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Geometrical and physical interpretation of evolution governed by general complex algebra

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Abstract

In this paper we explore a geometrical and physical matter of the evolution governed by the generator of General Complex Algebra, GC_2. The generator of this algebra obeys a quadratic polynomial equation. It is shown that the geometrical image of the GC_2-number is given by a straight line fixed by two given points on Euclidean plane. In this representation the straight line possesses the norm and the argument. The motion of the straight line conserving the norm of the line is described by evolution equation governed by the generator of the GC_2-algebra. This evolution is depicted on the Euclidean plane as rotational motion of the straight line around the semicircle to which this line is tangent. Physical interpretation is found within the framework of the relativistic dynamics where the quadratic polynomial is formed by mass-shell equation. In this way we come to a new representation for the momenta of the relativistic particle.

Keywords: Relativistic dynamics; Complex algebra; Hyperbolic geometry; Euclidean plane; Trigonometry; Evolution equation; Massless particle

1. Introduction

I.M. Yaglom was first who mentioned that beside of the complex numbers one may consider the General Complex Number $z = x + \mathbf{e}y$ with the generator obeying the quadratic equation [1]

$$\mathbf{e}^2 = a_1\mathbf{e} - a_0. \tag{1.1}$$

Some properties of this algebra had been studied by Fjelstad and Gal [2]. In Ref. [3] the idea of general complex algebras has been extended to the case of generators obeying $n \geqslant 2$-degree polynomial equations. These algebras were denominated by the symbol GC_n. The GC_n algebra is n-dimensional commutative algebra which generalize the multi-complex algebra MC_n. The latter is generated by $\mathbf{e}$ obeying the equation $\mathbf{e}^n = \pm 1$. The MC_n algebra and extensions of cosine–sine functions induced by this algebra had been used in n-order phase space formulation of the classical mechanics (see Ref. [4]). MC_n algebras form the commutative part of the Generalized Clifford algebras [5]. These algebras were used in quantum mechanics based on para-statistic description of elementary particles [6]. Also,

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on the basis of Generalized Clifford algebras had been constructed powerful method of solution of the system of non-linear algebraic equations [7]. In Ref. [3] the GC_n algebra has been used in order to build nth order oscillator model and nth order Hamilton dynamic equations. Our further investigations of these algebras pursue an object to find faithful geometrical representation of GC_n-numbers and their applications in the theory of generalized relativistic dynamics.

The link between hyperbolic complex algebra and the relativistic kinematics has been noted long time ago (see, for instance, [8–10] and references therein). This link is a natural consequence of the addition formula for the velocity defined with respect to the coordinate time. The velocity addition formula can be represented as an addition formula for the hyperbolic tangent [11]. Then, the components of the velocity defined with respect to the proper time are hyperbolic cosine–sine functions, which can be considered in quality of components of a hyperbolic complex number. The geometrical interpretation of the hyperbolic complex number is given by analogue with the interpretation of a complex number by using pseudo-Euclidean plane instead of two-dimensional Euclidean plane. In this case the unimodular multiplicative group of the hyperbolic numbers are treated as rotations and dilatation on the pseudo-Euclidean plane [12].

In this paper we shall restrict ourselves with the case of general complex algebra of second order and its geometrical representation. We show that the geometrical representation of GC_2 number is closely connected with the classical relativistic dynamics. The generator of GC_2-algebra induces an evolution equation. Thus, beside the evolution governed by the Lorentz-force equations in the relativistic dynamics one meets with a special kind of evolution generated by the mass-shell equation given by quadratic polynomial. The situation is quite unusual in the scope of the classical mechanics. The main task of the present paper is to find adequate geometrical and physical interpretations for the motions generated by the GC_2-algebra. We shall show that the GC_2-algebra admits geometrical interpretation on ordinary Euclidean plane: the image of the GC_2-number on the Euclidean plane is a straight line fixed by two points on the line. This idea leads to a new method of analytical description of the straight line: any straight line is associated with the quadratic polynomial and possesses a norm and an argument. In this way we come to the interesting link between Euclidean and Hyperbolic Geometries. According to the new analytical method the rotations of the line are described by evolution equation governed by the generator of GC_2 algebra. The physical usage of the geometrical interpretation of the general complex algebra is realized within the framework of the relativistic mechanics which implies a quadratic polynomial formed by mass-shell equation. The momenta of the relativistic particle are defined via trigonometry induced by GC_2 algebra. In this representation momenta of the relativistic particle possess regular behavior near the zero-mass point. The limit to the zero-mass has a crucial importance in the physical interpretation of the hyperbolic argument as a inverse value of the momentum of a massless particle.

The paper is presented by the following sections.

