibria for a one-person game.

THEOREM 3.2. *Let X be a non-empty paracompact convex subset of a topological vector space. Suppose A, B, P: $X \to 2^X$ are three mappings such that*

(1) *$A \cap P$ is of class L_C-majorized;*

(2) co $A(x) \subset B(x)$ *for each $x \in X$ and the set $A^{-1}(y)$ is compactly open in X for each $y \in X$;*

(3) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, we have* $\operatorname{co}(X_0 \cup \{y\}) \cap \operatorname{co}(P(y) \cap A(y)) \neq \emptyset$.

Then exists an $x \in K$ such that $x \in \overline{B}(x)$ and $A(x) \cap P(x) = \emptyset$.

Proof. Let $F := \{x \in X : x \in \overline{B}(x)\}$. Then F is closed in X. Define Ψ: $X \to 2^X$ by

$$\Psi(x) = \begin{cases} A(x) \cap P(x), & \text{if } x \in F, \\ A(x), & \text{if } x \notin F. \end{cases}$$

If $x \notin F$, then $X \setminus F$ is an open neighborhood of x such that for each $z \in X \setminus F$ we have $z \notin \overline{B}(z)$. Now define Φ_x: $X \to 2^X$ by $\Phi_x(z) = A(z)$ for each $z \in X$ and $N_x := X \setminus F$. It is clear that N_x is an open neighborhood of x in X. Moreover it is easy to see that Φ_x is an L_C-majorant of Ψ at x. Now if $x \in F$ and $\Psi(x) = A(x) \cap P(x) \neq \emptyset$, as $A \cap P$ is L_C-majorized, there exist an open neighborhood N_x of x in X and a mapping Φ_x: $X \to 2^X$ such that $\Psi(z) = A(z) \cap P(z) \subset \Phi_x(z)$, $z \notin \operatorname{co} \Phi_x(z)$ for each $z \in N_x$ and the set $\operatorname{co} \Phi_x^{-1}(y)$ is also compactly open in X for each $y \in X$. Define the mapping Φ'_x: $X \to 2^X$ by

$$\Phi'_x(z) := \begin{cases} A(z) \cap \Phi_x(z), & \text{if } z \in F, \\ A(z), & \text{if } z \notin F. \end{cases}$$

Note that $(A \cap P)(z) \subset \Phi_x(z)$ and $A(z) \subset A(z)$ for each $z \in N_x$. It follows that $\Psi(z) \subset \Phi'_x(z)$. Furthermore, it is clear that $z \notin \operatorname{co} \Phi'_x(z)$ for all $z \in X$ and the set $(\Phi'_x)^{-1}(y) := [\Phi_x^{-1}(y) \cup (X \setminus F)] \cap A^{-1}(y)$ is compactly open in X for any $y \in X$. Thus Φ'_x is an L_C-majorant of Ψ at point x and Ψ is of L_C-majorized. By condition (ii), it follows that for each $y \in X \setminus K$, there exists $x \in \operatorname{co}(X_0 \cup \{y\})$ such that $x \in \operatorname{co}(P(y) \cap A(y)) \subset \operatorname{co} \Psi(y)$. By Theorem 2.2, there exists an $x \in K$ such that $\Psi(x) = \emptyset$. Since $A(x) \neq \emptyset$ for all $x \in X$, we must have $x \in \overline{B}(x)$ and thus $\Psi(x) = A(x) \cap P(x) = \emptyset$. ∎

As another application of Theorem 2.2, we have the following existence of equilibria for a qualitative game in topological vector spaces.

THEOREM 3.3. *Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game such that $X = \Pi_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied for each $i \in I$:*

(*a*) *X_i is a non-empty convex subset of a topological vector space*;

(*b*) *P_i: $X \to 2^{X_i}$ is L_C-majorized*;

(*c*) $\bigcup_{i \in I}\{x \in X : P_i(x) \neq \emptyset\} = \bigcup_{i \in I} \operatorname{int}_X\{x \in X : P_i(x) \neq \emptyset\}$;

(*d*) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in \operatorname{co}(X_0 \cup \{y\})$ with $x_i \in \operatorname{co} P_i(y)$ for all $i \in I$.*

Then Γ has an equilibrium point in K.

Proof. For each $x \in X$, let $I(x) := \{i \in I : P_i(x) \neq \emptyset\}$. Define a mapping P: $X \to 2^X$ by

$$P(x) := \begin{cases} \bigcap_{i \in I(x)} \operatorname{co} P'_i(x), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{if } I(x) = \emptyset, \end{cases}$$

where $P'_i(x) = \Pi_{j \neq i,\, j \in I} X_j \otimes P_j(x)$ for each $x \in X$. Then it is clear that for each $x \in X$, we have $I(x) \neq \emptyset$ if and only if $P(x) \neq \emptyset$. We shall show that P is L_c-majorized. Let $x \in X$ be such that $P(x) \neq \emptyset$. Then there exists an $i \in I$ such that $x \in \operatorname{int}_X\{z \in X : P_i(z) \neq \emptyset\}$. By (b), there exists an open neighborhood $N(x)$ of x in X and an L_C-majorant ψ_i of P_i at x such that (i) for each $z \in N(x)$, $P_i(z) \subset \psi_i(z)$ and $z_i \notin \operatorname{co} \psi_i(z)$; (ii) for each $z \in X$, $\operatorname{co} \psi_i(z) \subset X_i$; and (iii) for each $y \in X_i$, $\psi_i^{-1}(y)$ is compactly open in X. By (c), without loss of generality, we may assume that $N(x) \subset \operatorname{int}_X\{z \in X : P_i(z) \neq \emptyset\}$, so that $P_i(z) \neq \emptyset$ for all $z \in N(x)$. Now define Ψ_i: $X \to 2^X$ by $\Psi_i(z) := \Pi_{j \neq i,\, j \in I} X_j \otimes \operatorname{co} \psi_i(z)$ for each $z \in X$. We claim that Ψ_i is an L_C-majorant of P at x. Indeed, for each $z \in N(x)$, by (i), $P(z) = \bigcap_{j \in I(z)} P'_i(z) \subset P'_i(z) \subset \Psi_i(z)$ and $z \notin \operatorname{co} \Psi_i(z)$ and for each $z \in X$, $\operatorname{co} \Psi_i(z) \subset \Pi_{j \neq i,\, j \in I} X_j \otimes \operatorname{co} \psi_i(z) \subset X$. Since for each $y \in X$,

$$\Psi_i^{-1}(y) = \begin{cases} (\operatorname{co} \psi_i)^{-1}(y_i), & \text{if } y_j \in X_j \text{ for all } j \neq i, \\ \emptyset, & \text{if } y_j \notin X_j \text{ for some } j \neq i \end{cases}$$

and $(\operatorname{co} \psi_i)^{-1}(y_i)$ is compactly open in X by Lemma 5.1 of Yannelis and Prabhakar [47], thus $\Psi_i^{-1}(y)$ is compactly open in X. Therefore, Ψ_i is an L_C-majorant of P at x. This shows that P is L_C-majorized. By condition (d), for each $y \in X \setminus K$, there exists $x \in \operatorname{co}(X_0 \cup \{y\})$ with $x_i \in \operatorname{co} P_i(y)$ for all $i \in I$ so that $x \in \operatorname{co} P(y)$ by the definitions of P and P_i'. Hence all hypotheses of Theorem 3.2 are satisfied. By Theorem 3.2, there exists an $\hat{x} \in K$ such that $P(\hat{x}) = \emptyset$. This implies $I(\hat{x}) = \emptyset$ and hence $P_i(\hat{x}) = \emptyset$ for all $i \in I$. Thus we complete the proof. ∎

Now we use the following approximation technique to study existence of equilibria for generalized games. The idea is as follows: for a given generalized game $\Gamma = (X_i; A_i, B_i; P_i)_{i \in I}$, we first construct an associated approximate generalized game $\Gamma_V = (X_i; (A_V)_i, (B_V)_i; (P_V)_i)_{i \in I}$ for each non-empty open neighborhood V of zero in locally convex topological vector space. Then for the approximate generalized game $\Gamma_V = (X_i; (A_V)_i, (B_V)_i; (P_V)_i)_{i \in I}$, there exists an associated qualitative game $\mathsf{T}_V = (X_i; (Q_V)_i)_{i \in I}$ which exhibits the same equilibrium points as the approximate generalized game $\Gamma_V = (X_i; (A_V)_i, (B_V)_i; (P_V)_i)_{i \in I}$. Finally the existence of equilibria for Γ follows by Lemma 3.1.

