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Arithmetic Properties of the Bell Polynomials*

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Dedicated to H. S. Vandiver on his eighty-third birthday

1. INTRODUCTION

Let $x, \alpha_1, \alpha_2, \alpha_3, \cdots$ denote indeterminates. The general Bell polynomial [3, Ch. 2]

$$\phi_n(x) = \phi_n(x; \alpha_1, \alpha_2, \alpha_3, \cdots) = Y_n(\alpha_1 x, \alpha_2 x, \alpha_3 x, \cdots) \tag{1.1}$$

may be defined by $\phi_0(x) = 1$ and

$$\phi_n(x) = \sum A_n(k_1, k_2, k_3, \cdots)\, \alpha_1^{k_1}\alpha_2^{k_2}\alpha_3^{k_3} \cdots x^k, \tag{1.2}$$

where $k = k_1 + k_2 + k_3 + \cdots$,

$$A_n(k_1, k_2, k_3, \cdots) = \frac{n!}{k_1!(1!)^{k_1}\, k_2!(2!)^{k_2}\, k_3!(3!)^{k_3} \cdots},$$

and the summation in the right member of (1.2) is over all nonnegative integers $k_1, k_2, k_3, \cdots$ such that

$$k_1 + 2k_2 + 3k_3 + \cdots = n. \tag{1.3}$$

Note in particular that (1.1) implies

$$Y_n(\alpha_1, \alpha_2, \alpha_3, \cdots) = \phi_n(1; \alpha_1, \alpha_2, \alpha_3, \cdots). \tag{1.4}$$

The coefficients $A_n(k_1, k_2, k_3, \cdots)$ are evidently positive integers and it is clear from (1.3) that, for fixed n, the number of $A_n(k_1, k_2, k_3, \cdots)$ is equal to $P(n)$, the number of unrestricted partitions of n. In [1], [2] the writer considered the following problem. Let p be a fixed prime and let $\theta(n)$ denote the number of coefficients $A_n(k_1, k_2, k_3, \cdots)$ that are prime to p so that

$$\theta(n) = P(n) \qquad (n < p).$$

The writer proved that

$$\theta(p^{r_1} + p^{r_2} + \cdots + p^{r_m}) = \sum_{n=0}^{m} \sigma_{m,m-n} B_n, \tag{1.5}$$

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where $\sigma_{m,n}$ is the nth elementary symmetric function of the distinct integer $r_1, r_2, \cdots, r_m$ and B_n is the Bell number defined by $B_0 = 1$ and

$$B_{n+1} = \sum_{s=0}^{n} \binom{n}{s} B_s .$$

In the general case it was proved that

$$\theta(a_1 p^{r_1} + a_2 p^{r_2} + \cdots + a_m p^{r_m})$$

$$= \sum_{j_1=0}^{a_1} \cdots \sum_{j_m=0}^{a_m} \binom{r_1 + j_1 - 1}{j_1} \cdots \binom{r_m + j_m - 1}{j_m} P(a_1 - j_1, \cdots, a_m - j_m) \quad (1.6)$$

where $r_1, r_2, \cdots, r_m$ are distinct integers,

$$0 \leqslant a_j < p \qquad (j = 1, \cdots, m),$$

and $P(a_1, a_2, \cdots, a_m)$ denotes the number of partitions of the "m-partite" number $(a_1, a_2, \cdots, a_m)$.

In the present paper we treat the following problem. For fixed n, k let $\theta(n, k)$ denote the number of coefficients $A_n(k_1, k_2, k_3, \cdots)$ with

$$k_1 + 2k_2 + 3k_3 + \cdots = n, \qquad k_1 + k_2 + k_3 + \cdots = k \qquad (1.7)$$

that are prime to p. It follows from (1.7) that

$$\theta(n, k) = P_k(n) \qquad (n < p),$$

where $P_k(n)$ denotes the number of partitions of n into k parts. We shall prove the following results. In the first place

$$\sum_{k=0}^{\infty} \theta(p^{r_1} + \cdots + p^{r_m}, k)\, x^k = \sum_{j=0}^{m} \sigma_{m,m-j}(x)\, B_j(x), \qquad (1.8)$$

where $r_1, \cdots, r_m$ are distinct integers $\sigma_{m,j}(x)$ denotes the jth elementary symmetric function of $u_1(x), \cdots, u_m(x)$,

$$u_j(x) = x^p + x^{p^2} + x^{p^2} + \cdots + x^{p^j}$$

and $B_n(x)$ is the single-variable Bell polynomial defined by $B_0(x) = 1$ and

$$B_{n+1}(x) = x \sum_{j=0}^{n} \binom{n}{j} B_j(x).$$

When $x = 1$, it is evident that (1.8) reduces to (1.5).

In the general case we show that

$$\sum_{k=0}^{\infty} \theta(a_1^{r_1} + \cdots + a_m p^{r_m}, k)$$

$$= \sum_{j_1=0}^{a_1} \cdots \sum_{j_m=0}^{a_m} D_{r_1}(j_1, x) \cdots D_{r_m}(j_m, x)\, P(a_1 - j_1, \cdots, a_m - j_m; x), \tag{1.9}$$

where $r_1, \cdots, r_m$ are distinct integers, $0 < a_s < p$ $(1 \leqslant s \leqslant m)$,

$$P(a_1, \cdots, a_m; x) = \sum_{k=0}^{\infty} P_k(a_1, \cdots, a_m)\, x^m,$$

$P_k(a_1, \cdots, a_m)$ is the number of partitions of $(a_1, \cdots, a_m)$ into exactly k parts and

$$D_r(a, x) = (x^p + x^p + \cdots + x^{p^r})^{(a)},$$

where in the expansion of the right side the multinomial coefficients are deleted.