In Section 2 we recall the basic notions of the General Complex Algebra, GC_2, and give an geometrical interpretation for GC_2-number on Euclidean plane: it is shown that on the Euclidean plane the GC_2-number is presented by a straight line. In Section 3 the relationship between hyperbolic argument and curvature of the hyperbolic planes is established. Section 4 presents elements of the relativistic dynamics. The relativistic motion is depicted as rotational motion of the straight line, tangent the semicircle with radius equal to mass of the particle, around this circle. A new representation for the momenta of the relativistic particle is found.

Lemmas, definitions and theorems are numerated, for instance, as follows *Lemma a.b*, where a means number of the section, b is item of the lemma.

2. General Complex Algebra and its geometrical interpretation

2.1. General complex algebra

The simplest (but important) generalization of the complex algebra, denominated as General Complex Algebra GC_2, is defined by unique generator **e** satisfying the quadratic equation

$$X^2 - 2P_0X + P^2 = 0, \quad P_0^2 \geqslant P^2. \tag{2.1}$$

The coefficients of this equation P_0, P^2 are real numbers, ordered by $P_0^2 \geqslant P^2 \geqslant 0$. Due to this condition the eigenvalues are defined by real positive numbers λ_1, λ_2. The matrix solution of Eq. (2.1) is given by the following (2×2) matrix [3]:

$$E := \begin{pmatrix} 0 & -P^2 \\ 1 & 2P_0 \end{pmatrix}. \tag{2.2}$$

This matrix can be considered as a natural matrix representation for $\mathbf{e}$. The isomorphism with the matrix algebra is used in order to define *modulus* of the general complex number. It can be shown that the determinant of the matrix $Z = r_0 + Er_1$ is a unique candidate to be the squared *modulus* of the $z = r_0 + \mathbf{e}r_1$. Define

$$|z|^2 \to \mathrm{Det}(Z) = \mathrm{Det}\begin{pmatrix} r_0 & -P^2 r_1 \\ r_1 & 2P_0 r_1 + r_0 \end{pmatrix} = r_0(r_0 + 2P_0 r_1) + P^2 r_1 r_1. \tag{2.3}$$

The conjugated GC_2-number $\bar{z} = \bar{r}_0 + \mathbf{e}\bar{r}_1$ in the matrix representation is given by adjoint to Z matrix Z^+, so that

$$ZZ^+ = \mathrm{Det}(Z)I,$$

where

$$Z^+ = \begin{pmatrix} r_0 + 2P_0 r_1 & P^2 r_1 \\ -r_1 & r_0 \end{pmatrix} \to \bar{z} = (r_0 + 2P_0 r_1) - \mathbf{e}r_1. \tag{2.4}$$

In the same way that the usual complex number system can be used to describe trigonometry, the general complex number system induces its trigonometric functions [2]. Let us start from Euler formulae

$$\exp(E\phi) = g_0(\phi) + Eg_1(\phi), \qquad \exp(\mathbf{e}\phi) = g_0(\phi) + \mathbf{e}g_1(\phi). \tag{2.5}$$

Let $[\lambda_1, \lambda_2]$ be the set of eigenvalues of the matrix E. The eigenvalue problem for E is formulated as follows

$$\begin{pmatrix} 0 & -P^2 \\ 1 & 2P_0 \end{pmatrix}\begin{pmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{2.6}$$

Consider evolution equation governed by the matrix E:

$$\frac{d}{d\phi}\begin{pmatrix} g_0(\phi) \\ g_1(\phi) \end{pmatrix} = \begin{pmatrix} 0 & -P^2 \\ 1 & 2P_0 \end{pmatrix}\begin{pmatrix} g_0(\phi) \\ g_1(\phi) \end{pmatrix}. \tag{2.7}$$

The solutions of the eigenvalue problem (2.6) help to find solutions of the evolution equation (2.7). The explicit form of these solutions with initial conditions $g_0(0) = 1$, $g_1(0) = 0$ are given by formulae

$$g_0(\phi) = \frac{\lambda_2 \exp(\lambda_1\phi) - \lambda_1 \exp(\lambda_2\phi)}{\lambda_2 - \lambda_1}, \qquad g_1(\phi) = \frac{\exp(\lambda_2\phi) - exp(\lambda_1\phi)}{\lambda_2 - \lambda_1}. \tag{2.8}$$

On the other hand, Eq. (2.7) one may consider as formulae of differentiation for g-functions.

