



Abelian subgroup structure of square complex groups and arithmetic of quaternions

Diego Rattaggi^a, Guyan Robertson^{b,*}

^a *Université de Genève, Section de mathématiques, 2–4, rue du Lièvre, CP 64, CH-1211 Genève 4, Switzerland*

^b *School of Mathematics and Statistics, University of Newcastle, NE1 7RU, UK*

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1. Introduction

A square complex is a 2-complex formed by gluing squares together. This article is concerned with the fundamental group Γ of certain square complexes of nonpositive curvature, related to quaternion algebras. The Abelian subgroup structure of Γ is studied in some detail. Before outlining the results, it is necessary to describe the construction of Γ .

In [5, Section 3], there is constructed a lattice subgroup $\Gamma = \Gamma_{p,l}$ of $G = PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$, where $p, l \equiv 1 \pmod{4}$ are two distinct primes. This restriction was made because -1 has a square root in \mathbb{Q}_p if and only if $p \equiv 1 \pmod{4}$, but the construction of Γ is generalized in [8, Chapter 3] to all pairs (p, l) of distinct odd primes.

The affine building Δ of G is a product of two homogeneous trees of degrees $(p+1)$ and $(l+1)$, respectively. The group Γ is a finitely presented torsion free group which acts freely and transitively on the vertices of Δ , with a finite square complex as quotient Δ/Γ .

Here is how Γ is constructed. Let

$$\mathbb{H}(\mathbb{Z}) = \{x = x_0 + x_1i + x_2j + x_3k; x_0, x_1, x_2, x_3 \in \mathbb{Z}\}$$

be the ring of integer quaternions where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$. Let $\bar{x} = x_0 - x_1i - x_2j - x_3k$ be the conjugate of x , and $|x|^2 = x\bar{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2$ its norm.

* Corresponding author.

E-mail addresses: rattaggi@math.unige.ch (D. Rattaggi), a.g.robertson@newcastle.ac.uk (G. Robertson).

Let $c_p, d_p \in \mathbb{Q}_p$ and $c_l, d_l \in \mathbb{Q}_l$ be elements such that $c_p^2 + d_p^2 + 1 = 0, c_l^2 + d_l^2 + 1 = 0$. Such elements exist by Hensel’s Lemma and [2, Proposition 2.5.3]. We can take $d_p = 0$, if $p \equiv 1 \pmod{4}$, and $d_l = 0$, if $l \equiv 1 \pmod{4}$. Define

$$\psi : \mathbb{H}(\mathbb{Z}) - \{0\} \rightarrow PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$$

by

$$\psi(x) = \left(\left(\begin{array}{cc} x_0 + x_1c_p + x_3d_p & -x_1d_p + x_2 + x_3c_p \\ -x_1d_p - x_2 + x_3c_p & x_0 - x_1c_p - x_3d_p \end{array} \right), \right. \\ \left. \left(\begin{array}{cc} x_0 + x_1c_l + x_3d_l & -x_1d_l + x_2 + x_3c_l \\ -x_1d_l - x_2 + x_3c_l & x_0 - x_1c_l - x_3d_l \end{array} \right) \right). \tag{1}$$

This formula abuses notation by identifying an element of $PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$ with its representative in $GL_2(\mathbb{Q}_p) \times GL_2(\mathbb{Q}_l)$.

Note that $\psi(xy) = \psi(x)\psi(y), \psi(\lambda x) = \psi(x)$, if $\lambda \in \mathbb{Z} - \{0\}$, and $\psi(x)^{-1} = \psi(\bar{x})$. Moreover the inverse image under ψ of the identity element in $PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$ is precisely

$$\mathbb{Z} - \{0\} = \{x \in \mathbb{H}(\mathbb{Z}); x_0 \neq 0, x_1 = x_2 = x_3 = 0\}.$$

Let

$$\tilde{\Gamma} = \{x \in \mathbb{H}(\mathbb{Z}); |x|^2 = p^r l^s, r, s \geq 0; \\ x_0 \text{ odd, } x_1, x_2, x_3 \text{ even, if } |x|^2 \equiv 1 \pmod{4}; \\ x_1 \text{ even, } x_0, x_2, x_3 \text{ odd, if } |x|^2 \equiv 3 \pmod{4}\}.$$

Then $\Gamma = \psi(\tilde{\Gamma})$ is a torsion free cocompact lattice in G . Let

$$\tilde{A} = \{x \in \tilde{\Gamma}; x_0 > 0, |x|^2 = p\}, \quad \tilde{B} = \{y \in \tilde{\Gamma}; y_0 > 0, |y|^2 = l\}.$$

