ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Differential Equations 226 (2006) 295–322
Journal of Differential Equations
www.elsevier.com/locate/jde

Semi-classical spectral estimates for Schrödinger operators at a critical level. Case of a degenerate maximum of the potential ☆

Brice Camus

Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätsstr. 150, D-44780 Bochum, Germany

Received 1 March 2005; revised 11 August 2005

Available online 14 November 2005

Abstract

We study the semi-classical trace formula at a critical energy level for a Schrödinger operator on $\mathbb{R}^n$. We assume here that the potential has a totally degenerate critical point associated to a local maximum. The main result, which establishes the contribution of the associated equilibrium in the trace formula, is valid for all time in a compact subset of $\mathbb{R}$ and includes the singularity in $t = 0$. For these new contributions the asymptotic expansion involves the logarithm of the parameter h. Depending on an explicit arithmetic condition on the dimension and the order of the critical point, this logarithmic contribution can appear in the leading term.

Keywords: Trace formula; Semi-classical analysis; Degenerate oscillatory integrals; Schrödinger operators

1. Introduction

Let us consider P_h a self-adjoint h-pseudodifferential operator, or more generally h-admissible (see [16]), acting on a dense subset of $L^2(\mathbb{R}^n)$. A classical and accessible

☆ This work was supported by the *SFB/TR12* project, *Symmetries and Universality in Mesoscopic Systems* and by the *IHP-Network, Analysis&Quantum* reference HPRN-CT-2002-00277.
E-mail addresses: brice.camus@univ-reims.fr, cmsbrice@yahoo.fr.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2005.10.003

problem is to study the asymptotic behavior, as h tends to 0, of the spectral distributions:

$$\gamma(E,h,\varphi)=\sum_{|\lambda_j(h)-E|\leqslant\varepsilon}\varphi\left(\frac{\lambda_j(h)-E}{h}\right), \tag{1}$$

where the $\lambda_j(h)$ are the eigenvalues of P_h, E is an energy level of the principal symbol of P_h and φ a function. Here we suppose that the spectrum is discrete in $[E-\varepsilon,E+\varepsilon]$, a sufficient condition for this is given below. If p_0 is the principal symbol of P_h we recall that an energy E is regular when $\nabla p_0(x,\xi)\neq 0$ on the energy surface:

$$\Sigma_E=\left\{(x,\xi)\in T^*\mathbb{R}^n\colon\ p_0(x,\xi)=E\right\}, \tag{2}$$

and critical when it is not regular. A classical result is the existence of a link between the asymptotics of (1), as h tends to 0, and the closed trajectories of the Hamiltonian flow of p_0 on the energy surface Σ_E, i.e.,

$$\lim_{h\to 0}\gamma(E,h,\varphi)\rightleftharpoons\left\{(t,x,\xi)\in\mathbb{R}\times\Sigma_E\colon\ \Phi_t(x,\xi)=(x,\xi)\right\},$$

where $\Phi_t=\exp(tH_{p_0})$ and $H_{p_0}=\partial_\xi p_0.\partial_x-\partial_x p_0.\partial_\xi$. This duality between spectrum and periodic orbits exists in a lot of various settings such as in the Selberg trace formula or for the trace of the wave operator on compact manifolds [10]. In the semi-classical regime this correspondence was initially pointed out in the physic literature: Gutzwiller [12], Balian and Bloch [1]. For a rigorous mathematical approach, and when E is a regular energy, a non-exhaustive list of references is Brummelhuis and Uribe [3], Paul and Uribe [14], and more recently Combescure et al. [8], Petkov and Popov [15].

Equilibriums are suspected to give special contributions in both sides of the trace formula. When E is no more a regular value, the asymptotic behavior of Eq. (1) depends on the nature of the singularities of p on Σ_E which is too complicated to be treated in general position. The case of a nondegenerate critical energy for p_0, that is such that the critical-set $\mathfrak{C}(p_0)=\{(x,\xi)\in T^*\mathbb{R}^n\colon\ dp_0(x,\xi)=0\}$ is a compact C^∞ manifold with a Hessian d^2p_0 transversely nondegenerate along this manifold, has been investigated first by Brummelhuis et al. in [2]. They treated this question for quite general operators but for some "small times," i.e., it was assumed that 0 was the only period of the linearized flow in $\mathrm{supp}(\hat{\varphi})$. Later, Khuat-Duy in [13] has obtained the contributions of the nonzero periods of the linearized flow for $\mathrm{supp}(\hat{\varphi})$ compact, but for Schrödinger operators with symbol $\xi^2+V(x)$ and a potential V with nondegenerate critical points. Our contribution to this subject was to generalize his result for some more general operators, always with $\hat{\varphi}$ of compact support and under some geometrical assumptions on the flow (see [4]). In [7] we have studied the case of a Schrödinger operator near a degenerate minimum of the potential and the objective of the present work is to investigate the situation near a degenerate maximum which leads to a totally different asymptotic problem.

After a reformulation, via the theory of Fourier integral operators of [11], the spectral distribution of Eq. (1) can be expressed in terms of oscillatory integrals whose phases are related to the classical flow of p_0. Moreover, the asymptotic behavior as h tends to 0 of these oscillatory integrals is related to closed orbits of this flow. When (x_0, ξ_0) is a critical point of p_0, and hence an equilibrium of the flow, it is well known that the relation

$$\mathfrak{F}_t = \operatorname{Ker}\bigl(d_{x,\xi}\Phi_t(x_0, \xi_0) - \mathrm{Id}\bigr) \neq \{0\}, \tag{3}$$

leads to the study of degenerate oscillatory integrals. In the present work we consider the case of a Schrödinger operator:

$$P_h = -h^2\Delta + V(x), \tag{4}$$

but a generalization to an h-admissible operator (in the sense of [16]) of principal symbol $\xi^2 + V(x)$ is outlined in the last section. In particular, we will consider the case of a potential V with a single and degenerate critical point x_0 attached to a local maximum. A typical example is the top of a polynomial double well. An immediate consequence is that the symbol admits a unique critical point $(x_0, 0)$ on the energy surface $\{\xi^2 + V(x) = V(x_0)\}$ and that the linearized flow at this point is given by the flow of the free Laplacian. A fortiori, Eq. (3) is automatically satisfied with

$$\mathfrak{F}_t = \bigl\{(\delta u, \delta v) \in T_{x_0,\xi_0}T^*\mathbb{R}^n\colon\ \delta v = 0\bigr\} \simeq \mathbb{R}^n, \quad t \neq 0,$$
$$\mathfrak{F}_0 = T_{x_0,\xi_0}\bigl(T^*\mathbb{R}^n\bigr) \simeq \mathbb{R}^{2n}.$$

In particular, the stationary phase method cannot be applied at all in a microlocal neighborhood of $t = 0$.

The core of the proof lies in establishing suitable local normal forms for the local phase functions of a Fourier integral operator approximating the propagator in the semi-classical regime and in a generalization of the stationary phase formula for these normal forms. This generalization, based on an analytic representation of the associated class of oscillatory integrals, is more complicated than in the case of a local minimum but however allows to compute, explicitly, the leading term of the related local trace formula.

2. Hypotheses and main result

Let $p(x,\xi) = \xi^2 + V(x)$ where the potential V is smooth on $\mathbb{R}^n$ and real-valued. To this symbol is attached the h-differential operator $P_h = -h^2\Delta + V(x)$ and by a classical result P_h is essentially autoadjoint, for h small enough, if V is bounded from below. Moreover, if E is an energy level of p satisfying:

($\mathbf{H_1}$). There exists $\varepsilon_0 > 0$ such that $p^{-1}([E - \varepsilon_0, E + \varepsilon_0])$ is compact,

then, by [16, Theorem 3.13] the spectrum $\sigma(P_h) \cap [E - \varepsilon, E + \varepsilon]$ is discrete and consists in a sequence $\lambda_1(h) \leqslant \lambda_2(h) \leqslant \cdots \leqslant \lambda_j(h)$ of eigenvalues of finite multiplicities, if ε and h are small enough. For example, (H_1) is certainly satisfied if V goes to infinity at infinity. More generally, this is true when $E < \liminf_\infty V$.

We want to study the asymptotic behavior of the spectral distribution:

$$\gamma(E_{\mathrm{c}}, h, \varphi) = \sum_{\lambda_j(h) \in [E_{\mathrm{c}}-\varepsilon, E_{\mathrm{c}}+\varepsilon]} \varphi\left(\frac{\lambda_j(h) - E_{\mathrm{c}}}{h}\right). \tag{5}$$

We use the subscript E_{c} to recall that this energy level is critical. To avoid any problem of convergence we impose the condition:

($\mathbf{H_2}$). We have $\hat{\varphi} \in C_0^\infty(\mathbb{R})$ with a sufficiently small support near the origin.

Remark 1. This extra condition on the size of the support is simply here to avoid contributions of nontrivial closed orbits and can be easily relaxed. But, if the support of $\hat{\varphi}$ is large, the form of the asymptotic expansion changes. An explicit characterization of $\operatorname{supp}(\hat{\varphi})$ is given in Lemma 8.

To simplify notations we write $z = (x, \xi)$ for any point of the phase space and let be $\Sigma_{E_{\mathrm{c}}} = p^{-1}(\{E_{\mathrm{c}}\})$. In the next condition degenerate means that the second derivative at the critical point x_0 is zero. We impose now the type of singularity of the potential:

($\mathbf{H_3}$). On $\Sigma_{E_{\mathrm{c}}}$ the symbol p has a unique critical point $z_0 = (x_0, 0)$. This critical point is degenerate and associated to a local maximum of the potential V of the form:

$$V(x) = E_{\mathrm{c}} + V_{2k}(x) + \mathcal{O}\big(\|x - x_0\|^{2k+1}\big), \tag{6}$$

where V_{2k} is homogeneous of degree $2k$ w.r.t. $(x - x_0)$. Also, $k \geqslant 2$ and V_{2k} is definite negative.