Notice, however, that the exponential function $\exp(\mathbf{e}\phi)$ defined in (2.5) is not the unit number of GC_2-algebra, i.e. $|\exp(\mathbf{e}\phi)|^2 \neq 1$. This fact is a consequence of the next formula

$$\mathrm{Det}\big(\exp(E\phi)\big) = \mathrm{Det}\begin{pmatrix} g_0 & -P^2 g_1 \\ g_1 & 2P_0 g_1 + g_0 \end{pmatrix} = \exp(2P_0\phi). \tag{2.9}$$

The unit GC_2-number $u(\phi)$ is obtained by multiplying $\exp(\mathbf{e}\phi)$ on the factor $\exp(-P_0\phi)$:

$$u(\phi) := \exp(-P_0\phi)\exp(\mathbf{e}\phi) = c(\phi) + \mathbf{e}s(\phi). \tag{2.10}$$

The explicit form of the s–c-functions are obtained from (2.8) by using the same factor:

$$c(\phi) = \frac{\lambda_2 \exp(-m\phi) - \lambda_1 \exp(m\phi)}{2m}, \qquad s(\phi) = \frac{\exp(m\phi) - \exp(-m\phi)}{2m}, \tag{2.11}$$

where $2m = \lambda_2 - \lambda_1$. The c–s-functions obey the identity

$$\big|u(\phi)\big|^2 = c(\phi)\big(c(\phi) + 2P_0 s(\phi)\big) + P^2 s^2(\phi) = 1, \tag{2.12}$$

which is an analogue of trigonometric identity $\cos^2(\phi) + \sin^2(\phi) = 1$. The formulae of differentiation for s–c-functions are derived from (2.7):

$$\frac{d}{d\phi}\begin{pmatrix} c(\phi) \\ s(\phi) \end{pmatrix} = \begin{pmatrix} -P_0 & -P^2 \\ 1 & P_0 \end{pmatrix}\begin{pmatrix} c(\phi) \\ s(\phi) \end{pmatrix}. \tag{2.13}$$

Lemma 2.1. *GC_2-number z with non-trivial modulus $|z|$ possesses a polar representation*:

$$z = |z|u(\phi).$$

2.2. Analytical representation of the straight line by GC_2 number

Consider straight line $\mathcal{L}$ on Euclidean plane $\mathcal{E}\sqcap$. Let us fix two points on $\mathcal{E}\sqcap$ with coordinates (x_1, y_1), (x_2, y_2) in the first quadrant, so that $0 \leqslant x_1 \leqslant x_2$. Through these points passes the line $\mathcal{L}$ which cuts X-axis in $x_0 \geqslant 0$. This line is described by the equation $y = ax + b$, where the coefficients a, b are defined by formulae

$$a = \frac{y_2 - y_1}{x_2 - x_1}, \qquad b = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}. \tag{2.14}$$

The coefficient a is related with the angle θ via tangent function: $a = \tan(\theta)$. The coordinates y_1, y_2 are linear functions of the coordinates x_1, x_2:

$$y_1 = ax_1 + b, \qquad y_2 = ax_2 + b. \tag{2.15}$$

The coordinates x_1, x_2 have a dimension of a distance and mark a certain length from the origin of the coordinate system O. By using the Vieta's formulae for each pair of the values $\lambda_1 = x_1$, $\lambda_2 = x_2$ one may put in correspondence the quadratic polynomial

$$X^2 - 2P_0 X + P^2 = 0, \quad \text{with } 2P_0 = x_1 + x_2,\ P^2 = x_1 x_2. \tag{2.16}$$

Let E be matrix solution of Eq. (2.16). Define GC_2-number by

$$Z = aE + b, \quad Z \in GC.$$

Consider diagonal matrix of the eigenvalues of the matrix Z,

$$\Lambda = \begin{pmatrix} y_1 = ax_1 + b & 0 \\ 0 & y_2 = ax_2 + b \end{pmatrix}. \tag{2.17}$$

Determinant of Λ, which equal to $|Z|^2$, is calculated by taking into account (2.16),

$$|Z|^2 = y_1 y_2 = b(b + 2P_0 a) + P^2 a^2.$$

This expression is positive if the inequality $x_0 \leqslant x_1 \leqslant x_2$ is satisfied.

Definition 2.1. The positive value

$$n := (y_1 y_2)^{1/2} = \big(b(b + 2P_0 a) + P^2 a^2\big)^{1/2} \tag{2.18}$$

is called the norm of the line $\mathcal{L}$ with respect to the parapets $x = x_1$, $x = x_2$.

If $y_1 \neq 0$, $y_2 \neq 0$, then according to Lemma 2.1 the number $Z = aE + b$, $Z \in GC$ possesses polar representation. Define c–s-functions of the argument ϕ by

$$c(\phi) = \frac{b}{n}, \qquad s(\phi) = \frac{a}{n}. \tag{2.19}$$

The elements of diagonal matrix Λ in (2.17) are expressed via c–s-functions

$$y_1 = n\big(s(\phi)x_1 + c(\phi)\big), \qquad y_2 = n\big(s(\phi)x_2 + c(\phi)\big). \tag{2.20}$$

By using formulae (2.11) for s–c-functions, we obtain

$$y_1 = n\exp(-m\phi), \quad y_2 = n\exp(m\phi), \quad \text{with } 2m = x_2 - x_1. \tag{2.21}$$

Definition 2.2. Analytical representation given by formulae (2.20), (2.21) is called a polar representation of the straight line, where n is a norm and the variable ϕ is an argument of the line $\mathcal{L}$.