THEOREM 3.4. *Let $\mathsf{G} = (X_i; A_i, B_i; P_i)_{i \in I}$ be a generalized game such that $X = \Pi_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied for each $i \in I$:*

(a) *X_i is a non-empty convex subset of a locally convex Hausdorff topological vector space E_i;*

(b) *$A_i\colon X \to 2^{X_i}$ is lower semicontinuous and for each $x \in X$, $A_i(x)$ is non-empty and* $\operatorname{co} A_i(x) \subset B_i(x)$;

(c) *$A_i \cap P_i$ is L_C-majorized;*

(d) *the set $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X; and*

(e) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$ there is an $x \in \operatorname{co}(X_0 \cup \{y\})$ such that $x_i \in \operatorname{co}(A_i(y) \cap P_i(y))$ for all $i \in I$.*

Then G has an equilibrium point in K, i.e., there exists a point $\hat{x} = (\hat{x}_i)_{i \in I} \in K$ such that for each $i \in I$,

$$\hat{x}_i \in \overline{B}_i(\hat{x}) \qquad \text{and} \qquad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

Proof. Let $V := \Pi_{i \in I} V_i$, where V_i is any open convex neighborhood of zero in E_i for each $i \in I$. Fix any $i \in I$, we define $A_{V_i}, B_{V_i}\colon X \to 2^{X_i}$ by $A_{V_i}(x) := (\operatorname{co} A_i(x) + V_i) \cap X_i$ and $B_{V_i}(x) := (B_i(x) + V_i) \cap X_i$ for each $x \in X$. By (a), $\operatorname{co} A_i$ is also lower semicontinuous by Proposition 2.6 of [33, p. 366]. By Lemma 4.1 of Chang [9, p. 244] (see also Tulcea [44, p. 7]) the mapping A_{V_i} has an open graph in $X \times X$. We set $F_{V_i} := \{x \in X : x_i \notin \overline{B_{V_i}}(x)\}$. Then F_{V_i} is open in X. Define a mapping $Q_{V_i}\colon X \to 2^{X_i}$ by

$$Q_{V_i}(x) = \begin{cases} (A_i \cap P_i)(x), & \text{if } x \notin F_{V_i}, \\ A_i(x), & \text{if } x \in F_{V_i}. \end{cases}$$

We shall prove that the qualitative game $\mathcal{T} = (X_i, Q_{V_i})_{i \in I}$ satisfies all conditions of Theorem 3.4. First we note that for each $i \in I$, the set

$$\begin{aligned} \{x \in X : Q_{V_i}(x) \neq \varnothing\} &= F_{V_i} \cup \{x \in X \setminus F_{V_i} : A_i(x) \cap P_i(x) \neq \varnothing\} \\ &= F_{V_i} \cup \left((X \setminus F_{V_i}) \cap E^i\right) = F_{V_i} \cup E^i \end{aligned}$$

is open in X by (c). Let $x \in X$ be such that $Q_{V_i}(x) \neq \varnothing$. We consider the following two cases:

Case 1. If $x \in F_{V_i}$, let $\Psi_x := A_{V_i}$ and $N_x = F_{V_i}$. Then N_x is an open neighborhood of x in X such that (i) $Q_{V_i}(z) \subset \Psi_x(z)$ and by (b), $z_i \notin \operatorname{co} \Psi_x(z)$ for each $z \in N_x$; (ii) $\operatorname{co} \Psi_x(z) \subset X_i$ for each $z \in X$ by (b); and (iii) $\Psi_x^{-1}(y) = A_{V_i}^{-1}(y)$ is open in X for all $y \in X_i$ since A_{V_i} has an open graph. Therefore, Ψ_x is an L_C-majorant of Q_{V_i} at x.

Case 2. If $x \notin F_{V_i}$, note that $Q_{V_i}(x) = (A_i \cap P_i)(x) \neq \varnothing$, $A_i \cap P_i$ is L_C-majorized, and there exist an open neighborhood N_x of x in X and a mapping $\phi_x\colon X \to 2^{X_i}$ such that (i) $(A_i \cap P_i)(z) \subset \phi_x(z)$ and $z_i \notin \operatorname{co} \phi_x(z)$ for each $z \in N_x$; (ii) $\operatorname{co} \phi_x(z) \subset X_i$ for each $z \in X$; and (iii) $\phi_x^{-1}(y)$ is compactly open in X for each $y \in X_i$. Define $\Psi_x\colon X \to 2^{X_i}$ by

$$\Psi_x(z) = \begin{cases} A_{V_i}(z) \cap \phi_x(z), & \text{if } z \notin F_{V_i}, \\ A_{V_i}(z), & \text{if } z \in F_{V_i}. \end{cases}$$

Note that as $(A_i \cap P_i)(z) \subset \phi_x(z)$ and $A_i(z) \subset A_{V_i}(z)$ for each $z \in N_x$, we have $Q_{V_i}(z) \subset \Psi_x(z)$ and $\operatorname{co} \Psi_x(z) \subset X_i$. It is easy to see that $z_i \notin \operatorname{co} \Psi_x(z)$ for all $z \in X$. Moreover, for any $y \in X_i$, the set

$$\begin{aligned} \Psi_x^{-1}(y) &= \{z \in X : y \in \Psi_x(z)\} \\ &= \{z \in X \setminus F_{V_i} : y \in \Psi_x(z)\} \cup \{z \in F_{V_i} : y \in \Psi_x(z)\} \\ &= \{z \in X \setminus F_{V_i} : y \in A_{V_i}(z) \cap \phi_x(z)\} \cup \{z \in F_{V_i} : y \in A_{V_i}(z)\} \\ &= \left[(X \setminus F_{V_i}) \cap A_{V_i}^{-1}(y) \cap \phi_x^{-1}(y)\right] \cup \left[F_{V_i} \cap A_{V_i}^{-1}(y)\right] \\ &= \left[\phi_x^{-1}(y) \cup F_{V_i}\right] \cap A_{V_i}^{-1}(y) \end{aligned}$$

is compactly open in X. Thus Ψ_x is an L_C-majorant of Q_{V_i} at point x. Therefore Q_{V_i} is an L_C-majorized mapping. By our assumption, for each $y \in X \setminus K$, there is an $x \in \operatorname{co}(X_0 \cup \{y\})$ with $x_i \in \operatorname{co}(A_i(y) \cap P_i(y))$ for all $i \in I$. Note that if $y \notin F_{V_i}$, then $x_i \in \operatorname{co}(A_i(y) \cap P_i(y)) \subset \operatorname{co} Q_{V_i}(y)$ and if $y \in F_{V_i}$, then $x_i \in \operatorname{co}(A_i(y) \cap P_i(y)) \subset \operatorname{co}(A_i(x)) = \operatorname{co} Q_{V_i}(y)$. Thus for each $i \in I$, $x_i \in \operatorname{co} Q_{V_i}(y)$. Moreover the set $\{x \in X : Q_{V_i}(x) \neq \emptyset\} = F_{V_i} \cup \{x \in X \setminus F_{V_i} : (A_i \cap P_i)(x) \neq \emptyset\} = F_{V_i} \cup E^i$, which is open in X by condition (c). Therefore all hypotheses of Theorem 3.3 are satisfied. By Theorem 3.3, there exists a point $x_V = (x_{V_i})_{i \in I} \in K$ such that $Q_{V_i}(x_V) = \emptyset$ for all $i \in I$. For each $i \in I$, as $A_i(x)$ is non-empty, it follows that $x_{V_i} \in \overline{B_{V_i}}(x_V)$ and $A_i(x_V) \cap P_i(x_V) = \emptyset$.