Remark 2. Since all previous derivatives are 0 in x_0, the function V_{2k} does not depend on the choice of local coordinates near x_0. Contrary to the case exposed in [7], z_0 is not isolated on $\Sigma_{E_{\mathrm{c}}}$ and the connected component of z_0 in the energy surface can contain nontrivial closed trajectories. A generalization to more general maximums, e.g., to a sum of such homogeneous terms with different degrees, is possible but to simplify we only consider (H_3).

In this work, we are particulary interested in the contribution of the fixed point z_0. To understand the new phenomenon it suffices to study the localized problem:

$$\gamma_{z_0}(E_{\mathrm{c}}, h, \varphi) = \frac{1}{2\pi} \operatorname{Tr} \int_{\mathbb{R}} e^{i\frac{tE_{\mathrm{c}}}{h}} \hat{\varphi}(t) \psi^w(x, hD_x) \exp\left(-\frac{it}{h} P_h\right) \Theta(P_h)\, dt. \tag{7}$$

Here Θ is a function of localization near the critical energy surface $\Sigma_{E_{\mathrm{c}}}$, $\psi \in C_0^\infty(T^*\mathbb{R}^n)$ is micro-locally supported near z_0 and $\psi^w(x, hD_x)$ stands for the associated operator obtained by h-Weyl quantization. Rigorous justifications are given in Section 3 for the introduction of $\Theta(P_h)$ and in Section 4 for $\psi^w(x, hD_x)$. In [7] it was proven:

Theorem 3. *Under* (H$_1$)*, if* x_0 *is a local minimum of the potential, homogeneous as in* (H$_3$)*, then for all* φ *with* $\hat{\varphi} \in C_0^\infty(\mathbb{R})$ *we have*

$$\gamma_{z_0}(E_{\mathrm{c}}, h, \varphi) \sim h^{-n+\frac{n}{2}+\frac{n}{2k}} \sum_{j,l \in \mathbb{N}^2} h^{\frac{j}{2}+\frac{l}{2k}} \Lambda_{j,l}(\varphi),$$

where the $\Lambda_{j,l}$ *are some computable distributions. The leading coefficient is*

$$\Lambda_{0,0}(\varphi) = \frac{\mathrm{S}(\mathbb{S}^{n-1})}{(2\pi)^n} \int_{\mathbb{S}^{n-1}} \left|V_{2k}(\eta)\right|^{-\frac{n}{2k}} d\eta \int_{\mathbb{R}_+ \times \mathbb{R}_+} \varphi\left(u^2 + v^{2k}\right) u^{n-1} v^{n-1} \, du \, dv, \tag{8}$$

where $\mathrm{S}(\mathbb{S}^{n-1})$ *is the surface of the unit sphere of* $\mathbb{R}^n$.

These spectral estimates, near a local maximum of the potential, are related to a local problem. Precisely, the asymptotic behavior of

$$\frac{1}{(2\pi h)^n} \int_{T^*\mathbb{R}^n} \varphi\left(\frac{\xi^2 + V_{2k}(x)}{h}\right) dx \, d\xi, \quad h \to 0^+, \tag{9}$$

computes the coefficient $\Lambda_{0,0}$ and the degree w.r.t. h. This result can be interpreted as a scaling w.r.t. h of the trace of $\varphi(-\Delta + V_{2k})$, cf. the first term of the trace formula. A fortiori, at the critical energy level and for a local minimum, to replace V by V_{2k} is enough to compute the leading term of the expansion. In the present contribution, such an interpretation cannot hold since the trace of $\varphi(-\Delta + V_{2k})$ generally does not exists if V_{2k} is negative definite.

The main result of the present work is:

Theorem 4. *Under hypotheses* (H$_1$) *to* (H$_3$) *we have*

$$\gamma_{z_0}(E_{\mathrm{c}}, h, \varphi) \sim h^{-n+\frac{n}{2}+\frac{n}{2k}} \sum_{m=0,1} \sum_{j,l \in \mathbb{N}^2} h^{\frac{j}{2}+\frac{l}{2k}} \log(h)^m \Lambda_{j,l,m}(\varphi),$$

where $\Lambda_{j,l,m} \in \mathcal{S}'(\mathbb{R})$ *and* $\operatorname{sing\,supp}(\Lambda_{j,l,m}) \subset \{0\}$.

As concerns the leading term of the expansion, when $\frac{n(k+1)}{2k} \notin \mathbb{N}$*, the first nonzero coefficient of this local trace formula is given by*

$$h^{-n+\frac{n}{2}+\frac{n}{2k}} \langle T_{n,k}, \varphi\rangle \frac{\mathrm{S}(\mathbb{S}^{n-1})}{(2\pi)^n} \int_{\mathbb{S}^{n-1}} \left|V_{2k}(\eta)\right|^{-\frac{n}{2k}} d\eta. \tag{10}$$

The distributions $T_{n,k}$ are given by

$$\langle T_{n,k},\varphi\rangle = \int_{\mathbb{R}} \left(C_{n,k}^{+}|t|_{+}^{n\frac{k+1}{2k}-1} + C_{n,k}^{-}|t|_{-}^{n\frac{k+1}{2k}-1}\right)\varphi(t)\,dt, \quad \text{if } n \text{ is odd}, \tag{11}$$

$$\langle T_{n,k},\varphi\rangle = C_{n,k}^{-}\int_{\mathbb{R}} |t|_{-}^{n\frac{k+1}{2k}-1}\varphi(t)\,dt, \quad \text{if } n \text{ is even}. \tag{12}$$

But if $\frac{n(k+1)}{2k}\in\mathbb{N}$ and n is odd then the top-order term is

$$C_{n,k}\log(h)h^{-n+\frac{n}{2}+\frac{n}{2k}}\frac{\mathrm{S}(\mathbb{S}^{n-1})}{(2\pi)^n}\int_{\mathbb{S}^{n-1}}\left|V_{2k}(\eta)\right|^{-\frac{n}{2k}}d\eta\int_{\mathbb{R}}|t|^{n\frac{k+1}{2k}-1}\varphi(t)\,dt. \tag{13}$$

Finally, if $\frac{n(k+1)}{2k}\in\mathbb{N}$ and n is even, $C_{n,k}^{+}=C_{n,k}^{-}$ and we have

$$C_{n,k}^{\pm}h^{-n+\frac{n}{2}+\frac{n}{2k}}\frac{1}{(2\pi)^n}\int_{\mathbb{S}^{n-1}}\left|V_{2k}(\eta)\right|^{-\frac{n}{2k}}d\eta\int_{\mathbb{R}}|t|^{n\frac{k+1}{2k}-1}\varphi(t)\,dt. \tag{14}$$

In all expressions above $C_{n,k}$, $C_{n,k}^{\pm}$ are nonzero universal constants depending only on n and k.

For an explicit formulation of the numbers $C_{n,k}^{\pm}$, see Section 6. The arithmetical condition on k and n might be surprising at the first look. But this condition becomes clear when the oscillatory integrals of our spectral problem are analytically reformulated in Section 6. Viewing the top-order coefficient of the trace as a tempered distribution acting on the Schwartz function φ, i.e.,

$$\gamma_{z_0}(E_{\mathrm{c}},\varphi,h)\sim f(h)\langle\gamma,\varphi\rangle, \quad h\to 0^{+},$$

in any cases we obtain that $\operatorname{sing\,supp}(\gamma)=\{0\}$. Finally, for $n=1$ the contributions in Eqs. (10), (13) are bigger than the standard Weyl estimates:

$$\gamma(E,\varphi,h)\sim h^{1-n}(2\pi)^{-n}\hat{\varphi}(0)\,\mathrm{LVol}(\Sigma_E),$$

where E is a regular energy. Applying the results of [2, Sections 5 and 6], the local Weyl law is modified for $n=1$ and the counting function of eigenvalues in some interval $[E_{\mathrm{c}}-ah, E_{\mathrm{c}}+bh]$, $a,b>0$, changes according to Theorem 4. A geometrical interpretation is that $n=1$ is the only case where the Liouville measure LVol has a non-integrable singularity on $\Sigma_{E_{\mathrm{c}}}$ located in z_0.

3. Oscillatory representation of the spectral functions

The construction below is more or less classical and will be sketchy. For a more detailed exposition the reader can consult [2,4] or [13].

Let be $\varphi \in \mathcal{S}(\mathbb{R})$ with $\hat{\varphi} \in C_0^\infty(\mathbb{R})$, we recall that

$$\gamma(E_c, h, \varphi) = \sum_{\lambda_j(h) \in I_\varepsilon} \varphi\left(\frac{\lambda_j(h) - E_c}{h}\right),$$

where $I_\varepsilon = [E_c - \varepsilon, E_c + \varepsilon]$, with $0 < \varepsilon < \varepsilon_0$, and $p^{-1}(I_{\varepsilon_0})$ compact in $T^*\mathbb{R}^n$. We localize near the critical energy level E_c by inserting a cut-off function $\Theta \in C_0^\infty(]E_c - \varepsilon, E_c + \varepsilon[)$, such that $\Theta = 1$ in a neighborhood of E_c and $0 \leqslant \Theta \leqslant 1$ on $\mathbb{R}$. The corresponding decomposition is

$$\gamma(E_c, h, \varphi) = \gamma_1(E_c, h, \varphi) + \gamma_2(E_c, h, \varphi),$$

with

$$\gamma_1(E_c, h, \varphi) = \sum_{\lambda_j(h) \in I_\varepsilon} (1 - \Theta)\big(\lambda_j(h)\big)\varphi\left(\frac{\lambda_j(h) - E_c}{h}\right), \tag{15}$$

$$\gamma_2(E_c, h, \varphi) = \sum_{\lambda_j(h) \in I_\varepsilon} \Theta\big(\lambda_j(h)\big)\varphi\left(\frac{\lambda_j(h) - E_c}{h}\right). \tag{16}$$

Since $\varphi \in \mathcal{S}(\mathbb{R})$ a classical estimate, see, e.g., [5, Lemma 1], is

$$\gamma_1(E_c, h, \varphi) = \mathcal{O}\big(h^\infty\big), \quad \text{as } h \to 0. \tag{17}$$

By inversion of the Fourier transform we have

$$\Theta(P_h)\varphi\left(\frac{P_h - E_c}{h}\right) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\frac{tE_c}{h}} \hat{\varphi}(t) \exp\left(-\frac{it}{h} P_h\right) \Theta(P_h)\, dt.$$