Then \tilde{A} contains $p + 1$ elements and \tilde{B} contains $l + 1$ elements, by a result of Jacobi [4, Theorem 2.1.8]. The images $A = \psi(\tilde{A}), B = \psi(\tilde{B})$ of \tilde{A}, \tilde{B} in Γ generate free groups $\Gamma_p = \langle A \rangle = \langle a_1, \dots, a_{(p+1)/2} \rangle, \Gamma_l = \langle B \rangle = \langle b_1, \dots, b_{(l+1)/2} \rangle$ of ranks $(p + 1)/2, (l + 1)/2$, respectively, and Γ itself is generated by $A \cup B$. The 1-skeleton of Δ is the Cayley graph of Γ relative to this set of generators. The group Γ has a finite presentation with generators $\{a_1, \dots, a_{(p+1)/2}\} \cup \{b_1, \dots, b_{(l+1)/2}\}$ and $(p + 1)(l + 1)/4$ relations of the form $ab = \tilde{b}\tilde{a}$, where $a, \tilde{a} \in A, b, \tilde{b} \in B$. In fact, given any $a \in A, b \in B$, there are unique elements $\tilde{a} \in A, \tilde{b} \in B$ such that $ab = \tilde{b}\tilde{a}$. This follows from a special case of Dickson’s factorization property for integer quaternions [3, Theorem 8].

Proposition 1.1 [3]. *Let $x \in \tilde{\Gamma}$ such that $|x|^2 = pl$. Then there are uniquely determined $z, \tilde{z} \in \tilde{A}, y, \tilde{y} \in \tilde{B}$ such that $zy, \tilde{y}\tilde{z} = \pm x$.*

It is worth noting that $zy \neq \tilde{y}\tilde{z}$ in general, as demonstrated by the following example.

Example 1.2. Let $p = 3, l = 5$ and $x = 1 + 2i + j + 3k$. Then $(1 - j + k)(1 + 2i) = x$ and $(1 - 2k)(1 - j - k) = -x$.

We can now outline the contents of this article. A fundamental fact, upon which much else depends, is that Γ is *commutative transitive*, in the sense that the relation of commutativity is transitive on non-trivial elements of Γ . In particular Γ cannot contain a subgroup isomorphic to $F_2 \times F_2$, where F_2 denotes the free group of rank 2. Furthermore, Γ is a *CSA-group*, i.e. all its maximal Abelian subgroups Γ_0 satisfy $g\Gamma_0g^{-1} \cap \Gamma_0 = \{1\}$ for all $g \in \Gamma - \Gamma_0$.

Every nontrivial element $\gamma \in \Gamma$ is the image under ψ of a quaternion of the form $x_0 + z_0(c_1i + c_2j + c_3k)$ where $c_1, c_2, c_3 \in \mathbb{Z}$ are relatively prime. The element γ is contained in a unique maximal Abelian subgroup Γ_0 and the integer $n = n(\Gamma_0) = c_1^2 + c_2^2 + c_3^2$ depends only on Γ_0 rather than the particular choice of γ . We define a class of maximal Abelian subgroups of Γ isomorphic to \mathbb{Z}^2 , which we call period subgroups, and which are characterized by the condition $(\frac{-n}{p}) = (\frac{-n}{l}) = 1$. Every maximal Abelian subgroup $\Gamma_0 \cong \mathbb{Z}^2$ is conjugate in Γ to a period subgroup and, as the name suggests, period subgroups are closely related to periodic tilings of the plane. On the other hand, some maximal Abelian subgroups of Γ are isomorphic to \mathbb{Z} , and we show how to construct these. Several explicit examples and counterexamples are included.

2. The CSA property

Let $\tau : \mathbb{H}(\mathbb{Q}) - \mathbb{Q} \rightarrow \mathbb{P}^2(\mathbb{Q})$ be defined by $\tau(x) = \mathbb{Q}(x_1, x_2, x_3)$, which is a line in \mathbb{Q}^3 through $(0, 0, 0)$. By [5, Section 3], two quaternions $x, y \in \mathbb{H}(\mathbb{Q}) - \mathbb{Q}$ commute if and only if $\tau(x) = \tau(y)$. This directly implies the following lemma, which in turn has Proposition 2.2 as a consequence, see also [8, Chapter 3].

Lemma 2.1. *Elements $x, y \in \tilde{\Gamma}$ commute if and only if their images $\psi(x), \psi(y) \in \Gamma$ commute.*

A group is said to be *commutative transitive* if the relation of commutativity is transitive on its non-trivial elements.

Proposition 2.2. *The group Γ is commutative transitive.*

Wise has asked in [11, Problem 10.9] whether the fundamental group of any nonelementary complete square complex contains a subgroup isomorphic to $F_2 \times F_2$. We can give a negative answer of this question, since our group Γ belongs to this class of fundamental groups, and it is a direct consequence of Proposition 2.2 that Γ does not contain a $F_2 \times F_2$ subgroup. In fact, since Γ is torsion free, and a (free) Abelian subgroup of Γ has rank ≤ 2 [7, Lemma 3.2], we have a more precise result.

Corollary 2.3. *The only nontrivial direct product subgroup of Γ is $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.*

If $\gamma = \psi(x) \in \Gamma - \{1\}$ then the centralizer $\Gamma_0 = Z_\Gamma(\gamma)$ is the unique maximal Abelian subgroup of Γ containing γ . Moreover Γ_0 is determined by $\tau(x)$, independent of the choice of x .