Thus, for the line $\mathcal{L}$ passing through certain parapets $x = x_1 \geqslant x_0$, $x = x_2 \geqslant x_0$ one may find the norm n and the argument of the line ϕ by

$$y_1 y_2 = n^2, \qquad \frac{1}{x_1 - x_2}\log\left(\frac{y_1}{y_2}\right) = \phi. \tag{2.22}$$

2.3. Description of motions of the straight line in Euclidean plane defined by the norm and the argument

Within the framework of our description the line $\mathcal{L}$ possesses: (1) the norm and (2) the argument. Now let us explore the following tasks:

(1) Find a motion of $\mathcal{L}$ which preserves the argument but changes the norm of the line.
(2) Find a motion of $\mathcal{L}$ which preserves the norm but changes the argument of the line.

The answer to the first task is given by the following:

Lemma 2.2. *All lines passing through the point $M(x_0, 0)$ possess same argument.*

Proof. Consider two lines: $y(1) = a_1 x + b_1$, $y(2) = a_2 x + b_2$, passing through the point $M(x_0, 0)$. The coefficients a_i, b_i, $i = 1, 2$ obey the following relationships

$$-x_0 = \frac{b_1}{a_1} = \frac{b_2}{a_2}, \qquad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{\tan\theta_1}{\tan\theta_2}. \tag{2.23}$$

Formula for the first line in polar representation is given by

$$y(1) = a_1 x + b_1 = n_1\big(s(\phi)x + c(\phi)\big), \quad \text{where } c(\phi) = \frac{b_1}{n_1},\ s(\phi) = \frac{a_1}{n_1}. \tag{2.24}$$

For the second line we write

$$y(2) = a_2 x + b_2 = n_2\big(\tilde{s}(\phi)x + \tilde{c}(\phi)\big), \quad \text{where } \tilde{c}(\phi) = \frac{b_2}{n_2},\ \tilde{s}(\phi) = \frac{a_2}{n_2}. \tag{2.25}$$

By using relationships (2.23)–(2.25) we come to the conclusion that

$$n_2 = n_1\frac{\tan\theta_2}{\tan\theta_1}, \qquad \tilde{s}(\phi) = s(\phi), \qquad \tilde{c}(\phi) = c(\phi).$$

Thus all these lines passing through point $M(x_0, 0)$ are described by the same s–c-functions $s(\phi), c(\phi)$. Hence, they have same argument ϕ and differ by the norms. □

The second task is to find motion of the line which changes the argument of the line but preserves its norm.

Draw line $\mathcal{L}$ (Fig. 1) and take on X-axis a point P. Throughout this point draw the line parallel to Y-axis. The intersection of this line with $\mathcal{L}$ denote by N. Let N be a center of the circle with radius $r = NP$ which acroses $\mathcal{L}$ in two points: A and B. Through the points A and B draw two lines parallel to Y-axis. Intersections of these lines with X-axis denote by C and D, correspondingly. Draw semicircle $\mathcal{C}$ with the center in P and with the diameter $2m = CD$. The angle $\angle APB$ is rectangle. Denote $\psi_1 = \angle APC$, $\psi_2 = \angle BPD$. Rotate rectangle $\angle APB$ around the point P. The sides of the rectangle will across the lines $x = CO$, $x = DO$ at the points A' and B'. Denote by $\mathcal{L}_1$ the line passing through A', B'.

Lemma 2.3. *The lines $\mathcal{L}$ and $\mathcal{L}_1$ possess the same norm and differ with the arguments.*

Proof. $DB = m\tan(\psi_2)$, $CA = m\tan(\psi_1) = m\cot(\psi_2)$, so that $y_1 y_2 = m^2$ for any $0 < \psi_2 < \pi/2$. Hence, $n(\mathcal{L}) = n(\mathcal{L}_1)$, i.e. the norms of the lines $\mathcal{L}$ and $\mathcal{L}_1$ are same whereas they are described by different arguments. □

Notice, in the present choice of the parapets the norm of the line is equal to one-half interval between parapets. Obviously, this not true in general case.