The trace of the left-hand side is precisely $\gamma_2(E_c, h, \varphi)$ and Eq. (17) gives

$$\gamma(E_c, h, \varphi) = \frac{1}{2\pi} \mathrm{Tr} \int_{\mathbb{R}} e^{i\frac{tE_c}{h}} \hat{\varphi}(t) \exp\left(-\frac{it}{h} P_h\right) \Theta(P_h)\, dt + \mathcal{O}\big(h^\infty\big). \tag{18}$$

If $U_h(t) = \exp(-\frac{it}{h} P_h)$ is the evolution operator, we can approximate $U_h(t)\Theta(P_h)$ by a Fourier integral-operator depending on a parameter h. Let Λ be the Lagrangian manifold associated to the flow of p:

$$\Lambda = \big\{(t, \tau, x, \xi, y, \eta) \in T^*\mathbb{R} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n \colon\ \tau = p(x, \xi),\ (x, \xi) = \Phi_t(y, \eta)\big\}.$$

We recall that $I(X, \Lambda)$ is the class of oscillatory integrals based on X and whose Lagrangian manifold is Λ. The next result is a semi-classical version of a well-known more general result:

Theorem 5. *The operator $U_h(t)\Theta(P_h)$ is an h-FIO associated to Λ. For each $N \in \mathbb{N}$ there exists $U^{(N)}_{\Theta,h}(t)$ with integral kernel in $I(\mathbb{R}^{2n+1}, \Lambda)$ and $R^{(N)}_h(t)$ bounded, with a L^2-norm uniformly bounded for $0 < h \leqslant 1$ and t in a compact subset of $\mathbb{R}$, such that*

$$U_h(t)\Theta(P_h) = U^{(N)}_{\Theta,h}(t) + h^N R^{(N)}_h(t).$$

For a proof we refer, e.g., to Duistermaat [9]. Next, the remainder associated to $R^{(N)}_h(t)$ is controlled by the classical trick:

Corollary 6. *Let $\Theta_1 \in C_0^\infty(\mathbb{R})$, with $\Theta_1 = 1$ on $\operatorname{supp}(\Theta)$ and $\operatorname{supp}(\Theta_1) \subset I_\varepsilon$, then $\forall N \in \mathbb{N}$*

$$\operatorname{Tr}\left(\Theta(P_h)\varphi\left(\frac{P_h - E_{\mathrm{c}}}{h}\right)\right) = \frac{1}{2\pi}\operatorname{Tr}\int_{\mathbb{R}} \hat{\varphi}(t) e^{\frac{i}{h}tE_{\mathrm{c}}} U^{(N)}_{\Theta,h}(t)\Theta_1(P_h)\,dt + \mathcal{O}\big(h^{N-n}\big).$$

The proof is easy by cyclicity of the trace (see [5] or [16]).

For the particular case of a Schrödinger operator the BKW ansatz shows that the integral kernel of $U^{(N)}_{\Theta,h}(t)$ can be recursively constructed as

$$K^{(N)}_h(t,x,y) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} b^{(N)}_h(t,x,y,\xi) e^{\frac{i}{h}(S(t,x,\xi) - \langle y,\xi\rangle)}\,d\xi,$$

$$b^{(N)}_h = b_0 + hb_1 + \cdots + h^N b_N,$$

where S satisfies the Hamilton–Jacobi equation:

$$\begin{cases} \partial_t S(t,x,\xi) + p(x, \partial_x S(t,x,\xi)) = 0, \\ S(0,x,\xi) = \langle x,\xi\rangle. \end{cases}$$

In particular, we obtain that

$$\big\{\big(t, \partial_t S(t,x,\eta), x, \partial_x S(t,x,\eta), \partial_\eta S(t,x,\eta), -\eta\big)\big\} \subset \Lambda_{\mathrm{flow}},$$

and that the function S is a generating function of the flow, i.e.,

$$\Phi_t\big(\partial_\eta S(t,x,\eta), \eta\big) = \big(x, \partial_x S(t,x,\eta)\big). \tag{19}$$

Inserting this approximation in Eq. (18) we find that, modulo an error $\mathcal{O}(h^{N-n})$, the trace $\gamma(E_{\mathrm{c}}, h, \varphi)$ can be written for all $N \in \mathbb{N}$ as

$$\gamma(E_{\mathrm{c}}, h, \varphi) = \sum_{j<N} \frac{h^j}{(2\pi h)^n} \int_{\mathbb{R}\times T^*\mathbb{R}^n} e^{\frac{i}{h}(S(t,x,\xi) - \langle x,\xi\rangle + tE_{\mathrm{c}})} a_j(t,x,\xi)\hat{\varphi}(t)\,dt\,dx\,d\xi, \tag{20}$$

where $a_j(t,x,\eta) = b_j(t,x,x,\eta)$ is the evaluation of b_j on the diagonal $\{x = y\}$.

Remark 7. By [16, Theorem 3.11 and Remark 3.14], $\Theta(P_h)$ is h-admissible with a symbol supported in $p^{-1}(I_\varepsilon)$. This allows to consider only oscillatory-integrals with compact support.

4. Classical dynamics near the equilibrium

The critical points of the phase function of Eq. (20) are given by

$$\begin{cases} E_c = -\partial_t S(t,x,\xi), \\ x = \partial_\xi S(t,x,\xi), \\ \xi = \partial_x S(t,x,\xi), \end{cases} \quad \Leftrightarrow \quad \begin{cases} p(x,\xi) = E_c, \\ \Phi_t(x,\xi) = (x,\xi), \end{cases}$$

where the right-hand side defines a closed trajectory of the flow inside Σ_{E_c}. Since we are mainly interested in the contribution of the critical point, we choose a function $\psi \in C_0^\infty(T^*\mathbb{R}^n)$, with $\psi = 1$ near z_0, hence

$$\begin{aligned} \gamma_2(E_c,h,\varphi) &= \frac{1}{2\pi}\operatorname{Tr}\int_{\mathbb{R}} e^{i\frac{tE_c}{h}}\hat{\varphi}(t)\psi^w(x,hD_x)\exp\left(-\frac{i}{h}tP_h\right)\Theta(P_h)\,dt \\ &\quad + \frac{1}{2\pi}\operatorname{Tr}\int_{\mathbb{R}} e^{i\frac{tE_c}{h}}\hat{\varphi}(t)\big(1-\psi^w(x,hD_x)\big)\exp\left(-\frac{i}{h}tP_h\right)\Theta(P_h)\,dt. \end{aligned}$$

Under the additional hypothesis of having a clean flow, the asymptotic expansion of the second term is given by the semi-classical trace formula on a regular level. We also observe that the contribution of the first term, which is precisely the distribution $\gamma_{z_0}(E_c,h,\varphi)$ of Theorem 4, is local. Hence this allows to introduce local coordinates near z_0. As pointed out in Remark 2, $\{z_0\}$ is not a connected component of Σ_{E_c} and elements

$$\big\{(T,z)\colon\ T\neq 0,\ z\in\Sigma_{E_c}\cap\operatorname{supp}(\psi),\ \Phi_T(z)=z\big\}$$

could contribute in the asymptotic expansion of $\gamma_{z_0}(E_c,h,\varphi)$. In fact, the next lemma will solve this problem:

Lemma 8. *There exists a $T>0$, depending only on V, such that $\Phi_t(z)\neq z$ for all $z\in\Sigma_{E_c}\setminus\{z_0\}$ and all $t\in\,]-T,0[\,\cup\,]0,T[$.*

Proof. If H_p is our Hamiltonian vector field and $z=(x,\xi)$ we have

$$\big\|H_p(z_1)-H_p(z_2)\big\|^2 = 4\|\xi_1-\xi_2\|^2 + \big\|\nabla_x V(x_1)-\nabla_x V(x_2)\big\|^2.$$

Since our potential is smooth, when z_1 and z_2 are in the energy surface Σ_{E_c}, which is compact by assumption, there exists $M>0$ such that

$$\big\|H_p(z_1)-H_p(z_2)\big\| \leqslant M\|z_1-z_2\|.$$

The main result of [17] shows that any periodic trajectory inside Σ_{E_c} has a period p such that $p \geqslant \frac{2\pi}{M}$. $\square$

This result allows to distinguish out the contribution of elements $\{t, z_0\}$ in the local object $\gamma_{z_0}(E_c, \varphi, h)$. Precisely, if we choose $\hat{\varphi} \in C_0^\infty(]-p, p[)$, where p is chosen as in Lemma 8 and depends only on V, any periodic orbit of Σ_{E_c} has a period greater than p and a fortiori does not contribute. Now, we restrict our attention to the singular contribution generated by the critical point. Since z_0 is an equilibrium of the flow we obtain that

$$d_{x,\xi}\Phi_t(z_0) = \exp(tH_{-\Delta}), \quad \forall t.$$

The computation of this linear operator is easy and gives:

$$d_{x,\xi}\Phi_t(z_0)(u, v) = (u + 2tv, v), \quad \forall (u, v) \in T_{z_0}T^*\mathbb{R}^n.$$

From classical mechanics we know that the singularities, in the sense of the Morse theory, of the function $S(t, x, \xi) - \langle x, \xi\rangle$ are supported in the set

$$\mathfrak{F}_t = \operatorname{Ker}\bigl(d_{x,\xi}\Phi_t(z_0) - \mathrm{Id}\bigr),$$

see, e.g., [4, Lemma 9]. As mentioned in the introduction we obtain

$$\mathfrak{F}_0 = T_{z_0}\bigl(T^*\mathbb{R}^n\bigr) \simeq \mathbb{R}^{2n},$$
$$\mathfrak{F}_t = \bigl\{(u, v) \in T_{z_0}(T^*\mathbb{R}^n):\ v = 0\bigr\} \simeq \mathbb{R}^n, \quad t \neq 0.$$

To simplify notations, and until further notice, all derivatives will be taken with respect to the initial conditions (x, ξ). The next nonzero terms of the Taylor expansion of the flow are computed via the technical result:

Lemma 9. *Let be z_0 an equilibrium of the C^∞ vector field X and Φ_t the flow of X. Then for all $m \in \mathbb{N}^*$, there exists a polynomial map P_m, vector-valued and of degree at most m, such that*

$$d^m\Phi_t(z_0)\bigl(z^m\bigr) = d\Phi_t(z_0)\int_0^t d\Phi_{-s}(z_0)P_m\bigl(d\Phi_s(z_0)(z), \ldots, d^{m-1}\Phi_s(z_0)\bigl(z^{m-1}\bigr)\bigr)\,ds.$$

For a proof we refer to [4] or [5]. Here we shall use that our vector field is

$$H_p = 2\xi\frac{\partial}{\partial x} - \partial_x V(x)\frac{\partial}{\partial \xi}.$$

We identify the linearized flow in z_0 with a matrix multiplication operator:

$$d\Phi_t(z_0) = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}.$$

Clearly, with the hypothesis (H_3) we obtain the polynomials:

$$P_j = 0, \quad \forall j \in \{2, \dots, 2k-2\},$$
$$P_{2k-1}(Y_1, \dots, Y_{2k-2}) = \begin{pmatrix} 0 \\ d^{2k-1}\nabla_x V(x_0)(Y_1^{2k-1}) \end{pmatrix} \neq 0,$$

where the notation Y_1^l stands for $(Y_1, \dots, Y_1)$: l-times. Inserting the definition of $d\Phi_s(z_0)$ and integration from 0 to t yields

$$d^{2k-1}\Phi_t(z_0)\big((x,\xi)^{2k-1}\big) = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix} \int_0^t \begin{pmatrix} 2sd^{2k-1}\nabla_x V(x_0)((x+2s\xi)^{2k-1}) \\ -d^{2k-1}\nabla_x V(x_0)((x+2s\xi)^{2k-1}) \end{pmatrix} ds.$$

Terms of higher degree can be obtained similarly by successive integrations. Finally, if we assume that $z_0 = 0$, since there is no intermediate terms between terms of degree 1 and $2k-1$, the jet of order $2k-1$ of the flow is

$$\Phi_t(z) = d\Phi_t(0)(z) + \frac{1}{(2k-1)!} d^{2k-1}\Phi_t(0)\big(z^{2k-1}\big) + \mathcal{O}\big(\|z\|^{2k}\big), \tag{21}$$

and can be computed explicitly with a given V_{2k}.

5. Normal forms of the phase function

Since the contribution we study is local, cf. the introduction of ψ in Eq. (7), we work with some local coordinates (x,ξ) near the critical point z_0. Via these coordinates we identify locally $T^*\mathbb{R}^n \cap V(z_0)$ with an open of $\mathbb{R}^{2n}$. With $z = (x,\xi) \in \mathbb{R}^{2n}$, we define

$$\Psi(t,z) = \Psi(t,x,\xi) = S(t,x,\xi) - \langle x,\xi\rangle + tE_c. \tag{22}$$

We start by a more precise description of our phase function.

Lemma 10. *Near z_0, here supposed to be 0 to simplify, we have*

$$\Psi(t,x,\xi) = -t\|\xi\|^2 + S_{2k}(t,x,\xi) + R_{2k+1}(t,x,\xi), \tag{23}$$

where S_{2k} is homogeneous of degree $2k$ w.r.t. (x,ξ) and is uniquely determined by V_{2k}. Moreover, $R_{2k+1}(t,x,\xi) = \mathcal{O}(\|(x,\xi)\|^{2k+1})$, uniformly for t in a compact subset of $\mathbb{R}$.

Proof. With the particular structure of the flow in z_0, cf. Eq. (21), we search our local generating function as

$$S(t,x,\xi) = -tE_c + S_2(t,x,\xi) + S_{2k}(t,x,\xi) + \mathcal{O}\big(\big\|(x,\xi)\big\|^{2k+1}\big),$$

where the S_j are time-dependent and homogeneous of degree j w.r.t. (x,ξ). Starting from the implicit relation $\Phi_t(\partial_\xi S(t,x,\xi),\xi) = (x, \partial_x S(t,x,\xi))$ and with Eq. (21), we obtain that

$$S_2(t,x,\xi) = \langle x,\xi\rangle - t\|\xi\|^2.$$

Now we compute the term S_{2k}. To do so, we retain only terms homogeneous of degree $2k-1$ and we get

$$d\Phi_t(0)\big((\partial_\xi S,0)\big) + \frac{1}{(2k-1)!} d^{2k-1}\Phi_t(0)\big((\partial_\xi S_2,\xi)^{2k-1}\big) = (0, \partial_x S_{2k}).$$

If J is the matrix of the usual simplectic form σ on $\mathbb{R}^{2n}$, we have

$$J\nabla S_{2k}(t,x,\xi) = \frac{1}{(2k-1)!} d^{2k-1}\Phi_t(0)\big((x-2t\xi,\xi)^{2k-1}\big).$$

By homogeneity and with Eq. (21) we obtain

$$S_{2k}(t,x,\xi) = \frac{1}{(2k)!}\sigma\big((x,\xi), d^{2k-1}\Phi_t(0)\big((x-2t\xi,\xi)^{2k-1}\big)\big).$$

This gives the result since $d^{2k-1}\Phi_t(0)$ is well determined by V_{2k}. □

Fortunately, we will not have to compute the remainder explicitly because of some homogeneous considerations. To prepare the construction of our normal forms, we study carefully S_{2k}.

Corollary 11. *The function S_{2k} satisfies*:

$$S_{2k}(t,x,\xi) = -tV_{2k}(x) + t^2\big\langle \xi, \nabla_x V_{2k}(x)\big\rangle + \sum_{j,l=1}^{n} \xi_j\xi_l g_{j,l}(t,x,\xi), \tag{24}$$

where the functions $g_{j,l}$ are smooth and vanish in $x=0$.

Proof. If f is homogeneous of degree $2k>2$ w.r.t. (x,ξ) we can write

$$f(t,x,\xi) = f_1(t,x) + \big\langle \xi, f_2(t,x)\big\rangle + \sum_{j,l=1}^{n} \xi_j\xi_l f_3^{(jl)}(t,x).$$

It remains to compute the function f_1 and the vector field f_2. We have

$$S_{2k}(t,x,\xi) = \frac{1}{(2k)!}\sigma\big((x,\xi), d\Phi_t(0)\big)\int_0^t \begin{pmatrix} 2sd^{2k-1}\nabla_x V(0)((x+2(s-t)\xi)^{2k-1}) \\ -d^{2k-1}\nabla_x V(0)((x+2(s-t)\xi)^{2k-1}) \end{pmatrix} ds.$$

Clearly, the term of degree homogeneous of degree $2k$ w.r.t. x is given by

$$-\frac{1}{(2k)!}\int_0^t \big\langle x, d^{2k-1}\nabla_x V(0)\big(x^{2k-1}\big)\big\rangle\, ds = -\frac{t}{2k}\big\langle x, \nabla_x V_{2k}(x)\big\rangle = -tV_{2k}(x),$$

where the last result holds by homogeneity. As concerns the linear term w.r.t. ξ, by combinatoric and linear operations, this one can be written:

$$\begin{aligned}&\frac{t^2}{(2k)!}\Big((2k-1)\big\langle x, d^{2k-1}\nabla_x V(0)\big(\big(x^{2k-2},\xi\big)\big)\big\rangle + \big\langle \xi, d^{2k-1}\nabla_x V(0)\big(x^{2k-1}\big)\big\rangle\Big)\\ &\quad= \frac{t^2}{2k}\Big((2k-1)\big\langle \nabla_x V_{2k}(x), \xi\big\rangle + \big\langle \xi, \nabla_x V_{2k}(x)\big\rangle\Big) = t^2\big\langle \xi, \nabla_x V_{2k}(x)\big\rangle.\end{aligned}$$

This completes the proof. □

These two terms can be also derived by a heuristic method. Starting from the Hamilton–Jacobi equation we obtain $S(0,x,\xi) = \langle x, \xi\rangle$, $\partial_t S(0,x,\xi) = -p(x,\xi)$ and also

$$\partial^2_{t,t} S(t,x,\xi) = -\big\langle \partial_\xi p\big(x, \partial_x S(t,x,\xi)\big), \partial^2_{t,x} S(t,x,\xi)\big\rangle.$$

But, for our flow, in $t=0$ we have simply

$$\partial^2_{t,t} S(0,x,\xi) = 2\big\langle \xi, \partial_x V(x)\big\rangle.$$

Hence the Taylor expansion in $t=0$ provides a good result with few calculations. Unfortunately, this approach gives no information about the degree, w.r.t. (x,ξ), of the remainders $\mathcal{O}(t^d)$ for each $d \geqslant 3$.

We have enough material to build the normal form of our phase function. In the following, the notation $f \simeq g$ means that f and g are conjugated by a local diffeomorphism, apart perhaps on a set of zero measure.