It is easily verified that (1.9) contains (1.6).

2. Preliminaries

Let p be a fixed prime. It is familiar that the binomial coefficient $\binom{b}{a}$ is prime to p if and only if the following conditions are satisfied.

$$a = a_0 + a_1 p + a_2 p^2 + \cdots \qquad (0 \leqslant a_j < p)$$

$$b = b_0 + b_1 p + b_2 p^2 + \cdots \qquad (0 \leqslant b_j < p)$$

and

$$b_j \leqslant a_j \qquad (j = 0, 1, 2, \cdots). \tag{2.1}$$

By an *arithmetic function* we shall understand a mapping from the non-negative integers into the reals. If f, g are two arithmetic functions we define the *Lucas product* $h = f * g$ by means of

$$h(n) = \sum_{r=0}^{n}{}^{*} f(r)\, g(n-r) \qquad (n = 0, 1, 2, \cdots), \tag{2.2}$$

where the asterisk indicates that the summation is restricted to r such that

$\binom{n}{r}$ is prime to p. The Lucas product is associative and commutative. The function u defined by

$$u(n) = \delta_{n0} \tag{2.3}$$

satisfies $f*u = f$ for all f. For given f, a function g exists satisfying

$$f * g = u \tag{2.4}$$

if and only if $f(0) \neq 0$. In particular for the function I defined by

$$I(n) = 1 \qquad (n = 0, 1, 2, \cdots)$$

we have $I * \mu = u$, where μ is defined by

$$\mu(a_0 + a_1 p + a_2 p^2 + \cdots) = \mu(a_0)\,\mu(a_1 p)\,\mu(a_2 p^2) \cdots (0 \leqslant a_j < p) \tag{2.5}$$

and

$$\mu(ap^j) = \begin{cases} 1 & (a = 0) \\ -1 & (a = 1) \\ 0 & (1 < a < p). \end{cases}$$

As an application we have

$$g(n) = \sum_{r=0}^{n}{}^{*} f(r)$$

if and only if

$$f(n) = \sum_{r=0}^{n}{}^{*} \mu(r)\, g(n - r).$$

We define the function

$$d_r = I^r = I * \cdots * I \qquad (r = 1, 2, 3, \cdots). \tag{2.6}$$

In particular we put $d = d_2 = I * I$ so that

$$d(n) = \sum_{r=0}^{n}{}^{*} 1. \tag{2.7}$$

If

$$n = a_0 + a_1 p + a_2 p^2 + \cdots \qquad (0 \leqslant a_j < p)$$

then we have

$$d_k(n) = \prod_{j=1}^{k} \binom{a_j + k - 1}{k - 1} \qquad (k = 1, 2, 3, \cdots). \tag{2.8}$$

It is easily verified that $d_k(n)$ is equal to the number of k-nomial coefficients

$$\frac{n!}{n_1! \cdots n_k!} \qquad (n_1 + \cdots + n_k = n)$$

that are prime to p.

3. Some Properties of $\phi_n(x)$

If we put

$$A = A(t) = \sum_{n=1}^{\infty} \alpha_n \frac{t^n}{n!} \tag{3.1}$$

then

$$e^{Ax} = \sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!} \tag{3.2}$$

and

$$D^n e^{Ax} = Y_n(A'x, A''x, A'''x, \cdots)\, e^{Ax}, \tag{3.3}$$

where

$$D = \frac{d}{dt}, \qquad A' = \frac{dA}{dt}, \qquad A'' = \frac{d^2A}{dt^2}, \qquad \cdots,$$

and, as in the Introduction,

$$Y_n(\alpha_1, \alpha_2, \alpha_3, \cdots) = \phi_n(1; \alpha_1, \alpha_2, \alpha_3, \cdots).$$

In particular, when $n = p^r$, (3.3) becomes

$$D^{p^r} e^{Ax} = Y_{p^r}(A'x, A''x, A'''x, \cdots)\, e^{Ax}. \tag{3.4}$$

Now

$$D^p e^{Ax} \equiv (A'^p x^p + A^{(p)} x)\, e^{Ax} \qquad (\mathrm{mod}\ p). \tag{3.5}$$

(We recall that the statement

$$\sum_0^{\infty} A_n \frac{t^n}{n!} \equiv \sum_0^{\infty} B_n \frac{t^n}{n!} \qquad (\mathrm{mod}\ p)$$

means

$$A_n \equiv B_n\ (\mathrm{mod}\ p) \qquad (n = 0, 1, 2, \cdots),$$

where the A_n, B_n are polynomials with integral coefficients.) Since in what follows all congruences are (mod p), we shall usually omit the modulus.