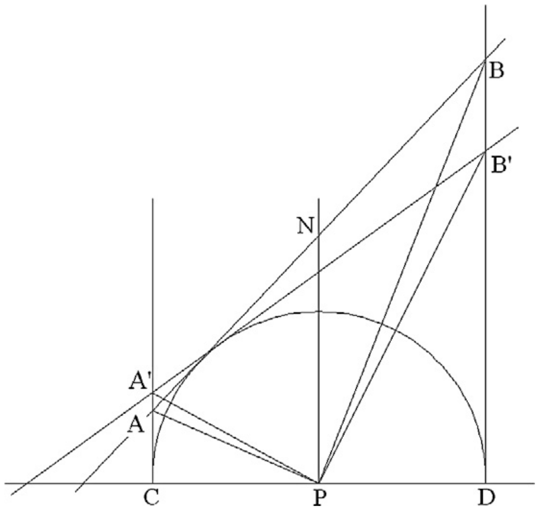

Fig. 1.

Lemma 2.4. *The lines $\mathcal{L}$ and $\mathcal{L}_1$ are tangent to the semicircle $\mathcal{C}$.*

The proof is fulfilled by using the elementary operations of Euclidean geometry.

The results of this section let us summarize by the following:

Theorem 2.1. *Rotational motion of $\mathcal{L}$ around semicircle $\mathcal{C}$, to which the line remains tangent, is described by functions*

$$y(x_1) = m\exp(-m\phi), \qquad y(x_2) = m\exp(m\phi)$$

where $2m = x_2 - x_1$, m is the radius of the semicircle, and x_1, x_2 are the end points of the semicircle on the X-axis.

3. Relationship between hyperbolic argument and curvature of hyperbolic geometry

3.1. Relationship between Euclidean angle and hyperbolic argument

Let $\mathcal{L}$ be a line passing trough the origin O and tangent to the semicircle $\mathcal{C}$ with center at the point P on X-axis. Let $\mathcal{L}_1$ be a line passing through point O' with coordinates $(x_0, 0)$ and tangent to the same semicircle $\mathcal{C}$. Draw parapets from end points of the semicircle. The coordinates of the parapets will $x = X_2$, $x = X_1$, if consider with respect to the origin O, or $x = x_2$, $x = x_1$, if they are measured from the point O', so that, $X_1 = x_1 + x_0$, $X_2 = x_2 + x_0$. Let us explore rotational motion of the line around the semicircle $\mathcal{C}$. At the initial point the line $\mathcal{L}$ is associated with the following polynomial:

$$X^2 - 2\overset{0}{P}_0 X + \overset{0}{P}{}^2 = 0, \tag{3.1}$$

where

$$\overset{0}{P}_0 = \frac{1}{2}(X_2 + X_1), \qquad \overset{0}{P}{}^2 = X_1 X_2, \qquad m = \frac{1}{2}(X_2 - X_1),$$

and

$$P^2(\phi_0) = \overset{0}{P}{}^2, \qquad P_0(\phi_0) = \overset{0}{P}_0.$$

During the motion the distances P and P_0 change, m is a constant. The set of dynamic quantities $\overset{0}{P}{}^2$, $\overset{0}{P}_0$ as coefficients of the polynomial (3.1) form the matrix E which generates the evolution with respect to the parameter ϕ. According to Theorem 2.1 this geometrical motion is governed by the evolution equation

$$\frac{d}{d\phi}\begin{pmatrix} c(\phi) \\ s(\phi) \end{pmatrix} = \begin{pmatrix} -\overset{0}{P}_0 & -\overset{0}{P}{}^2 \\ 1 & \overset{0}{P}_0 \end{pmatrix}\begin{pmatrix} c(\phi) \\ s(\phi) \end{pmatrix}. \tag{3.2}$$

At the initial point the line passes from the origin, $b = 0$. Hence $c(\phi_0) = 0$. According to formulae (2.11) this gives

$$c(\phi_0) = \frac{1}{2m}\big(X_2 \exp(-m\phi_0) - X_1 \exp(m\phi_0)\big) = 0, \quad \text{or} \quad \exp(2m\phi_0) = \frac{X_2}{X_1}. \tag{3.3}$$

Let the argument ϕ corresponds to the position of line $\mathcal{L}_1$. Then, $2P_0 = x_1 + x_2$, $P^2 = x_1x_2$, so that, $\overset{0}{P}_0 = P_0 + x_0$. The function $c(\phi)$ is defined by

$$c(\phi) = \frac{1}{2m}\big(X_2 \exp(-m\phi) - X_1 \exp(m\phi)\big) = \frac{1}{2m}\big(x_2 \exp(-m\phi) - x_1 \exp(m\phi)\big) - x_0 s(\phi). \tag{3.4}$$