As described in [6, Remark 4], a group is commutative transitive if and only if the centralizer of any non-trivial element is Abelian. A third equivalent condition (called *SA-property* in [6]) is proved for Γ in the following lemma. It is used to show in Proposition 2.6 that Γ is a *CSA-group*, i.e. all its maximal Abelian subgroups are malnormal, where a subgroup Γ_0 of Γ is *malnormal* (or *conjugate separated*) if $g\Gamma_0g^{-1} \cap \Gamma_0 = \{1\}$ for all $g \in \Gamma - \Gamma_0$. Any CSA-group is commutative transitive, but the converse is not true, see [6].

Lemma 2.4. *If $\Gamma_1 \neq \Gamma_2$ are maximal Abelian subgroups of Γ then $\Gamma_1 \cap \Gamma_2 = \{1\}$.*

Proof. Suppose that there exists a nontrivial element $\gamma \in \Gamma_1 \cap \Gamma_2$. If $\gamma_i \in \Gamma_i - \{1\}$, $i = 1, 2$, then $\gamma\gamma_1 = \gamma_1\gamma$ and $\gamma\gamma_2 = \gamma_2\gamma$ which implies $\gamma_1\gamma_2 = \gamma_2\gamma_1$ by Proposition 2.2. Since Γ_1, Γ_2 are maximal Abelian, $\Gamma_1 = \Gamma_2$. \square

It is well known that there is a (surjective) homomorphism

$$\theta : \mathbb{H}(\mathbb{Q}) - \{0\} \rightarrow SO_3(\mathbb{Q})$$

defined by $\theta(y)x = yxy^{-1}$ for any $x = x_1i + x_2j + x_3k \in \mathbb{H}(\mathbb{Q})$, also identified with $(x_1, x_2, x_3) \in \mathbb{Q}^3$.

If $y \in \mathbb{H}(\mathbb{Q}) - \mathbb{Q}$ then the axis of rotation of $\theta(y)$ is $\tau(y)$. This is an immediate consequence of the fact that

$$\theta(y)(y - y_0) = y(y - y_0)y^{-1} = y - y_0.$$

Moreover the angle of rotation is 2α where $\cos \alpha = y_0/|y|$ [10, Chapitre I, Section 3]. In particular, the angle of rotation is a multiple of π only if $y_0 = 0$.

Lemma 2.5. (a) *Suppose that $x, y \in \mathbb{H}(\mathbb{Q}) - \mathbb{Q}$ and $y_0 \neq 0$. Then yxy^{-1} commutes with x if and only if y commutes with x .*

(b) *If $a, b \in \Gamma$, then bab^{-1} commutes with a if and only if b commutes with a .*

Proof. (a) If yxy^{-1} commutes with x , then the rotations $\theta(yxy^{-1})$ and $\theta(x)$ have the same axis. However, the axis of $\theta(yxy^{-1}) = \theta(y)\theta(x)\theta(y)^{-1}$ is $\theta(y)\tau(x)$. Therefore $\theta(y)\tau(x) = \tau(x)$: in other words, $\theta(y)(x_1, x_2, x_3) = \pm(x_1, x_2, x_3)$. If $\theta(y)(x_1, x_2, x_3) = -(x_1, x_2, x_3)$ then $\theta(y)$ is a rotation by angle π with axis perpendicular to (x_1, x_2, x_3) . This cannot happen since $y_0 \neq 0$. Therefore $\theta(y)$ has axis $\tau(x)$. That is, $\tau(y) = \tau(x)$ and consequently y commutes with x . The converse is clear.

(b) If $a = 1$ or $b = 1$, the statement is obvious. If $a, b \in \Gamma - \{1\}$ and bab^{-1} commutes with a , then representatives x, y for a, b in $\mathbb{H}(\mathbb{Q}) - \mathbb{Q}$ have nonzero real parts and satisfy the same relation, by Lemma 2.1. The assertion follows from (a). Again, the converse is clear. \square

Proposition 2.6. Γ is CSA.

Proof. Suppose that Γ_0 is a maximal Abelian subgroup of Γ and that $b \in \Gamma$, with $b\Gamma_0b^{-1} \cap \Gamma_0 \neq \{1\}$. We must show that $b \in \Gamma_0$.

By Lemma 2.4, $b\Gamma_0b^{-1} = \Gamma_0$. Let $a \in \Gamma_0$. Then bab^{-1} commutes with a and so, by Lemma 2.5, b commutes with a . Since Γ_0 is maximal Abelian, $b \in \Gamma_0$. \square

We now recall the following known result.