Now for each $i \in I$, let $\mathcal{B}_i$ be the collection of all open convex neighborhoods of zero in E_i and $\mathcal{B} = \Pi_{i \in I} \mathcal{B}_i$. For any $V \in \mathcal{B}$, let $V := \Pi_{j \in I} V_j$, where $V_j \in \mathcal{B}_j$ for each $j \in I$. By the argument above, there exists $\hat{x}_V \in K$ such that $\hat{x}_{V_i} \in \overline{B_{V_i}}(\hat{x}_V)$ and $A_i(\hat{x}_V) \cap P_i(\hat{x}_V) = \emptyset$ for all $i \in I$, where $B_{V_i}(x) = (B_i(x) + V_i) \cap X_i$ for each $x \in X$. It follows that the set $Q_V := \{x \in K : x_i \in \overline{B_{V_i}}(x) \text{ and } A_i(x) \cap P_i(x) = \emptyset\}$ is a non-empty and closed subset of K by the condition (d).

In order to finish the proof, it suffices to prove that the family $\{Q_V\}_{V \in \mathcal{B}}$ has the finite intersect property. Let $\{V_1, \dots, V_n\}$ be any finite subset of $\mathcal{B}$. For each $i = 1, \dots, n$, let $V_i = \Pi_{j \in I} V_{ij}$, where $V_{ij} \in \mathcal{B}_i$ for each $j \in I$ and let $V := \Pi_{j \in I}(\bigcap_{i=1}^n V_{ij})$. Then $Q_V \neq \emptyset$. Clearly $Q_V \subset \bigcap_{i=1}^n Q_{V_i}$, and we have that $\bigcap_{i=1}^n Q_{V_i} \neq \emptyset$. Therefore the family $\{Q_V : V \in \mathcal{B}\}$ has the finite intersect property. Note that K is compact, and it follows that $\bigcap_{V \in \mathcal{B}} Q_V \neq \emptyset$. Taking any $\hat{x} \in \bigcap_{V \in \mathcal{B}} Q_V$, we have $\hat{x}_i \in \overline{B_{V_i}}(\hat{x})$ for each $V_i \in \mathcal{B}_i$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for each $i \in I$. By Lemma 3.1, it follows that $\hat{x}_i \in \overline{B_i}(\hat{x})$ for each $i \in I$ and thus the proof is completed. ∎

COROLLARY 3.5. *Let $\mathcal{G} := (X_i; A_i, B_i; P_i)_{i \in I}$ be a generalized game such that $X = \Pi_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied for each $i \in I$:*

(a) *X_i is a non-empty convex subset of a locally convex Hausdorff topological vector space E_i;*

(b) *$A_i(x)$ is non-empty and* $\operatorname{co} A_i(x) \subset B_i(x)$ *for each $x \in X$;*

(c) *A_i has an open graph in $X \times X_i$ (resp., is lower semicontinuous) and P_i is lower semicontinuous* (resp., has an open graph in $X \times X_i$);

(d) *$A_i \cap P_i$ is L_C-majorized;*

(e) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$ there is an $x \in \operatorname{co}(X_0 \cup \{y\})$ such that $x_i \in \operatorname{co}(A_i(y) \cap P_i(y))$ for all $i \in I$.*

Then $\mathcal{G}$ has an equilibrium point in K, i.e., there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in \overline{B_i}(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

Proof. For each $i \in I$, as A_i has an open graph in $X \times X_i$ (resp., is lower semicontinuous) and P_i is lower semicontinuous (resp., has an open graph in $X \times X_i$), it follows that the mapping $A_i \cap P_i$: $X \to 2^{X_i}$ is lower semicontinuous by Lemma 4.2 of [46, p. 103]. Thus the set $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X. Therefore the conclusion follows from Theorem 3.4. ∎

Remark 3.6. In this section, we have proved existence theorems of equilibria for generalized games with non-compact and infinite dimensional strategy spaces, an infinite number of agents, and nontotal-nontransitive constraint and preference mappings which may not have open graphs or open lower (resp., upper) sections. Since it is well known that if a mapping has an open graph, it then has open upper and lower sections (e.g., see Bergstrom *et al.* [5, p. 266]), and thus mappings with open lower sections are lower semicontinuous. However, a continuous mapping may not hold open lower (or upper) section properties in general (e.g., see Yannelis and Prabhakar [47, p. 237]). We also know that in the infinite settings, the set of feasible allocations generally is not compact in any topology of the commodity spaces. Thus our results generalize many results in the literature by relaxing the compactness of strategy spaces and the openness of graphs or lower (resp., upper) sections of constraint mappings. For example, Theorem 3.4 (and hence also Corollary 3.5) generalizes Corollary 3 of Borglin and Keiding [6, p. 315], Theorem 4.1 of Chang [9, p. 247], Theorem of Shafer and Sonnenschein [36, p. 374], and Theorem 5 of Tulcea [44, p. 10]. Also Corollary 3.5 (and hence Theorem 3.4) improves Theorem 6.1 of Yannelis and Prabhakar [47] in the following ways: (i) the index I need not be countable; (ii) the set X_i need not be metrizable for each $i \in I$; and (iii) $A_i \cap P_i$ need not be L-majorized for each $i \in I$.

Remark 3.7. Professors Yannelis and Prabhaker in [47, p. 243]) asked: *If there is an equilibrium point for the generalized game* $\Gamma = (X_i; A_i; P_i)_{i \in I}$ *when* X_i *is a non-empty compact and convex subset of locally convex topological vector space and both* A_i *and* P_i *have open lower sections, can the set of agents I be assumed to be any* (*finite or infinite*) *set*? By Remark 3.6, it follows that our Theorem 3.4 (thus Corollary 3.5) not only shows that Theorem 6.1 of Yannelis and Prabhakar above can be extended to non-metrizable subsets and the non-compact case without introducing an additional assumption, it shows that the question raised by Yannelis and Prabhakar can be answered in the affirmative; but in fact, some of the assumptions of their question above can be further weakened.

In what follows, as applications of Lemma 2.4 and Theorem 3.4, we will prove existence theorems of the generalized game $G = (X_i; A_i, B_i; P_i)_{i \in I}$ in which the constraint mapping is upper semicontinuous.

THEOREM 3.8. *Let* $G = (X_i, A_i, B_i, P_i)_{i \in I}$ *be an abstract economy such that* $X = \Pi_{i \in I} X_i$ *is paracompact. Suppose the following conditions are satisfied for each* $i \in I$.

(a) X_i *is a non-empty convex subset of locally convex topological vector space* E_i *and* X_i *has the property* (K);

(b) A_i, B_i: $X \to 2^{X_i}$ *are such that* B_i *is compact and upper semicontinuous with non-empty compact and convex values and* $A_i(x) \subset B_i(x)$ *for each* $x \in X$;

(c) P_i: $X \to 2^{X_i}$ *is lower semicontinuous and* L_C-*majorized*;

(d) *the set* $E^i := \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ *is open in* X;

(e) *there exist a non-empty compact convex subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $y \in X \setminus K$ *there is an* $x \in \mathrm{co}(X_0 \cup \{y\})$ *such that* $x_i \in \mathrm{co}(A_i(y) \cap P_i(y))$ *for all* $i \in I$.