Lemma 3.5. *The following relationship holds true*

$$x_2 \exp(-m\phi) = x_1 \exp(m\phi). \tag{3.5}$$

Proof. By definition $a = ms(\phi)$, $b = mc(\phi)$, and $a = \tan\theta$, $-\frac{b}{x_0} = \tan\theta$. Also, $m = P\tan\theta$. Hence,

$$s(\phi) = \frac{\tan\theta}{m} = \frac{1}{P} \quad \text{and} \quad c(\phi) = \frac{b}{m} = -\frac{x_0 \tan\theta}{m} = -x_0 s(\phi). \tag{3.6}$$

By comparing (3.4) with (3.6) we come to Eq. (3.5). □

From (3.5) it follows that

$$\exp(2m\phi) = \frac{x_2}{x_1}. \tag{3.7}$$

This formula can be also deduced from formulae (2.21). In fact, it is easily seen that

$$\exp\big(2m\phi(b)\big) = \frac{y_2}{y_1} = \frac{aX_2 + b}{aX_1 + b} = \frac{X_2 - x_0}{X_1 - x_0} = \frac{x_2}{x_1}.$$

Formulae (2.11) and (2.16), (3.1) lead to the following relations between s–c-functions and hyperbolic cosine–sine functions:

$$ms(\phi) = \sinh(m\phi), \qquad s(\phi)\overset{0}{P}_0 + c(\phi) = \cosh(m\phi).$$

From these equations one finds

$$P_0(\phi) = \overset{0}{P}_0 - x_0 = m\coth(m\phi), \qquad P(\phi) = m\frac{1}{\sinh(m\phi)}. \tag{3.8}$$

Now, our purpose is to establish relationship between the argument ϕ and the angle θ. By taking into account that

$$m = P_0 \sin\theta, \tag{3.9}$$

formula (3.7) can be transformed as follows:

$$\exp(2m\phi) = \frac{P_0 + P_0\sin\theta}{P_0 - P_0\sin\theta} = \exp(2m\phi) = \frac{1 + \sin\theta}{1 - \sin\theta}, \quad \text{or} \quad \exp(m\phi) = \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}. \tag{3.10}$$

It is seen that this formula connects the angle θ with the hyperbolic argument ϕ.

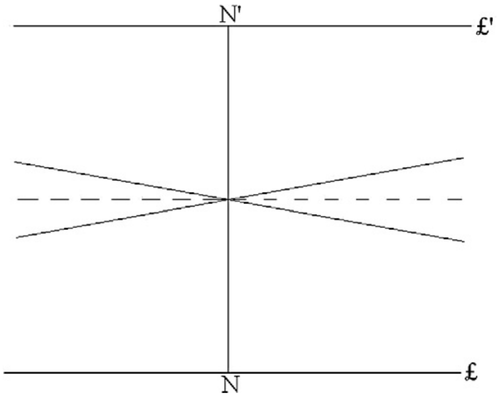

Fig. 2.

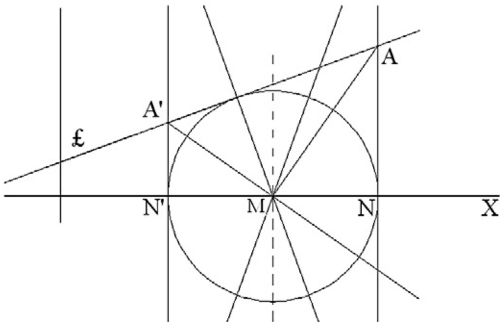

Fig. 3.

3.2. Relationship between hyperbolic argument and curvature of hyperbolic plane

Consider a straight line $\mathcal{L}$ and a point N not on the line (Fig. 2). Let MN be perpendicular to $\mathcal{L}$, and take any point P on $\mathcal{L}$. The line MP cuts $\mathcal{L}$ in P. As the point P moves along $\mathcal{L}$ away from N there two possibilities to consider [13]:

(1) P may return to its starting point after having traversed a finite distance. This is the hypothesis of Elliptic Geometry.

(2) P may continue moving, and the distance NP tend to infinity. This hypothesis is true in ordinary geometry.

The ray MP then tends to a definite limiting position ML, and ML is said to be *parallel* to NA. If P moves along $\mathcal{L}$ in the opposite sense, MP will tend to another limiting position, MK, and $MK \parallel NB$. In Euclidean Geometry, the two rays KM and ML form one line, and the angles $\angle NML$ and $\angle NMK$ are right angles. The hypothesis of Hyperbolic Geometry is that the rays MK, ML are distinct, so that Playfair's axiom is contradicted [13].