Lemma 2.7. (a) [6, Proposition 9(5)] *A non-Abelian CSA-group has no non-Abelian solvable subgroups.*

(b) [6, Proposition 10(3)] *Subgroups of CSA-groups are CSA.*

Corollary 2.8. *Let $a \in \Gamma_p - \{1\}$ and $b \in \Gamma_l - \{1\}$. Then either $\langle a, b \rangle \cong \mathbb{Z}^2$ or $\langle a, b \rangle$ contains a free subgroup of rank 2.*

Proof. If a, b commute, then $\langle a, b \rangle \cong \mathbb{Z}^2$, since Γ is torsion free and $\langle a, b \rangle$ is not cyclic. Assume that a, b do not commute. We will show that $\langle a, b \rangle$ is not virtually solvable. The Tits Alternative for finitely generated linear groups (see [9]) then implies that $\langle a, b \rangle$ contains a free subgroup of rank 2. Note that Γ is linear, see [8, Section 3.2] for an explicit injective homomorphism $\Gamma \rightarrow SO_3(\mathbb{Q})$. Let U be a finite index subgroup of $\langle a, b \rangle$, in particular there are $r, s \in \mathbb{N}$ such that $a^r, b^s \in U$. The elements a^r and b^s do not commute since otherwise also a and b would commute by Proposition 2.2. It follows that U is not Abelian. By Proposition 2.6 and Lemma 2.7(b), $\langle a, b \rangle$ is CSA. Lemma 2.7(a) shows that U is not solvable. \square

3. Maximal Abelian subgroups and period subgroups

Recall that the group Γ acts freely and transitively on the vertex set of the affine building Δ of $PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$. The building Δ is a product of two homogeneous trees and the apartments (maximal flats) in Δ are copies of the Euclidean plane tessellated by squares.

Notation 3.1. *If n is an integer and p is an odd prime, then the Legendre symbol is*

$$\left(\frac{n}{p}\right) = \begin{cases} 0 & \text{if } p \mid n, \\ 1 & \text{if } p \nmid n \text{ and } n \text{ is a square mod } p, \\ -1 & \text{if } p \nmid n \text{ and } n \text{ is not a square mod } p. \end{cases}$$

Any element of $\Gamma - \{1\}$ is the image under ψ of a quaternion of the form

$$x = x_0 + z_0(c_1i + c_2j + c_3k), \tag{2}$$

where $c_1, c_2, c_3 \in \mathbb{Z}$ are relatively prime, $z_0 \neq 0$, $(c_1, c_2, c_3) \neq (0, 0, 0)$, and

$$|x|^2 = x_0^2 + (c_1^2 + c_2^2 + c_3^2)z_0^2 = p^r l^s, \quad r, s \geq 0.$$

Recall that $\tau(x) = \mathbb{Q}(c_1, c_2, c_3) \in \mathbb{P}^2(\mathbb{Q})$ and that elements $\psi(x), \psi(y) \in \Gamma - \{1\}$ commute if and only if $\tau(x) = \tau(y)$. Moreover the centralizer $\Gamma_0 = Z_\Gamma(\psi(x))$ is the unique maximal Abelian subgroup of Γ containing $\psi(x)$. Let

$$n(x) = n(\psi(x)) = n(\Gamma_0) = c_1^2 + c_2^2 + c_3^2.$$

An Abelian subgroup of Γ has rank ≤ 2 [7, Lemma 3.2]. Since Γ is torsion free, a non-trivial Abelian subgroup Γ_0 of Γ is isomorphic to either \mathbb{Z} or \mathbb{Z}^2 . If $\Gamma_0 \cong \mathbb{Z}^2$ then there is a unique apartment \mathcal{A}_{Γ_0} which is stabilized by Γ_0 [7, 6.8], and Γ_0 acts cocompactly by translation on this apartment. We call \mathcal{A}_{Γ_0} a *periodic* apartment.

Definition 3.2. A maximal Abelian subgroup $\Gamma_0 \cong \mathbb{Z}^2$ will be called a *period subgroup* if the apartment \mathcal{A}_{Γ_0} contains the vertex O of Δ whose stabilizer in G is $PGL_2(\mathbb{Z}_p) \times PGL_2(\mathbb{Z}_l)$.

Since the action of Γ on Δ is vertex transitive, every maximal Abelian subgroup $\Gamma_0 \cong \mathbb{Z}^2$ is conjugate in Γ to a period subgroup. We want to show that $n(x)$ determines when $Z_\Gamma(\psi(x))$ is a period subgroup of Γ .

Recall that Γ is generated by free groups Γ_p, Γ_l , of ranks $(p + 1)/2, (l + 1)/2$, respectively. If $\gamma \in \Gamma$, let $\ell(\gamma)$ denote the natural word length of γ , in terms of the generators of Γ_p, Γ_l . The condition $\ell(\gamma^2) = 2\ell(\gamma)$, which is used in the next lemma, is equivalent to the assertion that γ has an axis containing O , upon which γ acts by translation.

Lemma 3.3. *Let $a = \psi(x) \in \Gamma_p - \{1\}$ and let $n = n(x)$. The following statements are equivalent:*

- (a) $p \nmid n$;
- (b) $\ell(a^2) = 2\ell(a)$;
- (c) $\left(\frac{-n}{p}\right) = 1$.

Similar equivalent assertions hold, if p is replaced by l .

Before giving the proof, we note that

$$\left(\frac{-n}{p}\right) = \begin{cases} \left(\frac{n}{p}\right), & \text{if } p \equiv 1 \pmod{4}, \\ -\left(\frac{n}{p}\right), & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. (a) \Leftrightarrow (b). The idea for this comes from the proof of [5, Proposition 3.15]. Write x as in (2) with $|x|^2 = x_0^2 + nz_0^2 = p^r, r > 0$. Extracting a common factor, if necessary, we may assume $\gcd(x_0, z_0) = 1$. This means that $r = \ell(a)$ [8, Corollary 3.11(4), Theorem 3.30(1)].