Lemma 12. *In a neighborhood of $z = z_0$, there exists local coordinates χ such that*

$$\Psi(t,z) \simeq -\chi_0\big(\chi_1^2 - \chi_2^{2k}\big). \tag{25}$$

Proof. We can here assume that z_0 is the origin. We proceed in two steps. First we want to eliminate terms of high degree. Starting form Eq. (24) we define

$$E(t,x,\xi) = \sum_{j,l} \xi_j \xi_l g_{j,l}(t,x,\xi).$$

Since $S(t,x,\xi) - \langle x,\xi\rangle = \mathcal{O}(t)$, we have $\Psi(t,z) = \mathcal{O}(t)$. Hence, all terms of the expansion are $\mathcal{O}(t)$ and this allows to write $E = t\tilde{E}$. Similarly, we write $R_{2k+1} = t\tilde{R}_{2k+1}$ in Eq. (23). To obtain a blow-up of the singularity, we use polar coordinates $x = r\theta$, $\xi = q\eta$, $\theta, \eta \in$

$\mathbb{S}^{n-1}(\mathbb{R})$, $q, r \in \mathbb{R}_+$. This induces naturally a Jacobian $r^{n-1}q^{n-1}$. By construction, there exists a function F vanishing in $(q,r) = (0,0)$ such that

$$\tilde{E}(t, r\theta, q\eta) = q^2 F(t, r, \theta, q, \eta). \tag{26}$$

With Lemma 10 and Corollary 11, near z_0 the phase $\Psi(t,z)$ can be written:

$$-t\big(q^2 + r^{2k}V_{2k}(\theta) - tqr^{2k-1}\langle \eta, \nabla V_{2k}(\theta)\rangle + q^2 F(t,r,\theta,q,\eta) + \tilde{R}_{2k+1}(t, r\theta, q\eta)\big).$$

Thanks to the Taylor formula, the remainder $\tilde{R}_{2k+1}$ can be written as

$$\tilde{R}_{2k+1}(t, r\theta, q\eta) = q^2 R_1(t,r,\theta,q,\eta) + r^{2k} R_2(t,r,\theta,q,\eta),$$

where R_1 vanishes in $r = 0$ and R_2 vanishes in $q = 0$. We obtain

$$\Psi(t,z) \simeq -t\big(q^2\alpha_1(t,r,\theta,q,\eta) - r^{2k}\alpha_2(t,r,\theta,q,\eta)\big) + t^2 qr^{2k-1}\langle \eta, \nabla V_{2k}(\theta)\rangle,$$

where we have defined:

$$\begin{aligned}
\alpha_1(t,r,\theta,q,\eta) &= (1 + R_1 + F)(t,r,\theta,q,\eta),\\
\alpha_2(t,r,\theta,q,\eta) &= \big|V_{2k}(\theta)\big| + R_2(t,r,\theta,q,\eta).
\end{aligned}$$

Since $|V_{2k}(\theta)| > 0$ on $\mathbb{S}^{n-1}$, we can eliminate α_1 and α_2 by a local change of coordinates:

$$\Big(q\alpha_1^{\frac{1}{2}}, r\alpha_2^{\frac{1}{2k}}\Big) \to (Q, R), \tag{27}$$

near $(q,r) = (0,0)$. This is acceptable since the corresponding Jacobian is

$$\big|J(Q,R)\big|(0,0) = \big|V_{2k}(\theta)\big|^{\frac{1}{2k}} \neq 0. \tag{28}$$

In these local coordinates, and still using (r,q) instead of (R,Q), we obtain

$$\Psi(t,z) \simeq -t\big(q^2 - r^{2k}\big) + t^2 qr^{2k-1}\varepsilon(t,r,\theta,q,\eta),$$

where

$$\varepsilon(t,r,\theta,q,\eta) = \langle \eta, \nabla V_{2k}(\theta)\rangle\Big(\alpha_1^{-\frac{1}{2}}\alpha_2^{\frac{1-2k}{2k}}\Big)(t,r,\theta,q,\eta).$$

In a second time, we eliminate the nonlinear term in t. To do so, we write

$$-t\big(q^2 - tqr^{2k-1}\varepsilon\big) = -t\left(q - \frac{t}{2}r^{2k-1}\varepsilon\right)^2 + \frac{1}{4}t^3 r^{4k-2}\varepsilon^2.$$

Now, we can factor out the last term if we use

$$\alpha_3(t,r,\theta,q,\eta) = \left(1 - \frac{t^2}{4} r^{4k-2} \varepsilon^2(t,r,\theta,q,\eta)\right).$$

Then the change of variables $r \to \alpha_3^{-\frac{1}{2k}} r$ gives

$$\Psi(t,z) \simeq -t\left(\left(q - \frac{t}{2} r^{2k-1} \tilde{\varepsilon}(t,r,\theta,q,\eta)\right)^2 - r^{2k}\right),$$

where

$$\tilde{\varepsilon}(t,r,\theta,q,\eta) = \alpha_3^{\frac{1-2k}{2k}} \varepsilon\left(t, \alpha_3^{-\frac{1}{2k}} r, t, \theta, q, \eta\right).$$

Finally, if we define:

$$(\chi_0, \chi_2, \chi_3, \ldots, \chi_{2n})(t,r,\theta,q,\eta) = (t,r,\theta,\eta), \tag{29}$$

$$\chi_1(t,r,\theta,q,\eta) = q - \frac{t}{2} r^{2k-1} \tilde{\varepsilon}(t,r,\theta,q,\eta), \tag{30}$$

the phase is $-\chi_0(\chi_1^2 - \chi_2^{2k})$ which is the desired result. $\square$

If we use these local coordinates we obtain a simpler problem:

$$\int e^{-\frac{i}{h}\chi_0(\chi_1^2 - \chi_2^{2k})} A(\chi_0, \chi_1, \chi_2)\, d\chi_0\, d\chi_1\, d\chi_2, \tag{31}$$

where the amplitude A is obtained via pullback and integration, i.e.,

$$A(\chi_0, \chi_1, \chi_2) = \int \chi^*\big(a|J\chi|\big) d\chi_4 \ldots d\chi_{2n}. \tag{32}$$

Now, we make several comments on this construction:

Remark 13. Near (x_0, ξ_0) we have the relations $\chi_2 = 0 \Leftrightarrow x = x_0$ and $\chi_1(t, 0, \theta, q, \eta) = 0 \Leftrightarrow \xi = 0$. Also, because of the introduction of the polar coordinates, our amplitude satisfies:

$$A(\chi_0, \chi_1, \chi_2) = \mathcal{O}\big(\chi_l^{n-1}\big), \quad l = 1, 2.$$

Finally, each diffeomorphism used has Jacobian 1 in z_0 excepted the correction w.r.t. α_2, cf. Eq. (28), which induces by pullback of the measure $r^{n-1}\, dr$ a multiplication by

$$\big|V_{2k}(\theta)\big|^{-\frac{n}{2k}}. \tag{33}$$

These facts will be useful, in the next section, to express the main coefficient w.r.t. h of $\gamma_{z_0}(E_c, \varphi, h)$ independently of the coordinates χ.

6. Proof of the main result

Let be $2k$ the even integer attached to our potential. We state two technical results which allow to compute the asymptotic expansion of our oscillatory integrals. The first one proves the existence of a total asymptotic expansion for the family of oscillatory integrals attached to our normal forms. The second result, which is the hardest part of this work, computes the main coefficients of this expansion w.r.t. h in the trace formula.

Lemma 14. *For $a \in C_0^\infty(\mathbb{R} \times [0, \infty]^2)$, the oscillatory integrals*

$$J(\lambda) = \int\limits_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+} e^{-i\lambda\chi_0(\chi_1^2-\chi_2^{2k})} A(\chi_0, \chi_1, \chi_2)\, d\chi_0\, d\chi_1\, d\chi_2, \tag{34}$$

admit, as $\lambda \to \infty$, the asymptotic expansion

$$J(\lambda) \sim \sum_{j=0}^{\infty} \lambda^{-\frac{l+1}{2k}} C_j(A) + \sum_{j=0}^{\infty} \lambda^{-j+1} \log(\lambda) D_j(A), \tag{35}$$

where C_j and D_j are universal (computable) distributions.

Remark 15. The result stated obviously also holds for integration on $\mathbb{R} \times \mathbb{R}^2$ if we split up the domain of integration and use the symmetry w.r.t. χ_1 and χ_2 of the phase. Also, note that terms with a logarithm of the parameter only occur when $(j/2k) \in \mathbb{N}^*$.

To compute the leading term of the trace formula we need a particular and explicit result which explains the effect of the dimension n in our spectral problem. For $A \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+^2)$ we define:

$$I(\lambda) = \int\limits_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+} e^{-i\lambda t(r^2-q^{2k})} A(t, r, q) r^{n-1} q^{n-1}\, dt\, dr\, dq. \tag{36}$$

We note $t_\pm = \max(\pm t, 0)$ and $\hat{A}$ the partial Fourier transform of A w.r.t. t.

Lemma 16. *If $n(k+1)/2k$ is not an integer then, as $\lambda \to \infty$, we have*

$$I(\lambda) = C_0(A)\lambda^{-n\frac{k+1}{2k}} + \mathcal{O}\Big(\log(\lambda)\lambda^{-\frac{n(k+1)+1}{2k}}\Big). \tag{37}$$

The distributional coefficients are given by:

$$C_0(A) = C^-_{n,k} \int_{\mathbb{R}} t_-^{n\frac{k+1}{2k}-1} \hat{A}(t,0,0)\,dt, \quad \textit{if } n \textit{ is even},$$

$$C_0(A) = \int_{\mathbb{R}} \Big(C^-_{n,k} t_-^{n\frac{k+1}{2k}-1} + C^+_{n,k} t_+^{n\frac{k+1}{2k}-1} \Big) \hat{A}(t,0,0)\,dt, \quad \textit{if } n \textit{ is odd}.$$

But if $n(k+1)/2k \in \mathbb{N}^*$ *and n is odd, then*

$$I(\lambda) = -C_0(A)\lambda^{-n\frac{k+1}{2k}} \log(\lambda) + \mathcal{O}\Big(\lambda^{-\frac{n(k+1)+1}{2k}}\Big), \tag{38}$$

with

$$C_0(A) = C_{n,k} \int_{\mathbb{R}} |t|^{n\frac{k+1}{2k}-1} \hat{A}(t,0,0)\,dt.$$

Finally, if $n(k+1)/2k \in \mathbb{N}^*$ *and n is even the asymptotic is given by Eq.* (37) *with*

$$C_0(A) = \int_{\mathbb{R}} \Big(\tilde{C}^-_{n,k} t_-^{n\frac{k+1}{2k}-1} + \tilde{C}^+_{n,k} t_+^{n\frac{k+1}{2k}-1} \Big) \hat{A}(t,0,0)\,dt.$$

Remark 17. We will not detail all the coefficients of the asymptotic expansion because these are given by lengthy formulae and it is, a priori, not possible to express invariantly their contributions to the trace formula. Anyhow, these can be explicitly computed with the procedure below. Also if $\frac{n(k+1)+1}{2k} \notin \mathbb{N}$ the remainder of Eq. (37) can be optimized to $\mathcal{O}(\lambda^{-\frac{n(k+1)+1}{2k}})$.