Thus, through any point M two parallels ML and MK can be drawn to a given line AB, so that $ML \parallel NA$ and $MK \parallel NB$ (Fig. 2). The angles NML and NMK are, by symmetry, equal, and this angle depends only on the length of the perpendicular $MN = r$. It is called the *angle of parallelism* or the *parallel-angle*, and is denoted by $\Pi(r)$. The dependence of the parallel-angle on the length r is given by the main formula of hyperbolic geometry [14]:

$$\exp\left(-\frac{r}{\mathcal{K}}\right) = \tan\frac{1}{2}\Pi, \tag{3.11}$$

where $\mathcal{K}$ is curvature of the hyperbolic plane.

Notice, the draught in Fig. 2 is represented in symmetric form. Now, choose the line NN' in quality of X-axis where the points N, N' will play the role of points x_2, x_1 (Fig. 3). Draw bisectrix of the angle Π, the angle $\angle N'MA'$. Denote this angle by $\Pi/2$. Draw the rectangle $\angle A'MA$. In Fig. 3 the draught we can look through two points of view. Firstly, we see the draught in the Euclidean plane, secondly, it can be realized as a projection of hyperbolic plane with curvature $\mathcal{K}$. Within notations of Fig. 3, $A'N' = y_1$, $AN = y_2$. Remembering formula (2.21), we write

$$A'N' = y_1 = N'M\exp(m\phi), \qquad AN = y_2 = NM\exp(-m\phi). \tag{3.12}$$

On the other hand,

$$A'N' = N'M\cot\left(\frac{\Pi}{2}\right), \qquad AN = NM\tan\left(\frac{\Pi}{2}\right). \tag{3.13}$$

From (3.13) and (3.12) we come to the equation

$$NA = NM\tan\left(\frac{\Pi}{2}\right) = NM\exp(-m\phi). \tag{3.14}$$

Here $NM = m = r$. Thus, by comparing (3.14) with (3.11) we get the following relationship:

$$\phi = \frac{1}{\mathcal{K}}. \tag{3.15}$$

This formula gives an interpretation of the hyperbolic argument via the curvature of the corresponding hyperbolic plane.

4. Relativistic motion in view of GC_2 algebra

In the previous sections we intentionally used the physical notations for geometric quantities, this provides an advantage to use directly formulae of the previous section. Only, we shall keep in mind that the quantities used above have to be given in appropriate physical units. For that purpose we will introduce the constant c-velocity of the light.

4.1. Relativistic Lorentz-force equations

Consider a motion of the relativistic particle with charge e in the external electromagnetic fields $(\vec{E}, \vec{B})$. The relativistic equations of motion with respect to the proper time τ are given by the Lorentz-force equations. These equations can be written in the Lorentz-covariant form [15], however in order to give the main idea we shall restrict ourselves only by consideration the projection of the Lorentz-force equations on the direction of motion (a covariant form of the full theory of such formulation of the Lorentz-force equations the reader may find in Ref. [16])

$$\frac{dP}{d\tau} = \frac{e}{mc}(\vec{E}\cdot\vec{n})P_0, \quad \frac{dP_0}{d\tau} = \frac{e}{mc}(\vec{E}\cdot\vec{n})P, \quad \vec{n} = \frac{\vec{P}}{P}. \tag{4.1}$$

These equations have to be complemented by connection between momenta and velocity

$$m\frac{dr}{d\tau} = (\vec{P}\cdot\vec{n}), \qquad mc\frac{dt}{d\tau} = P_0, \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{\vec{P}}{P_0}. \tag{4.2}$$

Equations (4.1) imply the first integral of motion

$$P_0^2 - P^2 = M^2c^2. \tag{4.3}$$

In the case of stationary potential field, i.e. when $e\vec{E} = -\vec{\nabla}V(r)$, the Lorentz-force equations imply the other constant of motion, the energy of the relativistic particle $\mathcal{E} = cP_0 + V(r)$. An interpretation of the constant of motion M^2 can be found by comparing with the non-relativistic equations of motion. The following two cases have to be considered separately: (1) $M \neq 0$, and (2) $M = 0$. Suppose that $M \neq 0$. In this case Eqs. (4.1) are reduced to Newtonian equations by supposing that $M = m$. Equations (4.1) admit *a polar representation*:

$$P_0 = A\big(\cosh(\psi) + B\sinh(\psi)\big), \qquad P = A\big(\sinh(\psi) + B\cosh(\psi)\big), \qquad \frac{d\psi}{d\tau} = \frac{e}{mc}(\vec{E}\cdot\vec{n}). \tag{4.4}$$