Suppose that $p \nmid n$. To prove $\ell(a^2) = 2\ell(a)$ we must show that p does not divide x^2 . Now if p divides

$$x^2 = (x_0^2 - nz_0^2) + 2x_0z_0(c_1i + c_2j + c_3k),$$

then p divides the real part $x_0^2 - nz_0^2$. Therefore p divides x_0 (since p divides $p^r = x_0^2 + nz_0^2$). But this implies that p divides z_0 , since $p \nmid n$. This contradicts the assumption $\gcd(x_0, z_0) = 1$.

Conversely, suppose that $\ell(a^2) = 2\ell(a)$. If p divides n , then p divides x_0 (since p divides $x_0^2 + nz_0^2$). Therefore p divides the real and imaginary parts of $x^2 = (x_0^2 - nz_0^2) + 2x_0z_0(c_1i + c_2j + c_3k)$. But this implies that $\ell(a^2) < 2r$, a contradiction.

(a) \Leftrightarrow (c). Suppose that $p \nmid n$. Note that p does not divide z_0 : otherwise p also divides x_0 . It follows that z_0 has a multiplicative inverse (mod p). That is, one can choose $t \in \mathbb{Z}$ such that $z_0t \equiv 1 \pmod{p}$. Then

$$0 \equiv (x_0^2 + nz_0^2)t^2 \equiv x_0^2t^2 + n \pmod{p}.$$

Since $p \nmid n$, this means that $(\frac{-n}{p}) = 1$. The converse is obvious. \square

Lemma 3.4. *If $\Gamma_0 \cong \mathbb{Z}^2$ is a period subgroup of Γ and $n = n(\Gamma_0)$, then $(\frac{-n}{p}) = (\frac{-n}{l}) = 1$.*

Proof. The group Γ_0 acts cocompactly by translation on the apartment \mathcal{A}_{Γ_0} containing the vertex O . It follows that Γ_0 contains elements $a \in \Gamma_p - \{1\}$, $b \in \Gamma_l - \{1\}$. These elements act freely by translation on the apartment, and so $\ell(a^2) = 2\ell(a)$, $\ell(b^2) = 2\ell(b)$. Therefore $(\frac{-n}{p}) = (\frac{-n}{l}) = 1$, by Lemma 3.3. \square

Lemma 3.5. *If $\gamma = \psi(x) \in \Gamma - (\Gamma_p \cup \Gamma_l)$ and $\gcd(n(x), pl) = 1$, then $Z_\Gamma(\gamma)$ is a period subgroup of Γ .*

Proof. Let $x = x_0 + z_0(c_1i + c_2j + c_3k)$ as in (2) and $n = n(x) = c_1^2 + c_2^2 + c_3^2$. We may assume $\gcd(x_0, z_0) = 1$ and $|x|^2 = x_0^2 + nz_0^2 = p^r l^s$, where $r, s \geq 1$ because $\psi(x) \notin \Gamma_p \cup \Gamma_l$.

The assumption $\gcd(n, pl) = 1$ implies that $\gcd(x_0z_0, pl) = 1$. For example, if $p \mid x_0$ then $p \mid z_0$, since $p \mid (x_0^2 + nz_0^2)$ and $p \nmid n$. This contradicts $\gcd(x_0, z_0) = 1$. Similarly, $p \nmid z_0$. It follows from the “if” part of the proof of [5, Proposition 3.15] (and an obvious generalization to the cases where $p \equiv 3 \pmod{4}$ or $l \equiv 3 \pmod{4}$) that $\gamma = \psi(x)$ lies in an Abelian subgroup Γ_0 of Γ , with $\Gamma_0 \cong \mathbb{Z}^2$. The same proof also shows that Γ_0 acts cocompactly by translation on an apartment \mathcal{A} containing O . (The essential point in the proof of Mozes is that $\ell(\gamma^2) = 2\ell(\gamma)$.) However, $Z_\Gamma(\gamma)$ is the unique maximal Abelian subgroup containing Γ_0 . Therefore $Z_\Gamma(\gamma)$ acts cocompactly by translation on the apartment \mathcal{A} , by the uniqueness assertion in [7, 6.8]. In other words, $Z_\Gamma(\gamma)$ is a period subgroup of Γ . \square

Now we can describe the period subgroups of Γ .

Proposition 3.6. *Let Γ_0 be a maximal Abelian subgroup of Γ , and let $n = n(\Gamma_0)$. Then Γ_0 is a period subgroup if and only if $\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1$.*

Before proceeding with the proof, we introduce some notation. There is a canonical Cartan subgroup C of $G = PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$ defined by

$$C = \left(\left(\begin{matrix} * & 0 \\ 0 & * \end{matrix} \right), \left(\begin{matrix} * & 0 \\ 0 & * \end{matrix} \right) \right) \cap G.$$

The group C acts by translation on an apartment \mathcal{A} , which contains the vertex O whose stabilizer in G is $PGL_2(\mathbb{Z}_p) \times PGL_2(\mathbb{Z}_l)$. The action of C is transitive on the vertices of \mathcal{A} .

