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The Use of Formal Power Series to Solve Finite Convolution Integral Equations

SANDY GRABINER

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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1. INTRODUCTION

Suppose d is a positive number and F and G are complex valued functions in $L^1[0, d]$, the space of Lebesgue summable functions on the interval $[0, d]$. By the *finite convolution* of F and G, we mean the "function" defined almost everywhere by:

$$F * G(x) = \int_0^x F(x - t)\, G(t)\, dt.$$

Under finite convolution, $L^1[0, d]$ becomes a commutative algebra ([4], p. 17f), ([7], pp. 8–10, p. 355).

Let $w(x)$ be the function identically equal to one on $[0, d]$. In this paper we will develop an operational calculus using formal power series $f = \sum_{n=1}^{\infty} a_n z^n$ for which the series $f(w) = \sum_{n=1}^{\infty} a_n w^n$ converges in the L^1, and uniform norms to a summable function which we denote by $f(w)$ (however, most of the formal power series will have *zero* radius of convergence). We will also see that this operational calculus easily gives formulas for the solutions, $X(x)$ of integral equations like:

$$F * X + Y = X, \tag{1}$$

$$F * X = Y, \tag{2}$$

$$X^n = X * X \cdots * X = F, \tag{3}$$

where $F(x)$ is the sum of a power series with radius of convergence greater than d, and $Y(x)$ is in $L^1[0, d]$. In general, these formulas will give sequential approximations to the solutions.

If $F(x)$ were an entire function of exponential type, our operational calculus would reduce to the Riesz operational calculus for the operator of convolution by $w(x)$. In this simple case, our solutions of the above integral equations can be obtained by the Riesz calculus ([9], p. 291), the Mikusiński calculus [1], ([3], pp. 39–41), ([7], p. 172f), or through the use of Laplace transforms ([2], Vol. I, pp. 356-358; Vol. III, pp. 133–136, 151, and 170).

The general problem of developing an operational calculus based on formal power series for an arbitrary quasi-nilpotent operator on a Banach space is considered in [5], and will be discussed elsewhere. The problem is greatly simplified for the operation of convolution by w, which we discuss in this paper, by the fact that

$$w * F = \int_0^x F(t)\, dt.$$

2. The Operational Calculus

We begin by relating power series in x to power series in w.

Lemma 1. *If $F(x)$ is the sum of a power series in x with radius of convergence greater than d, then there is a unique formal power series f for which $F = f(w)$, where $f(w)$ converges in the uniform and L^1 norms on the interval* $[0, d]$.

Proof. Let $F(x) = \sum_{n=0}^{\infty} b_n x^n$. This series converges in the uniform and L^1 norms on $[0, d]$. Moreover, for each positive integer n, $w^n = x^{n-1}/(n-1)!$. Hence, we may conclude:

$$F(x) = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} b_{n-1}(n-1)!\, w^n = f(w). \tag{4}$$

The formal power series f, of the above lemma, has a positive radius of convergence if, and only if, $F(x)$ is an entire function of exponential type. Moreover, in this case $f(1/s)$ is the Laplace transform of $F(x)$ ([2], Vol. I, pp. 356–359). As we pointed out above, for these $F(x)$ the problems considered in this paper could be solved by Laplace transforms, by the Riesz operational calculus, or by the Mikusiński operational calculus. For the more general $F(x)$ we consider, $f(1/s)$ is still the asymptotic development of the Laplace transform of any extension of $F(x)$ to a function in $L^1[0, \infty)$ ([2], Vol. II, p. 47). This fact can be used to give another version of the operational calculus which we describe in this paper ([5], pp. 134ff).

We let A be the set of all those formal power series f, with zero constant term, for which the series $f(w)$ converges to a function $F = f(w)$ which is analytic on the closed d disk. We will follow standard practice and denote by $A^\#$ and $L^1[0, d]^\#$ the algebras formed from A and $L^1[0, d]$ by adjoining an identity.

In Theorem 2 we discuss the nature of A and the map $f \to f(w)$. Before doing this let us see what properties we will require to solve Eqs. (1), (2), and (3). As a minimum, we will certainly need A to be an algebra and the map $f \to f(w)$ to be a homomorphism. By formula (4) the map would then be an isomorphism.

Equation (1) can be rewritten

$$(1 - f)(w) * X = Y.$$

To solve this, we will require that $(1 - f)^{-1}$ exist in $A^{\#}$, whenever f belongs to A; in other words, we require that A be a *radical* algebra. Then $X = (1 - f)^{-1}(w) * Y$, and $(1 - f)^{-1}$ can be calculated by the method of undetermined coefficients or the formula $\sum_{n=0}^{\infty} f^n$.