Then there exists $\bar{x} \in K$ *such that for each* $i \in I$, $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ *and* $\bar{x}_i \in \overline{B}_i(\bar{x})$.

Proof. By Lemma 2.4 for the upper semicontinuous mapping, for every $i \in I$, there exists a family $(B_{ij})_{j \in J}$ indeed by a filtering set J, consisting of regular mappings between X and X_i such that both $(B_{ij})_{j \in J}$ and $(\overline{B_{ij}})_{j \in J}$ are upper approximating families for B_i.

Now the game $G_j = (X_i; (B_{ij})_i, (\overline{B_{ij}})_i; P_i)_{i \in I}$ satisfies all hypotheses of Theorem 3.4 for each $j \in J$. Hence G_j has an equilibrium $x_j^* \in K$ for each $j \in J$ such that $B_{ij}(x_j^*) \cap P_i(x_j^*) = \emptyset$, and $\pi_i(x_j^*) \in \overline{B_{ij}}(x_j^*)$. Let U be an ultrafilter on J which is finer than the filter sections of J. Since $(x_j^*)_{j \in J} \subset K$, let $\bar{x} = \lim_{j, U} x_j^*$. Then for each $i \in I$, $\pi_i(\bar{x}) = \lim_{j, U} (x_j^*)_i$. Note that $A_i(x) \subset B_i(x) \subset B_{ij}(x)$ each $x \in X$, it follows that $A_i(x_j^*) \cap P_i(x_j^*) = \emptyset$ for all $i \in I$. By condition (d), we have that $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ for every $i \in I$. Since x_j^* is an equilibrium point of G_j and B_{ij} is regular, it follows that for each $x \in X$, $\mathrm{cl}\, B_{ij}(x) = \overline{B_{ij}}(x)$. Thus $(x_j^*)_i \in \mathrm{cl}(B_{ij}(x_j^*)) = \overline{B_{ij}}(x_j^*)$. As $\overline{B_{ij}}$ has a closed graph, $(\bar{x}, (\bar{x})_i) \in \mathrm{Graph}\, \overline{B_{ij}}$ for every $i \in I$. But we know for each $x \in X$, $\bigcap_{j \in J} B_j(x) \subset \overline{B_{ij}}(x)$. Therefore, we have $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ and $\pi_i(\bar{x}) \in \overline{B}_i(\bar{x})$ for each $i \in I$ and the proof is completed. ∎

Let $A_i := B_i$ for each $i \in I$ in Theorem 3.8. We have the following existence result which also generalizes Theorem 5 of Tulcea [45, p. 288].

THEOREM 3.9. *Let* $G = (X_i, A_i, P_i)_{i \in I}$ *be an abstract economy such that* $X = \Pi_{i \in I} X_i$ *is paracompact. Suppose the following conditions are satisfied for each* $i \in I$:

(a) X_i *is a non-empty convex subset of locally convex topological vector space* E_i *and* X_i *has the property* (K);

(b) A_i: $X \to 2^{X_i}$ *is compact and upper semicontinuous with non-empty compact and convex values for each* $x \in X$;

(c) P_i: $X \to 2^{X_i}$ *is lower semicontinuous and* L_C*-majorized*;

(d) *the set* $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ *is open in* X;

(e) *there exist a non-empty compact convex subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $y \in X \setminus K$, *there is an* $x \in \mathrm{co}(X_0 \cup \{y\})$ *such that* $x_i \in \mathrm{co}(A_i(y) \cap P_i(y))$ *for all* $i \in I$.

Then there exists $\bar{x} \in K$ *such that for each* $i \in I$, $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ *and* $\bar{x}_i \in \overline{A_i}(\bar{x}) = A_i(\bar{x})$.

The following example shows that the conclusion of Theorem 3.9 does not hold if we withdraw its condition (d).

EXAMPLE A. Let $I = \{1\}$ and $X = [0, 1]$. We define $A, P : X \to 2^X$ by

$$A(x) = \begin{cases} [1/2, 1], & \text{if } x \in [0, 1/2), \\ [0, 1], & \text{if } x = 1/2, \\ [0, 1/2], & \text{if } x \in (1/2, 1], \end{cases}$$

and

$$P(x) = \begin{cases} \emptyset, & \text{if } x = 0, \\ [0, x), & \text{if } x \in (0, 1]. \end{cases}$$

Then it is clear that A is upper semicontinuous with non-empty closed convex values and the fixed point set of A is a singleton $\{1/2\}$. The mapping P has convex values with open lower sections since for each $y \in [0, 1]$, $P^{-1}(y) = (y, 1]$ is open in X. Therefore A, P, and X satisfy all conditions except that the set "$E = \{x \in [0, 1] : A(x) \cap P(x) \neq \emptyset\} = [1/2, 1]$ *is closed*" instead of being open in $[0, 1]$. However, we have $A(1/2) \cap P(1/2) = \emptyset$, i.e., the generalized game $G = ([0, 1], A, P)$ has no equilibrium point as $A(1/2) \cap P(1/2) \neq \emptyset$.

If X_i is compact and closed convex in Theorem 3.9, we then have

COROLLARY 3.10. *Let* $G = (X_i, A_i, P_i)_{i \in I}$ *be an abstract economy and* $X = \Pi_{i \in I} X_i$. *Suppose the following conditions are satisfied for each* $i \in I$:

(a) X_i *is a non-empty closed compact convex subset of locally convex topological vector space* E_i;

(b) A_i: $X \to 2^{X_i}$ *is upper semicontinuous with non-empty compact and convex values for each* $x \in X$;

(c) P_i: $X \to 2^{X_i}$ *is lower semicontinuous and* L_C*-majorized*;

(d) *the set* $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ *is open in* X.

Then there exists an $\bar{x} \in K$ *such that for each* $i \in I$, $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ *and* $\bar{x}_i \in \overline{A_i}(\bar{x}) = A_i(\bar{x})$.

We also note that Corollary 3.10 generalizes Theorem 5 of Tulcea [45, p. 284] and Theorem of Shafer and Sonnenschein [36, p. 374] in several aspects. Indeed, as applications of Corollary 3.10, we have the following well-known Fan and Glicksberg fixed point theorem for upper semicontinuous correspondence in locally convex topological vector space.

THEOREM 3.11. *Let X be a convex compact subset of locally convex topological vector space and $A: X \to 2^X$ be upper semicontinuous with non-empty closed and convex values for each $x \in X$. Then A has a fixed point.*

Proof. Let $I := \{1\}$ and $P_i(x) = \emptyset$ for each $x \in X$ in Corollary 3.10. Then the conclusion follows. ∎

4. NON-COMPACT QUASI-VARIATIONAL INEQUALITIES

In this section, as applications of equilibria of generalized games, we will study an existence theorem of non-compact quasi-variational inequalities for lower semicontinuous mappings in locally convex topological vector space. We first have the following:

THEOREM 4.1. *Let X_i be a non-empty convex subset of locally convex topological vector space E_i and $X = \Pi_{i \in I} X_i$ be also paracompact. Suppose that the following conditions are satisfied for each $i \in I$:*

(i) *A_i: $X \to 2^{X_i}$ is a lower semicontinuous correspondence with a closed graph and non-empty convex values;*

(ii) *ψ_i: $X \times X_i \to \mathbb{R} \cup \{-\infty, +\infty\}$ is an extended value function such that the mapping $x \mapsto \psi_i(x, y_i)$ is lower semicontinuous in each non-empty compact subset C of X for each fixed $y_i \in X_i$;*

(iii) *$x_i \notin \operatorname{co}(\{y_i \in X_i : \psi_i(x, y_i) > 0\})$ for each $x \in X$;*

(iv) *the set $\{x \in X : \sup_{y_i \in A_i(x)} \psi_i(x, y_i) > 0\}$ is open in X for each fixed $y_i \in X_i$; and*

(v) *there exist a non-empty compact and convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists $x = (x_i)_{i \in I} \in \operatorname{co}(X_0 \cup \{y\})$ with $x_i \in \operatorname{co}(\{A_i(y) \cap \{z_i \in X_i : \psi_i(y, z_i) > 0\}\})$.*

Then there exists $x^* \in K$ such that for each $i \in I$,

$$x_i^* \in A_i(x^*) \qquad \text{and} \qquad \sup_{y_i \in A_i(x^*)} \psi_i(x^*, y_i) \leq 0.$$

Proof. For each $i \in I$, define a mapping P_i: $X \to 2^{X_i}$ by $P_i(x) = \{y_i \in X_i : \psi_i(x, y_i) > 0\}$ for each $x \in X$. Then we want to show that the $\Gamma = (X_i; A_i; P_i)_{i \in I}$ satisfies all hypotheses of Theorem 3.4.