Since $A'^p \equiv \alpha_1{}^p$, it follows from (3.2) and (3.5) that

$$\phi_{n+p}(x) \equiv \alpha_1{}^p x^p \phi_n(x) + x \sum_{j=0}^{n} \binom{n}{j} \alpha_{j+p} \phi_{n-j}(x). \tag{3.6}$$

If we replace n by np in (3.6) we get

$$\phi_{(n+1)p}(x) \equiv \alpha_1{}^p x^p \phi_{np}(x) + x \sum_{j=0}^{n} \binom{n}{j} \alpha_{(j+1)p} \phi_{(n-j)p}(x). \tag{3.7}$$

Since

$$\phi_{n+1}(x) = x \sum_{j=0}^{n} \binom{n}{j} \alpha_{j+1} \phi_{n-j}(x),$$

it follows from (3.7) that

$$\phi_{np}(x) \equiv Y_n(\alpha_1{}^p x^p + \alpha_p x, \alpha_{2p} x, \alpha_{3p} x, \cdots),$$

or equivalently

$$Y_{np}(\alpha_1 x, \alpha_2 x, \alpha_3 x, \cdots) \equiv Y_n(\alpha_1{}^p x^p + \alpha_p x, \alpha_{2p} x, \alpha_{3p} x, \cdots). \tag{3.8}$$

Replacing n by np (3.8) becomes

$$Y_{np^2}(\alpha_1 x, \alpha_2 x, \alpha_3 x, \cdots) \equiv Y_n(\alpha_1^{p^2} x^{p^2} + \alpha_p{}^p x^p + \alpha_{p^2} x, \alpha_{2p^2} x, \alpha_{3p^2} x, \cdots).$$

The general formula is evidently

$$\begin{aligned} \phi_{np^r}(x) &= Y_{np^r}(\alpha_1 x, \alpha_2 x, \alpha_3 x, \cdots) \\ &\equiv Y_n(\alpha_1^{p^r} x^{p^r} + \cdots + \alpha_{p^r} x, \alpha_{2p^r} x, \alpha_{3p^r} x, \cdots). \end{aligned} \tag{3.9}$$

Since $Y_1(\alpha_1, \alpha_2, \alpha_3, \cdots) = \alpha_1$, (3.9) implies

$$\phi_{p^r}(x) \equiv \alpha_1^{p^r} x^{p^r} + \alpha_p^{p^{r-1}} x^{p^{r-1}} + \cdots + \alpha_{p^r} x. \tag{3.10}$$

It follows that

$$\begin{aligned} Y_{p^r}(A'x, A''x, A'''x, \cdots) &\equiv (A'x)^{p^r} + (A^{(p)}x)^{p^{r-1}} + \cdots + A^{(p^r)}x \\ &\equiv (\alpha_1 x)^{p^r} + (\alpha_p x)^{p^{r-1}} + \cdots + (\alpha_{p^{r-1}}\, x)^p + A^{(p^r)}x. \end{aligned}$$

Hence (3.4) yields

$$\phi_{n+p^r}(x) \equiv \sum_{j=0}^{r-1} (\alpha_{p^j} x)^{p^{r-j}} \phi_n(x) + x \sum_{j=0}^{n} \binom{n}{j} \alpha_{p^r+j} \phi_{n-j}(x). \tag{3.11}$$

It is convenient at this point to state a formula of a different nature, namely,

$$Y_n(\alpha_1 + y, \alpha_2, \alpha_3, \cdots) = \sum_{j=0}^{n} \binom{n}{j} y^{n-j} Y_{n-j}(\alpha_1, \alpha_2, \alpha_3 \cdots,), \tag{3.12}$$

which is an immediate consequence of (3.2).

In the next place, by (3.9) and (3.12)

$$Y_{ap^r}(A'x, A''x, A'''x \cdots,) \equiv Y_a(\alpha_1^{p^r} x^{p^r} + \cdots + \alpha_{p^{r-1}}^{p} x^p + A^{(p)}x, A^{(2p^r)}x, \cdots)$$

$$\equiv \sum_{j=0}^{a} \binom{a}{j} (\alpha_1^{p^r} x^{p^r} + \cdots + \alpha_{p^{r-1}} x)^{a-j} \cdot Y_j(A^{(p^r)}x, A^{(2p^r)}x, \cdots).$$

4. The Functions $\Theta(n, x)$ and $\Theta_j(n, x)$

We have defined $\theta(n, k)$ as the number of coefficients $A_n(k_1, k_2, k_3, \cdots)$ with

$$k_1 + 2k_2 + 3k_3 + \cdots = n, \qquad k_1 + k_2 + k_3 + \cdots = k \tag{4.1}$$

that are prime to p. We now define

$$\Theta(n, x) = \sum_{k=0}^{n} \theta(n, k)\, x^k, \tag{4.2}$$

where x is an indeterminate. Indeed it will be convenient to consider a slightly more general function. Put

$$\theta_j(n, k) = \sum_{r=0}^{n_*} d_j(r)\, \theta(n - j, k), \tag{4.3}$$

where the notation is that of Section 2. Thus $\theta_j(n, k)$ is the Lucas product of $d_j(n)$ and $\theta(n, k)$. The parameter k is held fixed; the Lucas product is with respect to n only.