Before entering in the proof we would like to ad a comment suggested by an interesting remark of D. Barlet. The Berstein–Sato polynomial of our phase $\chi_0(\chi_1^2 - \chi_2^{2k})$ can be explicitly computed as

$$\mathfrak{B}(z) = (z+1)^3 \prod_{l=0,\, l\neq k+1}^{2k-2} \Big(z + \frac{l+k-1}{2k} \Big). \tag{39}$$

In particular, we could expect to obtain terms $\log(\lambda)^2$ in the expansion since $1/\mathfrak{B}$ has a triple pole in $z = 1$. But we will see that if the amplitude is smooth there is no such contribution.

Proof of Lemma 14. To attain our objective we can restrict the proof of the lemma to an amplitude $A(t,r,q) = a(t)B(r,q)$. This is justified by a standard density argument in

C_0^∞ and the fact that the coefficients obtained below are linear continuous functionals. By integration w.r.t. t we obtain

$$J(\lambda) = \int_{\mathbb{R}_+^2} \hat{a}\big(\lambda\big(r^2 - q^{2k}\big)\big) B(r,q)\,dr\,dq.$$

This shows that the asymptotic is supported in the set $r = q^k$ since $\hat{a}$ decreases faster than any polynomial at infinities. Although the new problem looks simple it is still too complicated to obtain an explicit solution. First we split our integral as

$$J(\lambda) = J_+(\lambda) + J_-(\lambda), \tag{40}$$

with

$$J_+(\lambda) = \int_{0 \leqslant q^k \leqslant r < \infty} \hat{a}\big(\lambda\big(r^2 - q^{2k}\big)\big) B(r,q)\,dr\,dq,$$

$$J_-(\lambda) = \int_{0 \leqslant r \leqslant q^k < \infty} \hat{a}\big(\lambda\big(r^2 - q^{2k}\big)\big) B(r,q)\,dr\,dq.$$

Now we define the Melin transforms of $\hat{a}$:

$$M_+(z) = \int_0^\infty t^{z-1} \hat{a}(t)\,dt, \tag{41}$$

$$M_-(z) = \int_0^\infty t^{z-1} \hat{a}(-t)\,dt. \tag{42}$$

By Melin inversion formula we obtain that

$$J_+(\lambda) = \frac{1}{2i\pi} \int_{c+i\mathbb{R}} M_+(z) \int_{0 \leqslant q^k \leqslant r < \infty} \big(\lambda\big(r^2 - q^{2k}\big)\big)^{-z} B(r,q)\,dr\,dq\,dz. \tag{43}$$

Here $0 < c < (2k)^{-1}$ so that the Melin inversion makes sense. In order to desingularize the remaining part of the phase we introduce the new coordinates $r = sq^k$, $s > 0$ and $q > 0$. We accordingly obtain that

$$J_+(\lambda) = \frac{1}{2i\pi} \int_{c+i\mathbb{R}} M_+(z)\lambda^{-z} \int_{s=1}^{\infty} \int_{q=0}^{\infty} \big(s^2 - 1\big)^{-z} q^{-2kz} B\big(sq^k, q\big)\,ds q^k\,dq\,dz \tag{44}$$

and

$$J_-(\lambda) = \frac{1}{2i\pi} \int\limits_{c+i\mathbb{R}} M_-(z)\lambda^{-z} \int\limits_{s=0}^{1} \int\limits_{q=0}^{\infty} \left|s^2-1\right|^{-z} q^{-2kz} B\left(sq^k, q\right) dsq^k \, dq \, dz. \quad (45)$$

To control the remainders, and also to justify the changes of path below, we recall the following classical result:

Lemma 18. *If $a \in \mathcal{S}(\mathbb{R})$ then we have*

$$\forall c > 0, \quad M_\pm(c+iy) \in \mathcal{S}(\mathbb{R}_y), \quad (46)$$

a fortiori $M_\pm(c+iy) \in L^1(\mathbb{R}, dy)$.

Our integrals can be expanded via Cauchy's residue method by pushing of the complex path of integration to the right. The associated distributional factors are analytic (cf. Lemma 19 below) and if we choose $d > c$ conveniently we obtain

$$J_+(\lambda) = \sum_{c<z_i<d} \operatorname{Res}(z_i) + R(d, \lambda),$$

where the remainder satisfies:

$$\left|R(d,\lambda)\right| \leqslant C(B)\lambda^{-d} \left\| M_+(d+iy) \right\|_{L^1(\mathbb{R},dy)} = \mathcal{O}\left(\lambda^{-d}\right). \quad (47)$$

Here, for each d the constant $C(B)$ involves the L^1-norm of a finite number of derivatives of B. This will indeed lead to an asymptotic expansion.

The resulting asymptotics are related to poles of meromorphic distributions $z \mapsto M_\pm(z)\lambda^{-z}(s^2-1)^{-z}(q^{2k})^{-z}$ and it remains now to extend analytically these distributions.

Lemma 19. *The family of distributions on $C_0^\infty(\mathbb{R}_+^2): z \to |s^2-1|^{-z} q^{-2kz}$, initially defined in the domain $\Re(z) < 1/2k$, is meromorphic on $\mathbb{C}$ with poles located at the rational points*:

$$z_{j,k} = \frac{j}{2k}, \quad j \in \mathbb{N}^*. \quad (48)$$

These poles are of order 2 if $z_{j,k} \in \mathbb{N}^*$ and of order 1, otherwise.

Proof. We first observe that for $s \geqslant 0$ we can write

$$\left|s^2-1\right|^{-z} = (s+1)^{-z}|s-1|^{-z}.$$

Since we will integrate on $s \geqslant 0$ the first term defines clearly an entire distribution. Now we remark that for $s \geqslant 1$

$$\frac{\partial^{2k+1}}{\partial s \partial q^{2k}} (s-1)^{1-z} \left(q^{2k}\right)^{1-z} = \mathfrak{b}_0(z)(s-1)^{-z}\left(q^{2k}\right)^{-z},$$

$$\mathfrak{b}_0(z) = (1-z)\prod_{j=1}^{2k}(j-2kz).$$

Hence for $\Re(z) > 0$ we can write:

$$\begin{aligned}
&\int\limits_{s=1}^{\infty}\int\limits_{q=0}^{\infty} \left(s^2-1\right)^{-z} q^{-2kz} f(s,q)\, ds\, dq \\
&\quad = \frac{-1}{\mathfrak{b}_0(z)} \int\limits_{s=1}^{\infty}\int\limits_{q=0}^{\infty} (s-1)^{1-z} q^{2k(1-z)} \frac{\partial^{2k+1}}{\partial s \partial q^{2k}} (1+s)^{-z} f(s,q)\, ds\, dq. \qquad (49)
\end{aligned}$$

Now, the r.h.s. is meromorphic in $\Re(z) < 1 + 1/2k$ and we can iterate to get the analytic continuation in $\Re(z) < l + 1/2k$, $l \in \mathbb{N}$ arbitrary. The poles, and their order, can be read off the rational functions of z:

$$\mathfrak{R}_l(z) = \prod_{m=0}^{l-1} \frac{1}{\mathfrak{b}_0(z-m)}. \qquad (50)$$

Finally, a similar construction holds if we integrate w.r.t. $s \in [0, 1]$. □

With the residue method and classical estimates for the remainder (cf. Lemma 18) the last lemma proves the existence of a total asymptotic expansion for the integrals $J(\lambda)$. This ends the proof of Lemma 14. □

Proof of Lemma 16. With some slight modifications, we can apply the results above to our initial problem. Taking Remark 13 into account, we have to compute asymptotics of oscillatory integrals with amplitudes:

$$B(r,q) = b(r,q) r^{n-1} q^{n-1}. \qquad (51)$$

By substitution, we have to study the poles of

$$\mathfrak{g}_+(z) = M_+(z) \int\limits_{s=1}^{\infty}\int\limits_{q=0}^{\infty} \left(s^2-1\right)^{-z} q^{-2kz} b\left(sq^k, q\right) s^{n-1} q^{n(k+1)-1}\, ds\, dq, \qquad (52)$$

$$\mathfrak{g}_-(z) = M_-(z) \int\limits_{s=0}^{1}\int\limits_{q=0}^{\infty} \left|s^2-1\right|^{-z} q^{-2kz} b\left(sq^k, q\right) s^{n-1} q^{n(k+1)-1}\, ds\, dq. \qquad (53)$$

Here $b(sq^k, q)$ is in general no more of compact support but all expressions and manipulations will be legal because terms depending on z will decrease faster and faster when we will shift the path of integrations. To avoid unnecessary calculations and discussions below, we remark that we can commute the polynomial weights w.r.t. q via the relations:

$$\frac{\partial^{2k+1}}{\partial s \partial q^{2k}} (s-1)^{1-z} q^{2k(1-z)} q^{n(k+1)-1} = \mathfrak{b}(z)(s-1)^{-z} q^{-2kz} q^{n(k+1)-1},$$

$$\mathfrak{b}(z) = (z-1) \prod_{j=1}^{2k} \bigl(j - 2kz + n(k+1) - 1\bigr).$$

More generally, with the differential operator:

$$\mathrm{D} = \frac{\partial^{2k+1}}{\partial s \partial q^{2k}},$$

after l-iterations one obtains

$$\mathrm{D}^l (s-1)^{l-z} q^{2k(l-z)} q^{n(k+1)-1} = \mathfrak{B}_l(z)(s-1)^{-z} q^{-2kz} q^{n(k+1)-1},$$

$$\mathfrak{B}_l(z) = \prod_{i=0}^{l-1} \mathfrak{b}(z-i), \quad l \in \mathbb{N}^*.$$

A priori, this shows that there is poles of order 2 at positive integers and simple poles at rational points:

$$z_{p,j,n,k} = p + \frac{j + n(k+1) - 1}{2k} \notin \mathbb{N}, \quad p \in \mathbb{N},\ j \in [1, \ldots, 2k]. \tag{54}$$

For example, the analytic extension in the half-plane $\Re(z) < l$, $l \in \mathbb{N}^*$, of the complex function appearing in the first integral is explicitly given by

$$\frac{(-1)^l}{\mathfrak{B}_l(z)} M_+(z) \lambda^{-z} \int_{s=1}^{\infty} \int_{q=0}^{\infty} q^{2k(l-z)+n(k+1)-1} \mathrm{D}^l (1+s)^{-z} b\bigl(sq^k, q\bigr) s^{n-1}\, ds\, dq. \tag{55}$$

A similar relation for the term involving $M_-(z)$ is

$$\mathrm{D}|1-s|^{1-z} q^{n(k+1)-1-2k(1-z)} = -\mathfrak{b}(z)|1-s|^{-z} q^{n(k+1)-1-2kz}.$$

This change of sign, due to the modulus, will be important below because of some symmetries. After l-iterations we have

$$\frac{1}{\mathfrak{B}_l(z)} M_-(z) \lambda^{-z} \int_{s=0}^{1} \int_{q=0}^{\infty} q^{2k(l-z)+n(k+1)-1} \mathrm{D}^l (1+s)^{-z} b\bigl(sq^k, q\bigr) s^{n-1}\, ds\, dq. \tag{56}$$

Remark 20. The smallest double root of $\mathfrak{b}$, and a fortiori of each $\mathfrak{B}_l$, is greater than $z_{\min} = \frac{n(k+1)}{2k}$. In fact we will see below that there is no poles, and a fortiori no contributions, before this value. This insures that all integrals involved in the asymptotic expansion are absolutely convergent.