Proof of Proposition 3.6. In view of Lemma 3.4, it suffices to show that $\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1$ implies that Γ_0 is a period subgroup. Suppose therefore that $\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1$. Then $\gcd(n, pl) = 1$. The result will therefore follow from Lemma 3.5, if we can show that Γ_0 is not contained in $\Gamma_p \cup \Gamma_l$. By symmetry it is enough to prove that if Γ_0 contains an element $b = \psi(y) \in \Gamma_l - \{1\}$, then it also contains an element $a = \psi(x) \in \Gamma_p - \{1\}$. For then the element ba does not lie in $\Gamma_p \cup \Gamma_l$.

Write $y = y_0 + z_0(c_1i + c_2j + c_3k)$, where $c_1, c_2, c_3 \in \mathbb{Z}$ are relatively prime and $n = n(y) = c_1^2 + c_2^2 + c_3^2$. The quaternion y represents the element b of Γ_l of word length $\ell(b) = s > 0$. By Lemma 3.3, b acts by translation of distance s along an axis L_b containing O .

The element of $GL_2(\mathbb{Q}_p) \times GL_2(\mathbb{Q}_l)$ corresponding to y in the formula (1) has eigenvalues $y_0 \pm z_0\sqrt{-n}$. The assumption $\left(\frac{-n}{p}\right) = \left(\frac{-n}{l}\right) = 1$ implies that $\sqrt{-n}$ exists in both \mathbb{Q}_p and \mathbb{Q}_l and therefore that b is diagonalizable in G . In other words, there exists an element $h \in G$ such that $h^{-1}bh \in C$.

The group hCh^{-1} acts by translation on the apartment $h\mathcal{A}$. Also the element $b \in hCh^{-1} \cap \Gamma_l$ acts by translation on the apartment $h\mathcal{A}$, in a direction which will be called “vertical.” Now $h\mathcal{A}$ necessarily contains the axis L_b of b , by [1, Theorem II.6.8(3)]. In particular, $O \in h\mathcal{A}$.

Choose $g \in hCh^{-1}$ to act on $h\mathcal{A}$ by horizontal translation. Consider the horizontal strip H in $h\mathcal{A}$ obtained by translating the vertical segment $[O, bO]$ (see Fig. 1).

Since Γ acts freely and transitively on the vertices of Δ , each vertical segment $g^i[O, bO]$ of H lies in the Γ -orbit of precisely one segment of the form $[O, \gamma O]$, $\gamma \in \Gamma_l$, $\ell(\gamma) = s$. Moreover, there are only finitely many such segments $[O, \gamma O]$.

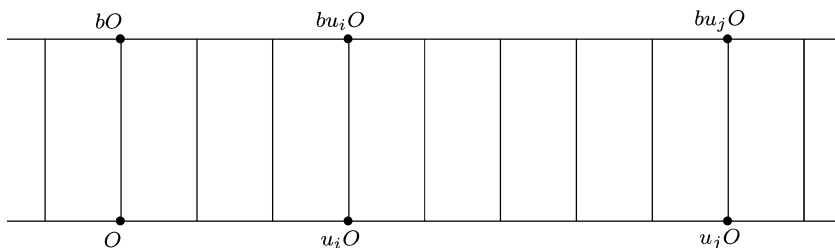


Fig. 1. The horizontal strip H .

If $i > 0$ then $g^i O = u_i O$, for some $u_i \in \Gamma_p - \{1\}$. Since b and g commute, we have $g^i b O = b g^i O = b u_i O$. That is, $g^i [O, b O] = [u_i O, b u_i O]$, which lies in the Γ -orbit of the segment $[O, u_i^{-1} b u_i O]$. By the finiteness assertion in the preceding paragraph, there exist integers $j > i > 0$ such that

$$[O, u_i^{-1} b u_i O] = [O, u_j^{-1} b u_j O].$$

By freeness of the action of Γ ,

$$u_i^{-1} b u_i = u_j^{-1} b u_j,$$

and $u_i \neq u_j$. Therefore $ab = ba$, where $a = u_i u_j^{-1} \in \Gamma_p - \{1\}$. \square

A maximal Abelian subgroup Γ_0 of Γ may be isomorphic to \mathbb{Z} . Here is a way of providing some examples.

Corollary 3.7. *Suppose that $a \in \Gamma_p - \{1\}$, and $n = n(a)$ satisfies*

$$\left(\frac{-n}{p}\right) = 1, \quad \left(\frac{-n}{l}\right) = -1.$$

Then $Z_\Gamma(a) < \Gamma_p$ is a maximal Abelian subgroup of Γ , and $Z_\Gamma(a) \cong \mathbb{Z}$. A similar assertion applies to elements of $\Gamma_l - \{1\}$.