We now define

$$\Theta_j(n, x) = \sum_{k=0}^{n} \theta_j(n, k)\, x^k. \tag{4.4}$$

It follows at once from (4.3) and (4.4) that

$$\Theta_j(n, x) = \sum_{r=0}^{n}{}^{*}\, d_j(r)\, \Theta(n - r, x). \tag{4.5}$$

Note in particular that

$$\theta_0(n, k) = \theta(n, k), \qquad \Theta_0(n, x) = \Theta(n, x).$$

Returning to (3.8) and applying (3.12) we get

$$Y_{np}(\alpha_1 x, \alpha_2 x, \alpha_3 x, \cdots) \equiv \sum_{j=0}^{n}{}^{*} \binom{n}{j} (\alpha_1 x)^{(n-j)p}\, Y_j(\alpha_p x, \alpha_{2p} x, \alpha_{3p} x, \cdots). \tag{4.6}$$

In counting the number of coefficients on the right side of (4.6) that are prime to p, it is evident that the external factor $(\alpha_1 x)^{(n-j)p}$ causes no overlapping. Hence we get

$$\theta(np, k) = \sum_{j=0}^{n}{}^{*}\, \theta(n - j, k - jp).$$

Then

$$\Theta(np, x) = \sum_{k=0}^{np} \theta(np, k)\, x^k = \sum_{k=0}^{np} x^k \sum_{j=0}^{n}{}^{*} \theta(n - j, k - jp)$$

$$= \sum_{j=0}^{n}{}^{*}\, x^{jp} \sum_{k=0}^{(n-j)p} \theta(n - j, k)\, x^k$$

and therefore

$$\Theta(np, x) = \sum_{j=0}^{n}{}^{*}\, x^{jp} \Theta(n - j, x). \tag{4.7}$$

Define

$$I(n, x) = x^n \qquad (n = 0, 1, 2, \cdots); \tag{4.8}$$

when $x = 1$ this function reduces to $I(n)$ as defined in Section 2. Thus (4.8) becomes

$$\Theta(np, x) = \sum_{j=0}^{n}{}^{*}\, I(j, x^p)\, \Theta(n - j, x). \tag{4.9}$$

Replacing n by np, (4.9) becomes

$$\Theta(np^2, x) = \sum_{j=0}^{n}{}^{*} I(j, x^{p^2})\, \Theta((n-j)p, x)$$

$$= \sum_{j=0}^{n}{}^{*} I(j, x^{p^2}) \sum_{k=0}^{n-j}{}^{*} I(k, x^p)\, \Theta(n-j-k, x)$$

$$= \sum_{m=0}^{n}{}^{*} \Theta(n-m, x) \sum_{j+k=m}{}^{*} I(j, x^{p^2})\, I(k, x^p).$$

Thus $\Theta(np^2, x)$ is the Lucas product of $\Theta(n, x)$, $I(n, x^p)$, $I(n, x^{p^2})$. The general formula of this type can be stated without any difficulty. When $x = 1$ it reduces to

$$\theta(np^r) = \Theta(np^r, 1) = \sum_{j=0}^{n}{}^{*} d_r(j)\, \theta(n-j) = \theta_r(n).$$

Making use of (4.5) we get

$$\Theta_j(np, x) = \sum_{r=0}^{n}{}^{*} d_j(rp)\, \Theta((n-r)p, x)$$

$$= \sum_{r=0}^{n}{}^{*} d_j(r) \sum_{s=0}^{n-r}{}^{*} I(s, x^p)\, \Theta(n-r-s, x)$$

$$= \sum_{s=0}^{n}{}^{*} I(s, x^p) \sum_{r=0}^{n-s} d_j(x)\, \Theta(n-r-s, x),$$

so that

$$\Theta_j(np, x) = \sum_{s=0}^{n}{}^{*} I(s, x^p)\, \Theta_j(n-s, x). \tag{4.10}$$

By means of (4.9) we can easily compute $\Theta(p^r, x)$. Indeed, (4.9) yields

$$\Theta(p^r, x) = \sum_{j=0}^{p^{r-1}}{}^{*} I(j, x)\, \Theta(p^{r-1} - j, x)$$

$$= \Theta(p^{r-1}, x) + I(p^{r-1}, x^p)$$

$$= \Theta(p^{r-1}, x) + x^{p^r}.$$

It follows at once that

$$\Theta(p^r, x) = \sum_{s=0}^{r} x^{p^s}. \tag{4.11}$$

More generally, (4.10) implies

$$\Theta_j(p^r, x) = \sum_{s=0}^{r} x^{p^s} + j \qquad (j = 0, 1, 2, \cdots). \tag{4.12}$$

5. Proof of (1.8)

Put

$$n = p^r + m, \qquad 0 \leqslant m < p^r. \tag{5.1}$$

Then (3.11) becomes

$$\phi_n(x) \equiv \sum_{j=0}^{r-1} (\alpha_{p^j} x)^{p^{r-j}} \phi_m(x) + x \sum_{j=0}^{m}{}^{*} \binom{m}{j} \alpha_{p^r+j} \phi_{m-j}(x). \tag{5.2}$$

In counting the number of coefficients in the right member of (5.2) that are prime to p, it is clear from (5.1) that there is no overlapping. It follows that

$$\theta(n, k) = \sum_{j=0}^{r-1} \theta(m, k - p^{r-j}) + \sum_{j=0}^{m}{}^{*} \theta(m - j, k - 1).$$

Then

$$\begin{aligned} \Theta(n, x) &= \sum_{k=0}^{n} x^k \sum_{j=0}^{r-1} \theta(m, k - p^{r-j}) + \sum_{k=1}^{n} x^k \sum_{j=0}^{m}{}^{*} \theta(m - j, k - 1) \\ &= \sum_{j=0}^{r-1} x^{p^{r-j}} \sum_{k=0}^{n-p^{r-j}} \theta(m, k)\, x^k + x \sum_{j=0}^{m}{}^{*} \sum_{k=0}^{n-1} \theta(m - j, k)\, x^k \\ &= \sum_{j=1}^{r} x^{p^j} \Theta(m, x) + x \sum_{j=0}^{m}{}^{*} \Theta(m - j, x). \end{aligned}$$