A carefully examination of the integral w.r.t q shows that all coefficients are zero until we reach the pole:

$$z_{\min} = n\frac{k+1}{2k} = \frac{n}{2} + \frac{n}{2k}. \tag{57}$$

This is justified by the fact that if $\alpha > 0$, $\beta \in \mathbb{N}^*$ and $\alpha + 1 > \beta$ we obtain

$$\int_0^\infty \partial_x^\beta \big(x^\alpha f(x)\big)\, dx = \big[\partial_x^{\beta-1}\big(x^\alpha f(x)\big)\big]_{x=0}^\infty = 0, \quad \forall f \in C_0^\infty(\mathbb{R}).$$

Since poles located at integers are of order 2 the attached residuum are computed via the elementary formula:

$$\lim_{z\to p} \frac{\partial}{\partial z}\big((z-p)^2 \mathfrak{g}_\pm(z)\lambda^{-z}\big), \quad p \in \mathbb{N}, \tag{58}$$

where $\mathfrak{g}_\pm$ are defined by Eqs. (52), (53). For h holomorphic and $\lambda > 0$ we have

$$\frac{\partial}{\partial z}\big(h(z)\lambda^{-z}\big) = \frac{\partial h}{\partial z}(z)\lambda^{-z} - \log(\lambda)\lambda^{-z}h(z), \tag{59}$$

and we can apply this to $h(z) = (z-p)^2\mathfrak{g}_\pm(z)$ near $z = p$. Hence, a generic double pole located at $z = p$ leads to a contribution:

$$c_p \log(\lambda)\lambda^{-p} M_+(z) \int_{s=1}^\infty \int_{q=0}^\infty q^{n(k+1)-1} \mathrm{D}^p (s+1)^{-p} b\big(sq^k, q\big) s^{n-1}\, ds\, dq. \tag{60}$$

Since $M_+(p)$ is well defined and

$$c_p = (-1)^p \lim_{z\to p} \frac{(z-p)^2}{\mathfrak{B}_p(z)} \in \mathbb{Q}, \tag{61}$$

our coefficients can be explicitly determined by the computation of the integrals. Also, cf. Remark 20, the c_p are zero until $p \geqslant z_{\min}$. By integrations by parts and up to a factorial number, the inner integral of Eq. (60) is

$$
\begin{aligned}
&-\int\limits_{s=1}^{\infty} \frac{\partial^p}{\partial s^p}(s+1)^{-p} s^{n-1}\left(\frac{\partial^{2kp-1}}{\partial q^{2kp-1}} b\big(sq^k,q\big) q^{n(k+1)-1}\right)\bigg|_{q=0} ds \\
&\quad = \left(\frac{\partial^{p-1}}{\partial s^{p-1}}(s+1)^{-p} s^{n-1}\left(\frac{\partial^{2kp-1}}{\partial q^{2kp-1}} b\big(sq^k,q\big) q^{n(k+1)-1}\right)\bigg|_{q=0}\right)\bigg|_{s=1}.
\end{aligned}
$$

But for all g smooth bounded, with bounded derivatives, and p large enough:

$$
\int\limits_{0}^{\infty} \frac{\partial^p}{\partial s^p}\big((1+s)^{-p} s^{n-1} g(s)\big)\, ds = 0. \tag{62}
$$

In particular, this is the case for all our coefficients since we have

$$
p \geqslant \frac{n}{2} + \frac{n(k+1)}{2k} > n.
$$

This trick shows that coefficients obtained by integration w.r.t. s on $[0, 1]$ and $[1, \infty]$ can be identified up to a sign and we can save some computations.

Computation of the leading term

Contrary to the case of nondegenerate critical points, the evaluation of the leading term is somehow technical and some computations will be left to the reader. According to the analysis above, we distinguish out the case where the first nonzero residue is attached to a simple or a double pole.

Case of $z_{\min}$ simple pole

Taking Remark 20 into account the first coefficient is given by

$$
\begin{aligned}
&\lim_{z\to z_{\min}} (-1)^l \lambda^{-z} \frac{(z-z_{\min})}{\mathfrak{B}_l(z)} M_+(z) \\
&\quad \times \int\limits_{s=1}^{\infty}\int\limits_{q=0}^{\infty} (s-1)^{l-z} q^{2k(l-z)+n(k+1)-1} D^l (s+1)^{-z} b\big(sq^k,q\big) s^{n-1}\, ds\, dq,
\end{aligned}
$$

with $l \in \mathbb{N}^*$ such that $l > z_{\min}$ (any such l is acceptable). With this choice we can take the limit under the integral to obtain:

$$
c_{l,k,n} \lambda^{-n\frac{k+1}{2k}} \int\limits_{s=1}^{\infty}\int\limits_{q=0}^{\infty} (s-1)^{l-n\frac{k+1}{2k}} q^{2kl-1} D^l (s+1)^{-n\frac{k+1}{2k}} b\big(sq^k,q\big) s^{n-1}\, ds\, dq, \tag{63}
$$

$$
c_{l,k,n} = (-1)^l M_+\left(n\frac{k+1}{2k}\right) \lim_{z\to z_{\min}} \frac{(z-z_{\min})}{\mathfrak{B}_l(z)}. \tag{64}
$$

By integrations by parts, the integral w.r.t. q of Eq. (63) is given by

$$\int_0^\infty q^{2kl-1} \frac{\partial^{2kl}}{\partial q^{2kl}} b\big(sq^k, q\big)\, dq = (2kl-1)!\, b(0,0).$$

Hence we obtain the asymptotic relation for the positive part of our integral:

$$I_+(\lambda) = c_1 b(0,0)\lambda^{-n\frac{k+1}{2k}} \int_1^\infty (s-1)^{l-n\frac{k+1}{2k}} \frac{\partial^l}{\partial s^l}\Big((1+s)^{-n\frac{k+1}{2k}} s^{n-1}\Big)\, ds + R_1(\lambda),$$
$$c_1 = (2kl-1)!\, c_{l,k,n}.$$

As concerns the convergence of the integral w.r.t. s, by construction the singularity in $s = 1$ is controlled. For the behavior at infinity we remark that the degree w.r.t. s is

$$l - n\frac{k+1}{2k} - n\frac{k+1}{2k} + n - 1 - l = -n\frac{k+1}{k} + n - 1 < -1, \tag{65}$$

so that the integral is absolutely convergent. To compute explicitly the values of our integrals we can choose $l = n$ since $n - z_{\min} > 0$. First, by induction on n and for $\frac{n}{2} < \Re(\alpha) < n+1$ we obtain that

$$E(n,\alpha) = \int_{s=1}^\infty (s-1)^{n-\alpha} \frac{\partial^n}{\partial s^n}\Big((1+s)^{-\alpha} s^{n-1}\Big)\, ds = 0, \quad \text{if } n \text{ is even}.$$

Next, for n odd, and always with $\frac{n}{2} < \Re(\alpha) < n+1$, we have

$$E(n,\alpha) = \prod_{j=1}^{\frac{n-1}{2}} (-2j-1) 2^{\frac{n+1}{2}-2\alpha} \frac{\Gamma(n+1-\alpha)\Gamma(-n+2\alpha)}{\Gamma\big(\frac{1-n}{2}+\alpha\big)}.$$

Hence for $\alpha = n(k+1)/2k$ and n odd, we obtain

$$\int_1^\infty (s-1)^{n-n\frac{k+1}{2k}} \frac{\partial^n}{\partial s^n}\Big((1+s)^{-n\frac{k+1}{2k}} s^{n-1}\Big)\, ds = c_{n,k} \frac{\Gamma\big(1+n\frac{k-1}{2k}\big)\Gamma\big(\frac{n}{k}\big)}{\Gamma\big(\frac{k+n}{2k}\big)}, \tag{66}$$

$$c_{n,k} = \prod_{j=1}^{\frac{n-1}{2}} (-2j-1) 2^{\frac{n+1}{2}-n\frac{k+1}{k}}. \tag{67}$$

For any integer $l > n(k+1)/2k$, similar computations show that

$$I_-(\lambda) = c_2 b(0,0)\lambda^{-n\frac{k+1}{2k}} \int_0^1 |s-1|^{l-n\frac{k+1}{2k}} \frac{\partial^l}{\partial s^l}\big((1+s)^{-n\frac{k+1}{2k}} s^{n-1}\big)\, ds + R_2(\lambda),$$

$$c_2 = (2kl-1)!\, c_p M_-\left(\frac{n(k+1)}{2k}\right).$$

Once more the choice of $l = n$ is admissible and we define:

$$a_{n,k} = \int_0^1 |s-1|^{n-n\frac{k+1}{2k}} \frac{\partial^n}{\partial s^n}\big((1+s)^{-n\frac{k+1}{2k}} s^{n-1}\big)\, ds,$$

since these integrals seem to have no formulation by mean of elementary functions, unless by mean of hypergeometric functions. These numbers $a_{n,k}$ are finite and nonzero in general position.