First we note that the condition (ii) implies that for each $i \in I$, P_i has compactly open lower sections in X which, in turn implies that P_i is L_C-majorized by (iii). The condition (iv) implies that for each $i \in I$, the set $\{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X. Thus all hypotheses of Theorem 3.4 are satisfied. By Theorem 3.4, there exists $x^* \in K$ such that for each $i \in I$, $A_i(x^*) \cap P_i(x^*) = \emptyset$ and $x_i^* \in A_i(x^*)$. As $\{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\} = \{x \in X : \alpha_i(x) = \sup_{y_i \in A_i(x)} \psi_i(x, y_i) > 0\}$, this implies that for each $i \in I$, we have $x_i^* \in A_i(x^*)$ and $\sup_{y_i \in A_i(x^*)} \psi_i(x^*, y_i) \leq 0$. Therefore the proof is completed. ∎

Let $I := \{1, 2\}$ and $X_1 := X_2 = X$ in Theorem 4.1. We have the following existence result of non-compact quasi-variational inequalities.

COROLLARY 4.2. *Let X be a non-empty convex subset of a locally convex topological vector space. Suppose $X \times X$ is paracompact such that:*

(i) *$A: X \to 2^X$ is lower semicontinuous with a closed graph and non-empty convex values;*

(ii) *$\psi: X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a mapping such that the function $x \mapsto \psi(x, y)$ is lower semicontinuous in each non-empty compact subset C of X for each fixed $y \in X$;*

(iii) *$x \notin \mathrm{co}(\{y \in X : \psi(x, y) > 0\})$ for each $x \in X$;*

(iv) *the set $\{x \in X : \sup_{y \in A(x)} \psi(x, y) > 0\}$ is open in X; and*

(v) *there exist a non-empty convex and compact subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists $x \in \mathrm{co}(X_0 \cup \{y\})$ with $x \in \mathrm{co}(\{A(y) \cap \{z \in X : \psi(y, z) > 0\}\})$.*

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$ and $\sup_{y \in A(x^*)} \psi(x^*, y) \leq 0$.

5. GENERALIZED NON-COMPACT QUASI-VARIATIONAL INEQUALITIES

In this section, as applications of Theorem 4.1, we will investigate the existence of solutions for the following generalized variational inequality problems $(*)$ and $(**)$ under various conditions.

Let X be a non-empty convex subset of locally convex Hausdorff topological vector E, where E^* denotes the dual space of E. Suppose $F: X \to 2^X$, $T: X \to 2^{E^*}$, and $f: X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ are three given mappings. We want to prove the existence of a solution $\hat{x} \in X$ for the following generalized quasi-variational inequalities (in short, GQVI):

$$\begin{cases} \hat{x} \in F(\hat{x}), \\ \sup_{u \in T(\hat{x})} \langle u, \hat{x} - y \rangle + f(\hat{x}, y) \leq 0 \qquad \text{for any } y \in F(\hat{x}) \end{cases} \tag{$*$}$$

and

$$\begin{cases} \hat{x} \in F(\hat{x}), \\ \inf_{u \in T(\hat{x})} \langle u, \hat{x} - y \rangle + f(\hat{x}, y) \leq 0 \qquad \text{for any } y \in F(\hat{x}). \end{cases}$$

Or more generally, to find $\hat{x} \in X$ and $\hat{u} \in E^*$ such that

$$\begin{cases} \hat{x} \in F(\hat{x}) \quad \text{and} \quad \hat{u} \in T(\hat{x}) \\ \langle \hat{u}, \hat{x} - y \rangle + f(\hat{x}, y) \leq 0 \qquad \text{for any } y \in F(\hat{x}). \end{cases} \tag{$**$}$$

Now we need to recall some notions and definitions (e.g., see Zhou and Chen [50]).

Let X be a convex subset of topological vector space. A function $\psi(x, y): X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be

(1) diagonal quasi-convex (resp., quasi-concave) in y, in short DQCX (respectively, DQCV) in y, if for each $A \in \mathcal{F}(X)$ and $y \in \mathrm{co}(A)$, then $\psi(y, y) \leq \max_{x \in A} \psi(y, x)$ (respectively, $\psi(y, y) \geq \inf_{x \in A} \psi(y, x)$) where $\mathcal{F}(X)$ denotes the family of all non-empty finite subsets of X;

(2) γ-diagonal quasi-convex (resp., γ-diagonal quasi-concave) in y, in short γ-DQCX (res., γ-DQCV) in y, if for any $A \in \mathcal{F}(X)$ and each $y \in \mathrm{co}(A)$, $\gamma \leq \max_{x \in A} \psi(y, x)$ (resp., $\gamma \geq \inf_{x \in A} \psi(y, x)$);

(3) γ-diagonal convex (resp., γ-diagonal concave) in y, in short γ-DCX (resp., γ-DCV) in y, if for each $A \in \mathcal{F}(X)$ and each $y \in \mathrm{co}(A)$ with $y = \sum_{i=1}^{m} \lambda_i y_i$ ($\lambda_i \geq 0$, and $\sum_{i=1}^{m} \lambda_i = 1$), we have $\gamma \leq \sum_{i=1}^{m} \lambda_i \psi(y, y_i)$ (respectively, $\gamma \geq \sum_{i=1}^{m} \lambda_i \psi(y, y_i)$);

(4) diagonal convex (resp., diagonal concave) in y, in short DCX (resp., DCV) in y, if for each $A \in \mathcal{F}(X)$ and each $y \in \mathrm{co}(A)$ with $y = \sum_{i=1}^{m} \lambda_i y_i$ ($\lambda_i \geq 0$, and $\sum_{i=1}^{m} \lambda_i = 1$), we have $\psi(y, y) \leq \sum_{i=1}^{m} \lambda_i \psi(y, y_i)$ (resp., $\psi(y, y) \geq \sum_{i=1}^{m} \lambda_i \psi(y, y_i)$).

Let X and Y be two non-empty convex subsets of E. We also recall that an extended value function $\psi: X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be quasi-convex (resp., quasi-concave) in y if for each fixed $x \in X$, for any $A \in \mathcal{F}(Y)$ and $y \in \mathrm{co}(A)$, $\psi(x, y) \leq \max_{x \in A} \psi(x, z)$ (resp., $\psi(x, y) \geq \inf_{z \in A} \psi(x, z)$).

In general, we know that the sum of two quasi-convex functions does not remain quasi-convex, and the same holds for the DQCX property. However, it is easy to see that the following simple facts are true.