Proof. The hypothesis implies that $\gcd(n, pl) = 1$. If $Z_\Gamma(a) \not\subset \Gamma_p$, then $Z_\Gamma(a)$ contains an element $\gamma \notin \Gamma_p \cup \Gamma_l$. Therefore $Z_\Gamma(a) = Z_\Gamma(\gamma)$ is a period group, by Lemma 3.5. But this implies $\left(\frac{-n}{p}\right) = 1$, by Proposition 3.6, a contradiction. \square

Example 3.8. Let $\Gamma = \Gamma_{3,5}$. This group has a presentation with generators $\{a_1, a_2, b_1, b_2, b_3\}$ and relators

$$\{a_1 b_1 a_2 b_2, a_1 b_2 a_2 b_1^{-1}, a_1 b_3 a_2^{-1} b_1, a_1 b_3^{-1} a_1 b_2^{-1}, a_1 b_1^{-1} a_2^{-1} b_3, a_2 b_3 a_2 b_2^{-1}\},$$

where

$$\begin{aligned} a_1 &= \psi(1 + j + k), & a_1^{-1} &= \psi(1 - j - k), \\ a_2 &= \psi(1 + j - k), & a_2^{-1} &= \psi(1 - j + k), \\ b_1 &= \psi(1 + 2i), & b_1^{-1} &= \psi(1 - 2i), \\ b_2 &= \psi(1 + 2j), & b_2^{-1} &= \psi(1 - 2j), \\ b_3 &= \psi(1 + 2k), & b_3^{-1} &= \psi(1 - 2k). \end{aligned}$$

The subgroup $\langle a_1 \rangle = Z_\Gamma(a_1) < \Gamma_3$ is maximal Abelian in Γ by Corollary 3.7, since $n(a_1) = 2$, $\left(\frac{-2}{3}\right) = 1$ and $\left(\frac{-2}{5}\right) = -1$.

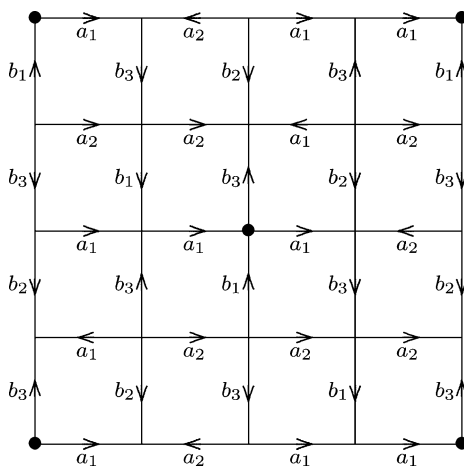


Fig. 2. Part of a periodic apartment for $\Gamma_0 < \Gamma_{3,5}$.

The subgroup $\langle a_1 a_2^{-1} a_1^2 \rangle = \langle \psi(-5 - 6i - 2j + 4k) \rangle$ is not maximal Abelian. It is contained in the period subgroup

$$\Gamma_0 = \langle a_1 a_2^{-1} a_1^2, b_3 b_2^{-1} b_3^{-1} b_1 \rangle \cong \mathbb{Z}^2.$$

Indeed, $n(\Gamma_0) = n(a_1 a_2^{-1} a_1^2) = 14$, $\left(\frac{-14}{3}\right) = 1$, $\left(\frac{-14}{5}\right) = 1$. Note that $b_3 b_2^{-1} b_3^{-1} b_1 = \psi(-11 + 18i + 6j - 12k)$. Part of the period lattice for Γ_0 is illustrated in Fig. 2.

Example 3.9. Let $\Gamma = \Gamma_{3,5}$. Consider $b_1 a_1 b_1^{-1} = \psi(5 - 7j + k)$. By Example 3.8, $\langle a_1 \rangle$ is maximal Abelian in Γ . Therefore so also is $\Gamma_0 = \langle b_1 a_1 b_1^{-1} \rangle = b_1 \langle a_1 \rangle b_1^{-1}$. Now $\gamma = b_1 a_1^6 b_1^{-1} = a_2 a_1^{-1} a_2^{-2} a_1^{-1} a_2 = \psi(5(23 + 14j - 2k)) = \psi(x) \in \Gamma_3$, with $|x|^2 = 5^2 \cdot 3^6$. Also $n(x) = n(\Gamma_0) = 50$, $\left(\frac{-50}{3}\right) = 1$ and $\left(\frac{-50}{5}\right) = 0$. There is a periodic horizontal strip of height 2 (Fig. 3), upon which γ acts by translation. This strip is the union of the axes of γ .

Example 3.10. Let $\Gamma = \Gamma_{3,5}$. Conjugating the period subgroup $\langle a_1 a_2^{-1} a_1^2, b_3 b_2^{-1} b_3^{-1} b_1 \rangle$ of Example 3.8 by a_2 gives the group

$$\begin{aligned} \Gamma_0 &= \langle a_2 a_1 a_2^{-1} a_1^2 a_2^{-1}, a_2 b_3 b_2^{-1} b_3^{-1} b_1 a_2^{-1} \rangle = \langle a_2 a_1 a_2^{-1} a_1^2 a_2^{-1}, b_2 b_1^{-1} b_2^2 \rangle \\ &= \langle \psi(-15 + 10i + 2j + 20k), \psi(-11 - 10i - 2j - 20k) \rangle \cong \mathbb{Z}^2, \end{aligned}$$

which is not a period subgroup since $n(\Gamma_0) = 126$, $\left(\frac{-126}{3}\right) = 1$ and $\left(\frac{-126}{5}\right) = 0$.