Hence if we put

$$u_r(x) = \sum_{j=1}^{r} x^{p^j}, \tag{5.3}$$

it is clear that

$$\Theta(n, x) = u_r(x)\, \Theta(m, x) + x(\Theta_1(m, x). \tag{5.4}$$

This formula admits of an immediate generalization, namely,

$$\Theta_j(n, x) = (u_r(x) + j)\, \Theta_j(m, x) + \Theta_{j+1}(m, x) \qquad (j = 0, 1, 2, \cdots). \tag{5.5}$$

It is convenient to put

$$n_k = p^{r_1} + p^{r_2} + \cdots + p^{r_k} \qquad (k = 1, 2, 3, \cdots), \tag{5.6}$$

where $0 \leqslant r_1 < r_2 < \cdots < r_k$. Then (5.5) becomes

$$\Theta_j(n_k, x) = (u_{r_k}(x) + j)\, \Theta_j(n_{k-1}, x) + x\Theta_{j+1}(n_{k-1}, x). \tag{5.7}$$

We have already computed $\Theta_j(p^r, x)$; in the present notation (4.12) becomes

$$\Theta_j(p^r, x) = u_r(x) + x + j. \tag{5.8}$$

Then by (5.7)

$$\begin{aligned}\Theta_j(p^{r_1} + p^{r_2}, x) &= (u_{r_2}(x) + j)(u_{r_1}(x) + x + j) + x(u_{r_1}(x) + x + j + 1)\\ &= (u_{r_1}(x) + j)(u_{r_2}(x) + j) + x(u_{r_1}(x) + u_{r_2}(x) + 2j)\\ &\qquad + x^2 + x,\\ \Theta_j(p^{r_1} + p^{r_2} + p^{r_3}, x) &= (u_{r_3}(x) + j)\, \Theta_j(p^{r_1} + p^{r_2}, x) + x\Theta_{j+1}(p^{r_1} + p^{r_2}, x)\\ &= (u_{r_1}(x) + j)(u_{r_2}(x) + j)(u_{r_3}(x) + j)\\ &\quad + x \sum (u_{r_1}(x) + j)(u_{r_2}(x) + j)\\ &\quad + (x^2 + x) \sum (u_{r_1}(x) + j) + x^3 + 3x^2 + x,\end{aligned}$$

where the sums on the right are with respect to the indices 1, 2, 3.

This suggests the general formula

$$\Theta_j(p^{r_1} + \cdots + p^{r_k}, x) = \sum_{s=0}^{k} \sigma_{k,k-s}^{(j)}(x)\, B_s(x), \tag{5.9}$$

where $\sigma_{k,m}^{(j)}(x)$ denotes the mth elementary symmetric function of the quantities

$$u_{r_1}(x) + j, \qquad u_{r_2}(x) + j, \qquad \cdots, \qquad u_{r_k}(x) + j$$

and $B_s(x)$ is a polynomial in x of degree s that is independent of $k, r_1, r_2, \cdots, r_k$.

It is, however, convenient to prove a more general result. Consider

$$\Theta_j(n_k + m, x) = \sum_{s=0}^{k} C_{k,k-s}(j, x)\, \Theta_{j+s}(m, x), \tag{5.10}$$

where now

$$0 \leqslant m < p^{r_k} < \cdots < p^{r_1}$$

and $C_{k,k-s}(j, x)$ is independent of m. Assuming that (5.10) holds up to and including the value k, we have, for $m < p^{r_{k+1}} < p^{r_k}$,

$$\begin{aligned}\Theta_j(n_{k+1} + m, x) &= \Theta_j(n_k + p^{r_{k+1}} + m, x)\\ &= \sum_{s=0}^{k} C_{k,k-s}(j, x)\, \Theta_{j+s}(p^{r_{k+1}} + m, x)\\ &= \sum_{s=0}^{k} C_{k,k-s}(j, x) \{(u_{r_{k+1}}(x) + j + s)\, \Theta_{j+s}(m, x)\\ &\qquad + x\Theta_{j+s+1}(m, x)\}\\ &= \sum_{s=0}^{k+1} \{(u_{r_{k+1}}(x) + j + s)\, C_{k,k-s}(j, x)\\ &\qquad + xC_{k,k-s+1}(j, x)\}\, \Theta_{j+s}(m).\end{aligned}$$

We note that if

$$\sum_{s=0}^{k} D_{ks}(x)\, \Theta_s(m, x) = 0 \qquad (m = 0, 1, \cdots, M),$$

where k is fixed, $D_{ks}(x)$ is independent of m and M is arbitrarily large, then

$$D_{ks} = 0 \qquad (0 \leqslant s \leqslant k).$$

It accordingly follows that

$$C_{k+1,k-s+1}(j, x) = (u_{r_{k+1}}(x) + j + s)\, C_{k,k-s}(j, x) + xC_{k,k-s+1}(j, x);$$

replacing s by $k - s + 1$ we get

$$C_{k+1,s}(j, x) = (u_{r_{k+1}}(x) + k - s + j + 1)\, C_{k,s-1}(j, x) + xC_{k,s}(j, x). \quad (5.11)$$