(1) if $\psi(x, y)$ is DCX (resp., γ-DCX, DCV, γ-DCV) in y, then $\psi(x, y)$ is DQCX (respectively, γ-DQCX, DQCV, γ-DCV) in y;

(2) if $\psi_i(x, y)$, $1 \leq i \leq n$, is a set of functionals, in which each of them is γ-DCX (resp., DCX, γ-DCV, DCV) in y, then $\psi(x, y) = \sum_{i=1}^{m} a_i(x)\psi_i(x, y)$ is still γ-DCX (resp., DCX, γ-DCV, DCV) in y, where a_i: $X \to \mathbb{R}$ with $a_i(x) \geq 0$ and $\sum_{i=1}^{m} a_i(x) = 1$ for each $x \in X$; and

(3) the function $\psi(x, y)$: $X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is 0-DQCV in y if and only if $x \notin \operatorname{co}(\{y \in X : \psi(x, y) > 0\})$ for each $x \in X$.

Now it is time for us to give existence of solutions for the problem $(*)$ in which T: $X \to 2^{E^*}$ is a monotone mapping.

THEOREM 5.1. *Let X be a non-empty convex subset of a locally convex Hausdorff topological vector space E. Suppose that*:

(i) *$X \times X$ is paracompact*;

(ii) *F: $X \to 2^X$ is lower semicontinuous with a closed graph and non-empty convex values*;

(iii) *T: $X \to 2^{E^*}$ is a monotone mapping with non-empty values such that for each one-dimensional flat $L \subset E^*$, $T|_{L \cap X}$ is lower semicontinuous from the topology of E into the weak*-topology $\sigma(E^*, E)$ of E^**;

(iv) *f: $X \times X \to \mathbb{R} \cup \{\infty, +\infty\}$ is a mapping such that the function $x \mapsto f(x, y)$ is lower semicontinuous in each non-empty compact subset C of X for each fixed $y \in X$, and for each fixed $x \in X$, $y \mapsto f(x, y)$ concave and $f(x, x) = 0$ for each $x \in X$*;

(v) *the set $\{x \in X : \sup_{y \in F(x)}[\sup_{u \in T(y)}\langle u, x - y\rangle + f(x, y)] > 0\}$ is open in X*;

(vi) *there exist a non-empty convex compact subset X_0 of X and a non-empty compact subset K of X such that for each $x \in X \setminus K$, there exists $y \in \operatorname{co}(X_0 \cup \{x\})$ such that $y \in \operatorname{co}(F(x) \cap \{z \in X : \sup_{u \in T(z)}\langle u, x - z\rangle + f(x, z) > 0\})$.*

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$ and

$$\sup_{u \in T(\hat{x})}[\langle u, \hat{x} - y) + f(\hat{x}, y\rangle] \leq 0 \qquad \text{for all } y \in F(\hat{x}).$$

Proof. Define a function ψ: $X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$\psi(x, y) = \sup_{u \in T(y)} \langle u, x - y\rangle + f(x, y)$$

for each $(x, y) \in X \times X$. Then we have that $x \mapsto \psi(x, y)$ is lower semicontinuous in each non-empty compact subset C of X for each $y \in X$. As T is monotone, by condition (iv) it is clear that the mapping $y \mapsto \psi(x, y)$ is 0-DCV for each fixed $x \in X$ by Proposition 3.2 of Zhou and Chen [50]. The conditions (ii), (v), and (vi) imply that ψ satisfies all hypotheses of

Corollary 4.2. By Corollary 4.2, there exists $\hat{x} \in K$ such that $\hat{x} \in F(\hat{x})$ and $\sup_{u \in T(y)}[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \le 0$ for all $y \in F(\hat{x})$. For each one-dimensional flat $L \subset E^*$, $T|_{L \cap X}$ is lower semicontinuous from the relative topology of X into the weak*-topology $\sigma(E^*, E)$ of E^*. By the same argument of Step 2 of Theorem 1 of Shih and Tan [38, pp. 337−338], we can prove that $\sup_{u \in T(\hat{x})}[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)] \le 0$ for all $y \in F(\hat{x})$. Thus the proof is completed. ∎

We shall now observe that in Theorem 5.1, the interaction between the correspondences T and F (namely, the condition (v)) can be achieved by imposing additional continuity conditions on T and F.

THEOREM 5.2. *Let E be a locally convex topological vector space and X be a bounded and non-empty convex subset in space E such that $X \times X$ is paracompact. If $F\colon X \to 2^X$ is lower semicontinuous with a closed graph and non-empty convex values and $T\colon X \to 2^{E^*}$ is a monotone mapping such that for each $x \in X$, $T(x)$ is a non-empty subset of E^*, then T is also lower semicontinuous from the relative topology of X to the strong topology of E^*. Suppose that the following conditions are satisfied*:

(i) $f\colon X \times X \to \mathbb{R} \cup \{\infty, +\infty\}$ *is a function such that* $(x, y) \mapsto f(x, y)$ *is lower semicontinuous and for each fixed $x \in X$, $y \mapsto f(x, y)$ is concave and $f(x, x) = 0$ for each $x \in X$; and*

(ii) *there exist a non-empty convex compact subset X_0 of X and a non-empty compact subset K of X such that for each $x \in X \setminus K$, there exists $y \in \mathrm{co}(X_0 \cup \{x\})$ such that $y \in \mathrm{co}(F(x) \wedge \{z \in X : \sup_{u \in T(z)}\langle u, x - z\rangle + f(x, z) > 0\})$.*

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$ and

$$\sup_{u \in T(\hat{x})}\big[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)\big] \le 0 \qquad \text{for all } y \in F(\hat{x}).$$

Proof. By Theorem 5.1, it suffices to prove that the set

$$\Sigma := \left\{x \in X : \sup_{y \in F(x)} \sup_{u \in T(y)} [\langle u, x - y\rangle + f(x, y)] > 0\right\}$$

is open in X. Note that X is a bounded subset of locally convex space E, and we equip E^* with the strong topology. Define a mapping $\psi_1\colon X \times X \times E^* \to \mathbb{R} \cup \{-\infty, +\infty\}$ by $\psi_1(x, y, u) = \langle u, x - y\rangle$ for each $(x, y, u) \in X \times X \times E^*$. Then ψ_1 is continuous. Since $T\colon X \to 2^{E^*}$ is lower semicontinuous with non-empty values from the relative topology of X to the strong topology of E^*, by Theorem 2 of Aubin [3, p. 69], it follows that the mapping $\psi_2\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by $\psi_2(x, y) =$

$\sup_{u\in T(x)}\langle u, x-y\rangle$ is lower semicontinuous for each $(x, y)\in X\times X$. Thus the mapping $(x, y)\mapsto \sup_{u\in T(x)}\langle u, x-y\rangle + f(x, y)$ is lower semicontinuous by (i). As $F\colon X\to 2^X$ is lower semicontinuous with non-empty values for each $x\in X$, by Theorem 2 of Aubin [3, p. 69] again, the mapping $x\mapsto \sup_{y\in F(x)}\sup_{u\in T(x)}[\langle u, x-y\rangle + f(x, y)]$ is also lower semicontinuous from X to $\mathbb{R}\cup\{-\infty, +\infty\}$, so that the set $\Sigma = \{x\in X : \sup_{y\in F(x)}\sup_{u\in T(x)}[\langle u, x-y\rangle + f(x, y)] > 0\}$ is open in X. Therefore F, T, and f satisfy all hypotheses of Theorem 5.1 and the conclusion follows from Theorem 5.1. ∎

Theorem 5.2 generalizes Theorem 2 of Shih and Tan [38] to a non-compact case. Theorems 5.1 and 5.2 also generalize the corresponding result of Joly and Moscos (see Theorem 15.2.2 of Aubin [3]) in several aspects.