For Eq. (2), suppose $F = f(w)$ has a zero of order $k - 1 \geqslant 0$ at the origin. Then from formula (4) we see that there is a complex number C and a formal power series g such that:

$$f = Cz^k(1 + g).$$

Thus, we will need to know that g belongs to A whenever gz belongs to A, and, hence, whenever gz^k belongs to A. Moreover, since w is not a divisor of zero in $L^1[0, d]$, no solution exists unless $Y = w^k * Y^{(k)}$, where $Y^{(k)}$ is the k-th derivative of Y, almost everywhere on $[0, d]$.

Equation (3) is similar. Our method of solution will require that F have a zero of order $kn - 1$ for some positive integer k; and that the n-th roots of $1 + g$ belong to $A^{\#}$, whenever g belongs to A. These n-th roots can be calculated from the binomial theorem applied to $1 + g$.

We now state and prove a theorem showing that the set A and the map $f \to f(w)$ have the desired properties.

THEOREM 2. *If A and w are defined as above, and if g is a formal power series without constant term, then*:

(a) *A is a radical algebra;*

(b) *if $gz \in A$, then $g \in A$;*

(c) *if $g \in A$, then the n-th roots of $1 + g$ belong to $A^{\#}$;*

(d) *the map $f \to f(w)$ is an algebra isomorphism from A into $L^1[0, d]$.*

Proof. We first prove (b). Let $f = gz$ and let $F(x) = f(w)$. Applying formula (1) to $F'(x)$ instead of $F(x)$, shows $F'(x) = g(w)$ and, in particular, $g \in A$.

Now let K be the set of all formal power series $f = \sum_{n=1}^{\infty} a_n z^n$ for which

$\bar{f}(w)$ converges absolutely in the $L^1[0, d]$ norm. Since the $L^1[0, d]$ norm of w^n is $d^n/n!$, f belongs to K if, and only if,

$$\|f\|' = \sum_{n=1}^{\infty} |a_n| \frac{d^n}{n!} < \infty. \tag{5}$$

Moreover, since $\lim_n (d^n/n!)^{1/n} = \lim_n (d/n) = 0$, K becomes a radical Banach algebra under the norm $\|f\|'$ ([4], p. 120), ([8], p. 317). Since K is a radical Banach algebra, the Riesz operational calculus allows us to calculate the n-th roots of $1 + g$ in $K^{\#}$ from the binomial formula, whenever g belongs to K.

It is clear that $A \subseteq K$ and that the map $f \to \bar{f}(w)$ is an isomorphism from K into $L^1[0, d]$. We complete the proof by showing that A is the union of a nested family of algebras each of which is isomorphic to K.

For each $t > 1$, let A_t be the set of all formal power series f, with zero constant term, for which $f(tz)$ belongs to K. The map $f(z) \to f(tz)$ is an isomorphism of the algebra of all formal power series, so its restriction to A_t is certainly an isomorphism from A_t onto K. From formula (5), it is clear that $A_s \subseteq A_t$, if $s > t$. Finally, a direct calculation using formula (4) shows $fz \in A$, and, hence, $f \in A$ if, and only if, $f \in A_t$, for some $t > 1$. This completes the proof.

Little extra effort would be needed to strengthen part (c), to show that any algebraic function analytic at the origin may be applied to g. Hence, more complicated nonlinear integral equations than (3) could be solved by the formal power series operational calculus. The statement and proof of Lemma 1 and Theorem 2 would be essentially unchanged by replacing $L^1[0, d]$ with $L^1[-d, d]$. In fact, we could consider the set of functions analytic in the open d disk and continuous in the closed d disk. This set becomes a radical Banach algebra under the uniform norm and finite convolution ([6], p. 70), ([8], p. 316). Notice also that formula (4) does not involve d. Thus, if $0 < b \leqslant \infty$ and if F is analytic on the open b disk, we may carry out calculations simultaneously in all $L^1[0, d]$ with $0 < d < b$.

We have used formula (4) and Theorem 2 to obtain information about convolutions from information about formal power series. The reverse is also possible. For instance, the fact that

$$\frac{1}{1-x} * \frac{1}{1+x} = \frac{-\ln(1-x^2)}{x}$$

yields

$$\left(\sum_{n=0}^{\infty} n! z^n\right)\left[\sum_{n=0}^{\infty} (-1)^n n! z^n\right] = \sum_{n=0}^{\infty} \frac{(2n+1)!\, z^{2n}}{n+1}.$$

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