The initial conditions are

$$C_{00}(j, x) = 1, \qquad C_{0s}(j, x) = 0 \qquad (s \neq 0).$$

Put

$$f_k^{(j)}(x, y) = \sum_{s=0}^{k} C_{ks}(j, x)\, y^{k-s}. \quad (5.12)$$

Then, using (5.11), we find that

$$f_{k+1}^{(j)}(x, y) = (u_{r_{k+1}}(x) + j + xy) f_k^{(j)}(x, y) + y \frac{\partial f_k^{(j)}(x, y)}{y}.$$

We rewrite this in the operational form

$$f_{k+1}^{(j)}(x, y) = (u_{r_{k+1}}(x) + j + xy + yD)\, d_k^{(j)}(x, y), \tag{5.13}$$

where $D = \partial/\partial y$. Then (5.13) implies

$$f_k^{(j)}(x, y) = \prod_{s=1}^{k} (u_{r_s}(x) + j + xy + yD) \cdot 1 = \sum_{s=0}^{k} \sigma_{k,s}^{(j)}(x)\,(xy + yD)^{k-s} \cdot 1,$$

where $\sigma_{k,s}^{(j)}(x)$ is the sth elementary symmetric function of the quantities

$$u_{r_1}(x) + j, \qquad u_{r_2}(x) + j, \qquad \cdots, \qquad u_{r_k}(x) + j.$$

Also it is easily verified that

$$(xy + yD)^k \cdot 1 = \sum_{n=0}^{k} S(k, n)\, x^n y^n, \tag{5.14}$$

where

$$S(k, n) = \frac{1}{n!} \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} s^k. \tag{5.15}$$

Thus

$$f_k^{(j)}(x, y) = \sum_{n=0}^{k} \sigma_{k,k-n}^{(j)}(x) \sum_{t=0}^{n} S(n, t)\, x^t y^t.$$

Comparison with (5.11) yields

$$C_{k,k-t}(j, x) = \sum_{n=t}^{k} \sigma_{k,k-n}^{(j)}(x)\, S(n, t)\, x^t$$

and therefore (5.10) becomes

$$\Theta_j(n_k + m, x) = \sum_{t=0}^{k} \Theta_{j+t}(m, x) \sum_{n=t}^{k} \sigma_{k,k-n}^{(j)}(x)\, S(n, t)\, x^t. \tag{5.16}$$

We may now state our first principle result.

THEOREM 1. *Let* $0 \leqslant m < p^{r_k} < \cdots < p^{r_1}$ *and let* $\sigma_{k,n}^{(j)}(x)$ *denote the* nth *elementary symmetric function of the quantities*

$$u_{r_1}(x) + j, \qquad u_{r_2}(x) + j, \qquad \cdots, \qquad u_{r_k}(x) + j,$$

where

$$u_r(x) = \sum_{s=1}^{r} x^{p^s}.$$

Then (5.16) holds for $j = 0, 1, 2, \cdots$.

When $m = 0$, (5.16) becomes, since $\Theta_j(0, x) = 1$,

$$\Theta_j(n_k, x) = \sum_{t=0}^{k} \sum_{n=t}^{k} \sigma_{k,k-n}^{(j)}(x)\, S(n, t)\, x^t$$

$$= \sum_{n=0}^{k} \sigma_{k,k-n}^{(j)}(x) \sum_{t=0}^{n} S(n, t)\, x^t.$$

Hence if we put

$$B_n(x) = \sum_{t=0}^{n} S(n, t)\, x^t, \tag{5.17}$$

it is easy to identity $B_n(x)$ with the Bell polynomial defined by means of $B_0(x) = 1$ and

$$B_{n+1}(x) = x \sum_{s=0}^{n} \binom{n}{s} B_s(x). \tag{5.18}$$

We may state

THEOREM 2. *Let* $r_1, r_2, \cdots, r_k$ *be distinct integers. Then we have*

$$\Theta_j(p^{r_1} + \cdots + p^{r_k}, x) = \sum_{n=0}^{k} \sigma_{k,k-n}^{(j)}(x)\, B_n(x), \tag{5.19}$$

where $\sigma_{k,k-n}^{(j)}(n)$ *has the same meaning as in the previous theorem and* $B_n(x)$ *is the Bell polynomial defined by (5.18).*

6. THE GENERAL CASE

Let

$$D_r(n, x) = I(n, x^p) * \cdots * I(n, x^{p^r}), \tag{6.1}$$

where

$$I(n, x) = x^n \qquad (n = 0, 1, 2, \cdots). \tag{6.2}$$

Then by the discussion following (4.9) we have

$$\Theta(np^r, x) = \sum_{s=0}^{n_*} D_r(s, x)\, \Theta(n - s, x). \tag{6.3}$$

In particular, since

$$\theta(a, k) = P_k(x) \qquad (a < p),$$

where $P_k(n)$ is the number of partitions of a into exactly k parts, we have

$$\Theta(ap^r, x) = \sum_{s=0}^{a} D_r(s, x)\, P(a - s, x) \qquad (a < p), \tag{6.4}$$

where

$$P(a, x) = \sum_{k=0}^{a} P_k(a)\, x^k. \tag{6.5}$$

Now put

$$n = ap^r + m \qquad (0 \leqslant a < p; 0 \leqslant m < p^r). \tag{6.6}$$

Then by (3.3), (3.12), and (3.13)

$$\begin{aligned} D^{ap^r} e^{Ax} &= Y_{ap^r}(A'x, A''x, A'''x, \cdots)\, e^{Ax} \\ &\equiv Y_a((A'x)^{p^r} + \cdots + A^{(p^r)}x, A^{(2p^r)}x, \cdots) \\ &\equiv \sum_{j=0}^{a} \binom{a}{j} \{(A'x)^{p^r} + \cdots + (A^{(p^{r-1})}x)^p\}^{a-j} \cdot Y_j(A^{(p^r)}x, A^{(2p^r)}x, \cdots) \\ &\equiv \sum_{j=0}^{a} \binom{a}{j} \{(\alpha_1 x)^{p^r} + \cdots + (\alpha_{p^{r-1}} x)^p\}^{a-j} \cdot Y_j(A^{(p^r)}x, A(^{2p^r})x, \cdots). \end{aligned}$$