One could conjecture that every maximal Abelian subgroup of Γ is conjugate to either a period subgroup or to a subgroup of Γ_p or Γ_l . The next example shows that this conjecture is not true. We need the following definition and Lemma 3.11:

If $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$, let $m(x) = |x|^2 - \Re(x)^2 = x_1^2 + x_2^2 + x_3^2$, where $\Re(x) = x_0$ denotes the real part of x . Observe that $m(x) = \lambda^2 n(x)$ for some integer λ .

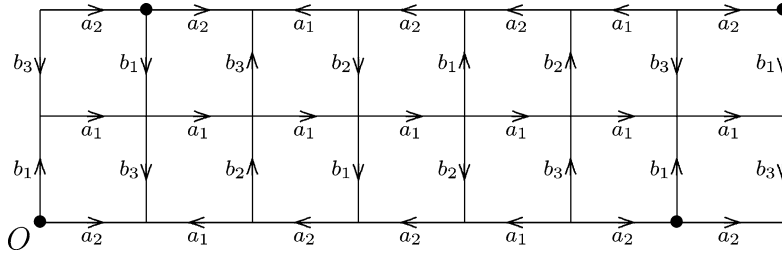


Fig. 3. Part of a periodic horizontal strip.

Lemma 3.11. Let $x, y \in \mathbb{H}(\mathbb{Z})$, then $m(xy\bar{x}) = (|x|^2)^2 m(y)$.

Proof. Using the rules $\Re(xy) = \Re(yx)$ and $|xy|^2 = |x|^2|y|^2$, we conclude:

$$m(xy\bar{x}) = |xy\bar{x}|^2 - \Re(xy\bar{x})^2 = (|x|^2)^2|y|^2 - (|x|^2\Re(y))^2 = (|x|^2)^2 m(y). \quad \square$$

Example 3.12. Let $\Gamma = \Gamma_{3,5}$ and $a_2b_3 = \psi(3 + 2i + j + k)$. The group $\Gamma_0 = Z_\Gamma(a_2b_3)$ is a maximal Abelian subgroup of Γ such that $n(\Gamma_0) = 6$. We fix any element $\gamma = \psi(x) \in \Gamma$.

The maximal Abelian subgroup $\gamma\Gamma_0\gamma^{-1}$ is not a subgroup of Γ_3 or Γ_5 , since $\gamma a_2b_3\gamma^{-1} \in \gamma\Gamma_0\gamma^{-1}$ is the ψ -image of $x(3 + 2i + j + k)\bar{x}$ whose norm is a product of an odd power of 3 and an odd power of 5.

We claim that $\gamma\Gamma_0\gamma^{-1}$ is not a period subgroup. If $|x|^2 = 3^r 5^s$, $r, s \geq 0$, then by Lemma 3.11

$$(3^r 5^s)^2 \cdot 6 = m(x(3 + 2i + j + k)\bar{x}) = \lambda^2 n(\gamma\Gamma_0\gamma^{-1})$$

for some integer λ . It follows that $3 \mid n(\gamma\Gamma_0\gamma^{-1})$, in particular

$$\left(\frac{-n(\gamma\Gamma_0\gamma^{-1})}{3} \right) = 0$$

and Proposition 3.6 proves the claim.

Since any maximal Abelian subgroup of rank 2 is conjugate to a period subgroup, it also follows that $\Gamma_0 \cong \mathbb{Z}$. See Fig. 4 for a periodic vertical strip of width 1 which is globally invariant under the action of a_2b_3 . Note that $(a_2b_3)^2 = b_2b_3$. Therefore a_2b_3 acts upon the strip by glide reflection and the unique axis of a_2b_3 is the vertical central line of the strip.

It is well known that period subgroups in Γ always exist. See for example [8, Proposition 4.2] for an elementary proof of this fact, using doubly periodic tilings of the Euclidean plane by unit squares. We mention a corollary of this in terms of integer quaternions.

Corollary 3.13. Given any pair (p, l) of distinct odd primes, there are $x, y \in \tilde{\Gamma}$ and $1 \leq r \leq 4(p + 1)^2(l + 1)^2$ such that $xy = yx$ and

$$|x|^2 = p^r, \quad |y|^2 = l^r, \quad \left(\frac{-n(x)}{p} \right) = \left(\frac{-n(y)}{l} \right) = 1.$$

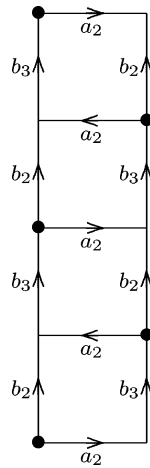


Fig. 4. Part of a periodic vertical strip.

The integer r in this corollary comes from the constructive proof of [8, Proposition 4.2], and its upper bound is certainly not optimal. In fact, if $p, l \equiv 1 \pmod{4}$, there is a direct proof of Corollary 3.13 (with $r = 1$), applying the Two Square Theorem.

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