Now we will consider the existence of solution for the problems $(*)$ and $(**)$ when $T\colon X\to 2^{E^*}$ is not a monotone mapping. We first have the following:

THEOREM 5.3. *Let X be a non-empty convex subset of a locally convex Hausdorff topological vector space E. Suppose that the following conditions are satisfied*:

(i) *$X\times X$ is paracompact*;

(ii) *$F\colon X\to 2^X$ is lower semicontinuous with a closed graph and non-empty convex values*;

(iii) *$T\colon X\to 2^{E^*}$ is a mapping with non-empty and convex values and such that for each fixed $y\in X$, $x\mapsto \inf_{u\in T(x)}\langle u, x-y\rangle$ is lower semicontinuous*;

(iv) *$f\colon X\times X\to \mathbb{R}\cup\{-\infty, +\infty\}$ is a function such that $x\mapsto f(x, y)$ is lower semicontinuous in each non-empty compact subset C of X for each fixed $y\in X$ and for each fixed $x\in X$, $y\mapsto f(x, y)$ is 0-diagonal concave*;

(v) *the set $\{x\in X : \sup_{y\in F(x)}[\inf_{u\in T(x)}\langle u, x-y\rangle + f(x, y)] > 0\}$ is open in X*;

(vi) *there exist a non-empty convex compact subset X_0 of X and a non-empty compact subset K of X such that for each $x\in X\setminus K$, there exists $y\in \mathrm{co}(X_0\cup\{x\})$ such that $y\in \mathrm{co}(F(x)\cap\{z\in X : \inf_{u\in T(x)}\langle u, x-z\rangle + f(x, z) > 0\})$.*

Then there exists $\hat{x}\in F(\hat{x})$ such that

$$\sup_{y\in F(\hat{x})}\inf_{u\in T(\hat{x})}\big[\langle u, \hat{x}-y\rangle + f(\hat{x}, y)\big] \le 0.$$

If in addition, for each fixed $x\in X$, $y\mapsto f(x, y)$ is concave and $T(x)$ is non-empty convex compact for each $x\in X$, then there exists $\hat{u}\in T(\hat{x})$ such that $\sup_{y\in F(\hat{x})}[\langle \hat{u}, \hat{x}-y\rangle + f(\hat{x}, y)] \le 0$.

Proof. Define a functional $\psi\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$\psi(x, y) := \inf_{u \in T(x)} [\langle u, x - y\rangle + f(x, y)],$$

for each $(x, y) \in X \times X$. Then we have

(1) for each fixed $y \in X$, $x \mapsto \psi(x, y)$ is lower semicontinuous in each non-empty compact subset of C of X and $x \notin \mathrm{co}(\{y \in X : \psi(x, y) > 0\})$ for each $x \in X$ by (iv);

(2) the set $\{x \in X : \sup_{y \in F(x)} \psi(x, y) > 0\}$ is open in X;

(3) there exist a non-empty convex and compact subset X_0 of X and a non-empty compact subset K of X such that for each $x \in X \setminus K$, there exists $y \in \mathrm{co}(X_0 \cup \{x\})$ with $y \in \mathrm{co}(\{F(x) \cap \{z \in X : \psi(x, z) > 0\}\})$.

Therefore F and ψ satisfy all conditions of Corollary 4.2. By Corollary 4.2, there exists an $\hat{x} \in K$ such that $\hat{x} \in F(\hat{x})$ and $\psi(\hat{x}, y) \le 0$ for all $y \in F(\hat{x})$.

If in addition, for each fixed $x \in X$, $y \mapsto f(x, y)$ is concave, define the function $f_1\colon F(\hat{x}) \times T(\hat{x}) \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$f_1(x, y) := \langle x, \hat{x} - y\rangle + f(\hat{x}, y)$$

for each $(x, y) \in F(\hat{x}) \times T(\hat{x})$. Then for each fixed $y \in X$, $x \mapsto f_1(x, y)$ is lower semicontinuous and for each fixed $x \in X$, $y \mapsto f_1(x, y)$ is concave; and $T(x)$ is non-empty convex compact for each $x \in X$. By the Kneser minimax theorem [31], it follows that

$$\begin{aligned}\inf_{u \in T(\hat{x})} \sup_{y \in F(\hat{x})} &\big[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)\big] \\ &= \sup_{y \in F(\hat{x})} \inf_{u \in T(\hat{x})} \big[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)\big] \le 0,\end{aligned}$$

so that there exists $\hat{u} \in T(\hat{x})$ such that $\sup_{y \in F(\hat{x})}[\langle \hat{u}, \hat{x} - y\rangle + f(\hat{x}, y)] \le 0$. ∎

If X is a bounded subset of locally convex topological vector space E and $T\colon X \times X \to \mathbb{R}$ is upper semicontinuous with non-empty compact and convex values from the topology of X into the strong topology of E^*, and we define a function $g\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by $g(x, y) := \inf_{u \in T(x)}\langle u, x - y\rangle$ for each $(x, y) \in X \times X$, then the mapping $x \mapsto g(x, y)$ is lower semicontinuous for each fixed $y \in X$ by Lemma 2 of Kim and Tan [29]. Thus, Theorem 5.3 above includes Theorem 3 of Shih and Tan [38] as a special case. Moreover, we have the following:

COROLLARY 5.4. *Let X be a bounded non-empty convex subset of a locally convex Hausdorff topological vector space E. Suppose the following conditions are satisfied*:

(i) *$X \times X$ is paracompact*;

(ii) *$F\colon X \to 2^X$ is lower semicontinuous with a closed graph and non-empty convex values*;

(iii) *$T\colon X \to 2^{E^*}$ is an upper semicontinuous mapping with non-empty convex (strong) compact values*;

(iv) *$f\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a function such that $x \mapsto f(x, y)$ is lower semicontinuous for each fixed $y \in X$ and for each $x \in X$, $y \mapsto f(x, y)$ is 0-diagonal concave*;

(v) *the set $\{x \in X : \sup_{y \in F(x)}[\inf_{u \in T(x)}\langle u, x - y\rangle + f(x, y)] > 0\}$ is open in X*;

(vi) *there exist a non-empty convex compact subset X_0 of X and a non-empty compact subset K of X such that for each $x \in X \setminus K$, there exists $y \in \mathrm{co}(X_0 \cup \{x\})$ such that $y \in \mathrm{co}(F(x) \cap \{z \in X : \inf_{u \in T(x)}\langle u, x - z\rangle + f(x, z) > 0\}$.*

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$ and

$$\sup_{y \in F(\hat{x})}\inf_{u \in T(\hat{x})}\big[\langle u, \hat{x} - y\rangle + f(\hat{x}, y)\big] \leq 0.$$

If in addition, for each fixed $x \in X$, $y \mapsto f(x, y)$ is concave, then there exists $\hat{u} \in T(\hat{x})$ such that $\sup_{y \in F(\hat{x})}[\langle \hat{u}, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0$.

Proof. Define a mapping $\psi\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$\psi(x, y) = \inf_{u \in T(x)}\big[\langle u, x - y\rangle + f(x, y)\big],$$

for each $(x, y) \in X \times X$. As X is bounded, Lemma 2 of Kim and Tan [29] implies that $x \mapsto \inf_{u \in T(x)}\langle u, x - y\rangle$ is lower semicontinuous for each fixed $y \in X$. Therefore ψ and F satisfy all hypotheses of Theorem 5.3. By Theorem 5.3, Corollary 5.4 follows and we complete the proof. ∎

If we impose continuity conditions to the correspondence F, we do have the following existence of solutions.