Put

$$(A^{(ip^r)}) = \sum_{m=0}^{\infty} C_j^{(i)}(m) \frac{t^m}{m!} \qquad (0 < i < p)$$

and let $\gamma_j^{(i)}(m)$ denote the number of terms in $C_j^{(i)}(m, x)$ with $p \nmid c$. Then if

$$m = a_0 + a_1 p + a_2 p^2 + \cdots \qquad (0 \leqslant a_t < p),$$

we have

$$\gamma_j^{(i)}(m) = P_j(a_0, a_1, a_2, \cdots), \tag{6.8}$$

where $P_j(a_0, a_1, a_2, \cdots)$ denotes the number of partitions of $(a_0, a_1, a_2, \cdots)$ into exactly k parts. If we now put

$$Y_j(A^{(p^r)}x, A^{(2p^r)}x, \cdots) = \sum_{m=0}^{\infty} B_j^{(r)}(m, x)\frac{t^m}{m!}$$

and $\beta_j(m, k)$ denotes the number of terms x^k in $B_j^{(r)}(m, x)$ with $p \nmid b$, the

$$\beta_j(m, k) = \sum \gamma_{j_1}^{(1)} * \gamma_{j_2}^{(2)} * \cdots, \tag{6.9}$$

where the summation is over all nonnegative $j_1, j_2, \cdots$ such that

$$j_1 + 2j_2 + \cdots = j, \qquad j_1 + j_2 + \cdots = k.$$

It follows at once from (6.9) that

$$\beta_j(m, x) = \sum \gamma_{j_1}^{(1)} * \gamma_{j_2}^{(2)} * \cdots x^{j_1+j_2+\cdots}, \tag{6.10}$$

where $j_1 + 2j_2 + \cdots = j$ and

$$\beta_j(m, x) = \sum_{k=0}^{m} \beta_j(m, k)\, x^k.$$

In the next place, since

$$\phi_{ap_r+m}(x) \equiv \sum_{j=0}^{a} \binom{a}{j} \{(\alpha_1 x)^{p^r} + \cdots + (\alpha_{p^r-1}x)^p\} \cdot \sum_{s=0}^{m_*} B_j^{(r)}(x, s)\, \phi_{m-s}(x),$$

we get

$$\Theta(ap^r + m, x) = \sum_{j=0}^{a} (u_r(x))^{(a-j)} \sum_{s=0}^{m_*} \beta_j(s, x)\, \Theta(m - s, x), \tag{6.11}$$

where $u_r(x)$ is defined by (5.3) and the notation $(u_r(x))^{(a-j)}$ indicates that after expansion the multinomial coefficients are deleted. It is easily verified that

$$(u_r(x))^{(a)} = D_r(a, x) \qquad (a < p). \tag{6.12}$$

We shall now show that

$$\Theta(a_1p^{r_1} + \cdots + a_zp^{r_z}, x)$$
$$= \sum_{j_s=0}^{a_s} D_{r_1}(j_1, x) \cdots D_{r_z}(j_z, x)\, Q(a_1 - j_1, \cdots, a_z - j_z; x), \tag{6.13}$$

where $Q(a_1 - j_1, \cdots, a_z - j_z; x)$ is independent of $r_1, \cdots, r_z$ and

$$0 \leqslant a_s < p \qquad (1 \leqslant s \leqslant z); \qquad r_1 < r_2 < \cdots < r_z .$$

For $z = 1$, (6.13) is in agreement with (6.4); indeed

$$Q(a; x) = P(a; x).$$

We assume that (6.13) holds up to and including the value z and apply (6.11) with $m = a_1 p^{r_1} + \cdot + a_z p^{r_z}$. Then if $0 < a < p$ and $r > r_k$ we have by (6.11) and (6.12)

$$\Theta(ap^r + m, x) = \sum_{j=0}^{a} D_r(a, x) \sum_{m'=0}^{m} \beta_j(m - m', x)\, \Theta(m', x).$$

By the inductive hypothesis

$$\sum_{m'=0}^{m} \beta_i(m - m', x)\, \Theta(m', x) = \sum_{b_s=0}^{a_s} \beta_j(m - m', x) \sum_{j_s=0}^{b_s} D_{r_1}(j_1, x) \cdots D_{r_z}(j_z, x)$$
$$\cdot Q(b_1 - j_1, \cdots, b_z - j_z; x),$$

where

$$m' = b_1 p^{r_1} + \cdots + b_z p^{r_z}.$$

Thus

$$\Theta(ap^r + m, x) = \sum_{j=0}^{a} D_r(j, x) \sum_{j_s=0}^{a_s} D_{r_1}(j_1, x) \cdots D_{r_z}(j_z, x)$$
$$\cdot \sum_{b_s=j_s}^{a_s} \beta_{a-j}(m - m', x)\, Q(b_1 - j_1, \cdots, b_z - j_z; x).$$