From the analysis of the poles above we know that the remainders R_1 and R_2 are of order $\mathcal{O}(\lambda^{-\frac{n(k+1)+1}{2k}})$ if the next pole is not an integer, respectively $\mathcal{O}(\log(\lambda)\lambda^{-\frac{n(k+1)+1}{2k}})$ if this is an integer (cf. Remark 17). By summation we obtain the leading term of the asymptotic expansion with a precise remainder.

Case of $z_{\min}$ pole of order 2

Starting from Eq. (59) we see that the associated coefficients are given by

$$-\log(\lambda)\lambda^{-z_{\min}}\big((z-z_{\min})^2 \mathfrak{g}_\pm(z) M_\pm(z)\big)\big|_{z=z_{\min}}. \tag{68}$$

But a great part of this limit was precisely computed above. Since $z_{\min}$ is by assumption an integer we will obtain some particular values. By induction on $p > 1$, and assuming recursively that $n < 2p$, we have

$$\int_1^\infty \partial_s^p\big((s+1)^{-p} s^{n-1}\big)\, ds = -\frac{1}{2^p}\prod_{j=0}^{p-1}(n-2j). \tag{69}$$

If n is odd this coefficient is not zero and we get the result. But if n is even we obtain that the associated contribution vanishes and there is no logarithm in the leading term. To obtain the top order coefficient we must compute the coefficient obtained by derivation of our meromorphic distributions.

Starting from Eqs. (55), (56), by Leibnitz rule, we have 3 possibilities: derivation of the rational function, of the Melin transform or of the analytic integrals in (q,s). Since the integral w.r.t. s vanishes in $z = z_{\min}$, the 2 first terms do not contribute. Similarly, the derivative of q^{-2kz} can be discarded. Hence, the only contribution comes from derivation of the distribution w.r.t. s and we have to use the modified constants:

$$\tilde{a}^+_{p,k} = -\int_1^\infty \log\bigl(s^2-1\bigr)\partial_s^p\bigl((s+1)^{-p}s^{n-1}\bigr)\,ds, \tag{70}$$

$$\tilde{a}^-_{p,k} = -\int_0^1 \log\bigl(s^2+1\bigr)\partial_s^p\bigl((s+1)^{-p}s^{n-1}\bigr)\,ds, \tag{71}$$

respectively for $I_+(\lambda)$ and $I_-(\lambda)$. By similar considerations as above, these integrals are absolutely convergent for any $p \geqslant z_{\min}$.

The other coefficients

To obtain a complete overview of the asymptotic expansion we show also how to compute the coefficients attached to logarithmic distributions. For $p \in \mathbb{N}^*$, $p \geqslant z_{\min}$, by Leibnitz rule, these derivatives are equal to

$$\partial_z M_\pm(p)\lim_{z\to p}(z-p)^2\mathfrak{g}_\pm(z) + M_\pm(p)\lim_{z\to p}\bigl(\partial_z(z-p)^2\mathfrak{g}_\pm(z)\bigr).$$

Hence the asymptotic expansion also involves the distributions:

$$\int_0^\infty \log(t)t^{p-1}\hat{a}(\pm t)\,dt, \qquad p \in \mathbb{N}^*,$$

and, by derivation of the meromorphic distributions, terms:

$$\int_{s=1}^\infty\int_{q=0}^\infty \log(q)q^\alpha D^p(1+s)^{-p}b\bigl(sq^k,q\bigr)s^{n-1}\,ds\,dq,$$

$$\int_{s=1}^\infty\int_{q=0}^\infty \log\bigl(s^2-1\bigr)q^\alpha D^p(1+s)^{-p}b\bigl(sq^k,q\bigr)s^{n-1}\,ds\,dq,$$

where the parameter α runs in the sequence of positive rational numbers $l/2k$. There is also similar terms with integration w.r.t. $s \in [0,1]$. Note that the singular support w.r.t. t is located in $t=0$ for all coefficients. Also, from the analysis above, we know that the logarithmic coefficients only occur when $p = l(k+1)/2k$ are integers.

Invariant formulation of the main coefficients

To complete the proof it remains to express our coefficients in terms of φ and V, the natural data of the problem. We apply Lemma 14 to our amplitude:

$$A(\chi_0,\chi_1,\chi_2) = \chi_1^{n-1}\chi_2^{n-1}\tilde{A}(\chi_0,\chi_1,\chi_2),$$

to prove the existence of the total asymptotic expansion. As concerns the leading term, in any of the cases at hand we have to evaluate $\widetilde{A}(\chi_0,0,0)$. By standard manipulations, already used in [5,6], we can inverse our diffeomorphism via an oscillatory representation of the delta-Dirac distribution by mean of a Schwartz kernel:

$$K(\delta_{\{\chi_1,\chi_2\}}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-i\chi_1 z_1} e^{-i\chi_2 z_2}\, dz_1\, dz_2.$$

There is no technical problem here since the amplitude is compactly supported. After integration w.r.t. $d\theta$, we accordingly obtain that

$$\tilde{A}(t,0,0) = a(t,z_0)\mathrm{S}\big(\mathbb{S}^{n-1}\big)\int_{\mathbb{S}^{n-1}} \big|V_{2k}(\eta)\big|^{-\frac{n}{2k}}\, d\eta. \tag{72}$$

We recall that these integrals over the spheres are simply given by the Jacobian of our coordinates on the blow-up of the critical point (cf. Remark 13). The principal symbol of $\Theta(P_h)$ is $\Theta(p)$ with $\Theta(p(z_0)) = 1$. Hence, at the critical point, the term homogeneous of degree 0 w.r.t. h of the amplitude of our FIO is given by $a(t,z_0) = \hat{\varphi}(t)$ (cf. Section 3). Substituting Eq. (72) in all integral formulas for the leading terms of the asymptotic expansion the Fourier inversion formula yields:

$$M_+\big(\hat{a}(t,z_0)\big)\left(n\frac{k+1}{2k}\right) = \int_0^\infty \varphi(t)t^{n\frac{k+1}{2k}-1}\, dt,$$

$$M_-\big(\hat{a}(t,z_0)\big)\left(n\frac{k+1}{2k}\right) = \int_0^\infty \varphi(-t)t^{n\frac{k+1}{2k}-1}\, dt.$$

Setting $\lambda = h^{-1}$, so that $\log(h) = -\log(\lambda)$, dividing by $(2\pi h)^n$ we obtain, via Lemma 16, the results stated in Theorem 4. □

Extension

We show here shortly how to extend the result of Theorem 4 to the case of an h-admissible operator P_h of symbol $p_h \sim \sum h^j p_j$ whose principal symbol is $p_0 = \xi^2 + V(x)$ with a nonvanishing subprincipal symbol p_1. In this case the Fourier integral operator approximating the propagator has the amplitude:

$$\tilde{a}(t,z) = a(t,z)\exp\left(i\int_0^t p_1\big(\Phi_s(z)\big)\, ds\right),$$

see Duistermaat [9] concerning the solution of the first transport equation. Since z_0 is an equilibrium we have simply $p_1(\Phi_s(z_0)) = p_1(z_0)$ and hence

$$\tilde{a}(t, z_0) = \hat{\varphi}(t)e^{itp_1(z_0)}. \tag{73}$$

If the subprincipal symbol vanishes at the critical point, which is the case in a lot of practical situations, the top order coefficient of the trace formula remains the same. Finally, when $p_1(z_0) \neq 0$ by Fourier inversion formula we replace $\varphi(t)$ by $\varphi(t + p_1(z_0))$ in all integral formulae of Theorem 4.

Acknowledgment

The author thanks Daniel Barlet for helpful comments on degenerate oscillatory integrals.

References

[1] R. Balian, C. Bloch, Solution of the Schrödinger equation in term of classical path, Ann. Physics 85 (1974) 514–545.
[2] R. Brummelhuis, T. Paul, A. Uribe, Spectral estimate near a critical level, Duke Math. J. 78 (3) (1995) 477–530.
[3] R. Brummelhuis, A. Uribe, A semi-classical trace formula for Schrödinger operators, Comm. Math. Phys. 136 (3) (1991) 567–584.
[4] B. Camus, A semi-classical trace formula at a nondegenerate critical level, J. Funct. Anal. 208 (2) (2004) 446–481.
[5] B. Camus, A semi-classical trace formula at a totally degenerate critical level. Contributions of extremums, Comm. Math. Phys. 207 (2) (2004) 513–526.
[6] B. Camus, Contributions of non-extremum critical points to the semi-classical trace formula, J. Funct. Anal. 217 (1) (2004) 79–102.
[7] B. Camus, Semi-classical spectral estimates for Schrödinger operators at a critical level. Case of a degenerate minimum of the potential, in preparation.
[8] M. Combescure, J. Ralston, D. Robert, A proof of the Gutzwiller semi-classical trace formula using coherent states decomposition, Comm. Math. Phys. 202 (2) (1999) 463–480.
[9] J.J. Duistermaat, Oscillatory integrals Lagrange immersions and unfolding of singularities, Comm. Pure Appl. Math. 27 (1974) 207–281.
[10] J.J. Duistermaat, V. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, Invent. Math. 29 (1975) 39–79.
[11] J.J. Duistermaat, L. Hörmander, Fourier integral operators, Acta Math. 128 (3–4) (1972) 183–269.
[12] M. Gutzwiller, Periodic orbits and classical quantization conditions, J. Math. Phys. 12 (1971) 343–358.
[13] D. Khuat-Duy, Semi-classical trace formula at a critical level, J. Funct. Anal. 146 (2) (1997) 299–351.
[14] T. Paul, A. Uribe, Sur la formule semi-classique des traces, C. R. Séances Acad. Sci. Sér. I 313 (5) (1991) 217–222.
[15] V. Petkov, G. Popov, Semi-classical trace formula and clustering of the eigenvalues for Schrödinger operators, Ann. Inst. H. Poincaré Phys. Théor. 68 (1) (1998) 17–83.
[16] D. Robert, Autour de l'approximation semi-classique, in: Progr. Math., vol. 68, Birkhäuser Boston, Boston, MA, 1987.
[17] J.A. Yorke, Periods of periodic solutions and the Lipschitz constant, Proc. Amer. Math. Soc. 69 (1969) 509–512.