THEOREM 5.5. *Let X be a non-empty convex bounded subset of a locally convex Hausdorff topological vector space E. Suppose that the following conditions are satisfied*:

(i) *$X \times X$ is paracompact*;

(ii) *$F\colon X \to 2^X$ is a continuous mapping with non-empty compact and convex values*;

(iii) $T\colon X \to 2^{E^*}$ *is upper semicontinuous with non-empty convex (strong) compact values*;

(iv) $f\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ *is a mapping such that* $(x, y) \mapsto f(x, y)$ *is lower semicontinuous and for each fixed* $x \in X$, $y \mapsto f(x, y)$ *is* 0-*diagonal concave*;

(v) *there exist a non-empty convex compact subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $x \in X \setminus K$, *there exists* $y \in \operatorname{co}(X_0 \cup \{x\})$ *such that* $y \in \operatorname{co}(F(x) \cap \{y \in X : \inf_{u \in T(x)} \langle u, x - y\rangle + f(x, y) > 0\})$.

Then there exists $\hat{x} \in X$ *such that* $\hat{x} \in F(\hat{x})$ *and*

$$\sup_{y \in F(\hat{x})}\Big[\inf_{u \in T(\hat{x})}\langle u, \hat{x} - y\rangle + f(\hat{x}, y)\Big] \leq 0.$$

If in addition, for each fixed $x \in X$, $y \mapsto f(x, y)$ *is concave, then there exists* $\hat{u} \in T(\hat{x})$ *such that* $\sup_{y \in F(\hat{x})}[\langle \hat{u}, \hat{x} - y\rangle + f(\hat{x}, y)] \leq 0$.

Proof. Since X is a bounded subset of locally convex space E, we equip E^* with the strong topology, we define a function $\psi_1\colon X \times X \times E^* \to \mathbb{R} \cup \{-\infty, +\infty\}$ by $\psi_1(x, y, u) = \langle u, x - y\rangle$ for each $(x, y, u) \in X \times X \times E^*$. Then ψ_1 is continuous. As $T\colon X \to 2^{E^*}$ is upper semicontinuous with one-empty (strong) compact and convex values, from Theorem 1 of Aubin [3, p. 67] it follows that the mapping $\psi_2\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by $\psi_2(x, y) = \inf_{u \in T(x)}\langle u, x - y\rangle$ for each $(x, y) \in X \times X$ is also lower semicontinuous. Therefore the mapping $(x, y) \mapsto \inf_{u \in T(x)}\langle u, x - y\rangle + f(x, y)$ is lower semicontinuous by condition (iv). Note that $F\colon X \to 2^X$ is lower semicontinuous with non-empty values and it follows that the mapping $x \mapsto \sup_{y \in F(x)}\inf_{u \in T(x)}[\langle u, x - y\rangle + f(x, y)]$ is lower semicontinuous from X to $\mathbb{R} \cup \{-\infty, +\infty\}$ by Theorem 2 of Aubin [3, p. 69]. Thus the set $\Sigma := \{x \in X : \sup_{y \in F(x)}\inf_{u \in T(x)}[\langle u, x - y\rangle + f(x, y)] > 0\}$ is open in X. Therefore F, T, and f satisfy all hypotheses of Corollary 5.4. By Corollary 5.4, the conclusion of Theorem 5.5 follows and we complete the proof. ∎

Remark 5.6. In Theorems 5.1, 5.2, 5.3, and 5.5, we assume that the mapping $T\colon X \to 2^{E^*}$ satisfies various kinds of continuity. In fact, under other appropriate conditions, the existence of solutions for problems $(*)$ and $(**)$ still hold without the continuous hypotheses. In this way, some results have been established by Ricceri [35], Cubiotti [10], and the references therein.

6. APPLICATION TO CONSTRAINED GAMES

Before we conclude this paper, as an application of Corollary 4.2, we derive one existence theorem of equilibria for constrained games. For the convenience of our study, we first recall some notations and definitions.

Let I be the set of agents which is any (countable or uncountable) set. Each agent (resp., player) has a choice (resp., strategy) set X_i. Denote by X_{-i} and X the products $\Pi_{j\in I,\, j\neq i} X_j$ and $\Pi_{j\in I} X_j$, respectively. A mapping (or say, feasible) $B_i\colon X_{-i} \to 2^{X_i}$ and a loss function $U_i : X = \Pi_{j\in I} X_j \to \mathbb{R} \cup \{-\infty, +\infty\}$ are given. We denote by B the product mapping of $\Pi_{(i\in I)} B_i$; and by x and x_{-i} an element of X and an element of $X_{(-i)}$, respectively.

A constrained game (e.g., see Aubin [3, p. 282]) $\Gamma = (X_i; B_i; U_i)_{i\in I}$ is defined as a family of triples $(X_i; B_i; U_i)_{i\in I}$. An equilibrium point for Γ is $x^* \in X$ such that

$$x^* \in B(x^*) = \Pi_{i\in I} B_i(x^*_{-i}) \qquad \text{and} \qquad U_i(x^*) \leq U_i(x^*_{-i}, x_i)$$

for all $x_i \in B_i(x^*_{-i})$ and $i \in I$.

We note that if $B_i(x_{-i}) := X_i$ for all $i \in I$, then the constrained game reduces to the usually conventional game $\Gamma = (X_i; U_i)_{i\in I}$ and the equilibrium is also called a Nash equilibrium.

In what follows, let $\psi(x, y) := \sum_{i\in I}[U_i(x) - U_i(x_{-i}, y_i)]$ and $B := \Pi_{j\in I} B_i$. Then we have the following existence result.

THEOREM 6.1. *Let $\Gamma = (X_i; B_i; U_i)_{i\in I}$ be a constrained game and X_i be a non-empty convex subset of a locally convex topological vector space. Suppose $X \times X$ is paracompact and the following conditions are satisfied*:

(i) $B\colon X = \Pi_{i\in I} X_i \to 2^{X_i}$ *is lower semicontinuous with a closed graph and non-empty convex values*;

(ii) $\psi\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ *is a mapping such that* $x \mapsto \psi_i(x, y)$ *is lower semicontinuous in each non-empty compact subset C of X for each fixed $y \in X$*;

(iii) $x \notin \operatorname{co}(\{y \in X : \psi(x, y) > 0\})$ *for each $x \in X$*;

(iv) *the set* $\{x \in X : \sup_{y\in B(x)} \psi(x, y) > 0\}$ *is open in X*;

(v) *there exist a non-empty convex and compact subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists* $x \in \operatorname{co}(X_0 \cup \{y\})$ *with* $x \in \operatorname{co}(\{B(y) \cap \{z \in X : \psi(y, z) > 0\})$.

Then there exists $x^* \in K$ such that for each $i \in I$,

$$x^*_i \in B_i(x^*_{-i}) \qquad \text{and} \qquad U_i(x^*) \leq \sup_{x_i\in B_i(x^*_{-i})} U_i(x^*_{-i}, x_i).$$

Proof. By hypotheses (i)–(v), it follows that $(X; B; \psi)$ satisfies all conditions of Corollary 4.2. By Corollary 4.2, there exists $x^* \in K$ such that $x^* \in B(x^*)$ and $\sup_{y \in B(x^*)} \psi(x^*, y) \leq 0$. Now let $y = (x_i^*, y_i)$. We have that $[U_i(x^*) - U_i(x_{-i}^*, y_i)] \leq 0$ for all $y_i \in B_i(x_{-i}^*)$ and $i \in I$. Hence x^* is an equilibrium point of the constrained game $\Gamma = (X_i; B_i; U_i)_{i \in I}$ and the proof is completed. ∎

Theorem 6.1 generalizes the corresponding results of Aubin [3, pp. 282–283] and Aubin and Ekeland [4, pp. 350–351] in the following ways: (i) I is any (countable or uncountable) set instead of a finite set; (ii) the feasible correspondence B_i is lower semicontinuous instead of continuous; and (iii) the strategy set X_i need not compact.

Remark 6.2. Finally, we point out that quasi-variational inequalities and generalized quasi-variational inequality theory have numerous applications in nonlinear problems, games theory, and economics theory; more details can be found in Aubin [3], Aubin and Ekeland [4], Border [8], Yuan [49], and the references therein.

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