This completes the proof of (6.13); moreover it shows that

$$Q(a_1 - j_1, \cdots a_z - j_z, a - j; x)$$
$$= \sum_{b_s=j_s}^{a_s} \beta_{a-j}(m - m', x)\, Q(b_1 - j, \cdots, b_z - j_z; x).$$

This may be replaced by

$$Q(a_1, \cdots, a_z, a; x) = \sum_{b_s=0}^{a_s} \beta_a(m - m', x)\, Q(b_1, \cdots, b_z; x). \qquad (6.14)$$

It remains to show that

$$Q(a_1, \cdots, a_z; x) = P(a_1, \cdots, a_z; x), \tag{6.15}$$

where

$$P(a_1, \cdots, a_z; x) = \sum_k P_k(a_1, \cdots, a_z)\, x^k$$

and $P_k(a_1, \cdots, a_z)$ is the number of partitions of $(a_1, \cdots, a_z)$ into exactly k parts. We recall that

$$\sum_{k=0}^{\infty} x^k \sum_{a_1, \ldots, a_z=0}^{\infty} P_k(a_1, \cdots, a_z)\, x_1^{a_1} \cdots x_z^{a_z} = \prod_{a_1, \ldots, a_z=0}^{\infty} (1 - x_1^{a_1} \cdots x_z^{a_z} x)^{-1}. \tag{6.16}$$

In proving (6.15) we drop the restriction $a_s < p$ and assume that (6.10) and (6.14) hold for all $a_s \geqslant 0$. Finally when (6.16) is applied to (6.13) the restriction is restored.

We have already seen that (6.15) holds when $z = 1$. We now assume that (6.15) holds up to and including the value z. Thus (6.14) becomes

$$Q(a_1, \cdots, a_z, a; x) = \sum_{b_s=0}^{a_s} \beta_a(m - m', x)\, P(b_1, \cdots, b_z; x),$$

so that

$$\sum_{a_1, \ldots, a_z, a=0}^{\infty} Q(a_1, \cdots, a_z, a; x)\, y_1^{a_1} \cdots a_z^{a_z} y^a$$

$$= \sum_{a_1, \ldots, a_z=0}^{\infty} \sum_{b_1, \ldots, b_z=0}^{\infty} \sum_{a=0}^{\infty} \beta_a(a_1 p^{r_1} + \cdots + a_z p^{r_z}; x)$$

$$\cdot P(b_1, \cdots, b_z; x)\, y_1^{a_1+b_1} \cdots y_z^{a_z+b_z} y^a$$

$$= \prod_{b_1, \ldots, b_z=0}^{\infty} (1 - y_1^{b_1} \cdots y_z^{b_z} x)^{-1}$$

$$\cdot \sum_{a_1, \ldots, a_z=0}^{\infty} \sum_{a=0}^{\infty} \beta_a(a_1 p^{r_1} + \cdots + a_z p^{r_z}; x)\, y_1^{a_1} \cdots y_z^{a_z} y^a.$$

By (6.10) and (6.8) the multiple sum on the right is equal to

$$\sum_{j_1, j_2, \ldots, =0}^{\infty} \sum_{a_{ts}=0}^{\infty} \prod_s P_{j_s}(a_{1s}, \cdots, a_{zs})\, y_1^{\Sigma a_{1s}} \cdots y_z^{\Sigma a_{zs}} y^{\Sigma s j_s} x^{\Sigma j_s}$$

$$= \prod_{s=1}^{\infty} \sum_{j_1, j_2, \ldots =0}^{\infty} \sum_{n_1, \ldots, n_z=0}^{\infty} P_{js}(n_1, \cdots, n_k)\, y_1^{n_1} \cdots y_z^{n_z} x^{j_s} y^{s j_s}$$

$$= \prod_{s=1}^{\infty} \prod_{n_1, \ldots, n_z=0}^{\infty} (1 - y_1^{n_1} \cdots y_z^{n_z} y^s x)^{-1}.$$

Therefore

$$\sum_{a_1, \ldots, a_z=0}^{\infty} Q(a_1, \cdots, a_z, a; x)\, y_1^{a_1} \cdots y_z^{a_z} y^a = \prod_{n_1, \ldots, n_z, n=0}^{\infty} (1 - y_1^{n_1} \cdots y_z^{n_z} y^n x)^{-1},$$

which evidently completes the induction.

We may now state

THEOREM 3. *Let $r_1, \cdots, r_z$ be distinct integers and let $0 < a_s < p$, $1 \leqslant s \leqslant z$. Then*

$$\Theta(a_1 p^{r_1} + \cdots + a_z p^z, x)$$

$$= \sum_{j_s=0}^{a_s} D_{r_1}(j_1, x) \cdots D_{r_z}(j_z, x)\, P(a_1 - j_1, \cdots, a_z - j_z; x),$$

where

$$P(a_1, \cdots, a_z; x) = \sum_{k=0}^{\infty} P_k(a_1, \cdots, a_z)\, x^z,$$

$P_k(a_1, \cdots, a_z)$ is the number of partitions of $(a_1, \cdots, a_z)$ into k parts and

$$D_r(a, x) = (x^p + x^{p^2} + \cdots + x^{p^r})^{(a)},$$

where in the expansion of the right member the multinomial coefficients are deleted.

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