These solutions will satisfy (4.3) if $A^2(1 - B^2) = M^2c^2$. Hence $B < 1$ for particles with non-vanishing mass. If ψ is chosen so that $\psi = 0$ for the particle in rest, then $B = 0$ and

$$P_0 = mc\cosh(\psi), \qquad P = mc\sinh(\psi).$$

Now suppose that $M = 0$. Then the correspondence with Newtonian mechanics breaks down. Nevertheless, in the case $M = 0$ the relativistic equations of motion (4.4) formally do not lose a sense because the condition $M = 0$ does not entail the equality of the m to zero. In this case m plays the role of a parameter of the equations. Equation of motion for this class of particles inside potential field $\varphi(r)$, $\dim(\varphi) = \dim(c)$, is written as [17]:

$$\frac{d}{d\tau} \log P = \pm(\vec{n} \cdot \vec{\nabla})\varphi(r), \tag{4.5}$$

which also admits the formal solutions

$$P_0 = A\big(\cosh(\psi) + B\sinh(\psi)\big), \qquad P = A\big(\sinh(\psi) + B\cosh(\psi)\big). \tag{4.6}$$

Here the constant B is defined by equation $B^2 = 1$.

4.2. Mass-shell equation

In the mass-shell equation (4.3) the constant of motion is defined by square of the mass, which implies that in this relationship the mass can be presented by a positive, as well, by a negative quantity. Evidently, by using only the square of the mass we ignore an information carried by the signs of the mass. In the relativistic mechanics the value $X = cP_0 - Mc^2$ is considered as kinetic energy of the massive particle. Noteworthy, in the relativistic mechanics we deal with two kinds of the kinetic energies corresponding two signs of the mass. They are solutions of the following quadratic equation:

$$X^2 - 2cP_0X + c^2P^2 = 0, \quad P_0^2 \geqslant P^2 > 0. \tag{4.7}$$

This equation has two real positive solutions which we will denote as follows:

$$q_1^2 = cP_0 - Mc^2, \qquad q_2^2 = cP_0 + Mc^2. \tag{4.8}$$

Here the value M (with capital letter) means the expression $Mc = +\sqrt{P_0^2 - P^2}$, whereas the value m means the mass of the particle. Evidently, one may equalize M with m if $P_0 \neq P$. By using Vieta's formulae from (4.7) and (4.8) we come to the following mapping:

$$c^2P^2 = q_1^2q_2^2, \qquad cP_0 = \frac{1}{2}\big(q_2^2 + q_1^2\big), \qquad Mc^2 = \frac{1}{2}\big(q_2^2 - q_1^2\big). \tag{4.9}$$

In stationary potential field for the pair of constants of motion $\mathcal{E}$, M we can put in correspondence the other pair of constants of motion:

$$\mathcal{E}_1 = q_1^2 + V(r), \qquad \mathcal{E}_2 = q_2^2 + V(r). \tag{4.10}$$

4.3. Evolution generated by mass-shell equation

The evolution generated by quadratic equation (4.7) geometrically is described as a motion of the line $\mathcal{L}$ tangent to the circle with radius m when the point O' runs from O to X_1 (see Fig. 1). In this motion the distances P and P_0 change, m is a constant.

The set of coefficients of the polynomial (4.7) form the matrix E which generates the evolution with respect to the parameter ϕ where $P^2(\phi_0)$, $P_0(\phi_0)$ are initial data of the evolution. Compare this geometrical motion with the physical motion governed by Lorentz-force equation (4.3). The relativistic equations imply two constants of motion, the energy $\mathcal{E}$ and mass M.

Thus, during the motion the constants of motion m, $\mathcal{E}$, as well as, $\mathcal{E}_p$, $\mathcal{E}_q$ are conserved whereas the kinetic energies q_1^2, q_2^2 will change. Hence $X_1 = \mathcal{E}_1$, $X_2 = \mathcal{E}_2$ and $x_0(\phi) = V(r)$. Then,

$$cP_0(\phi_0) = cP_0(\phi) + x_0(\phi),$$

which corresponds to the formula of the energy

$$\mathcal{E} = cP_0 + V(r).$$

Consequently, $cP_0(\phi_0) = \mathcal{E}$, $x_1 = q_1^2$, $x_2 = q_2^2$.

The relativistic motion, when in quality of parameter of evolution is taken the hyperbolic argument ϕ, is given by equations

$$\frac{d}{d\phi}P_0 = -P^2, \qquad \frac{d}{d\phi}P = -P_0 P. \tag{4.11}$$

Solutions of this system are given by

$$P_0(\phi) = mc\coth(mc\phi), \qquad P(\phi) = mc\frac{1}{\sinh(mc\phi)}. \tag{4.12}$$

Compare Eqs. (4.11) with Lorentz-force equations in polar representation (4.4). We come to the following relationship between the parameters of evolution

$$d\psi = P\,d\phi, \qquad P\,d\phi = e(\vec{n}\cdot\vec{E})\,d\tau.$$

Formulae (3.7) together with (4.8) give us the following useful relations between momenta and hyperbolic argument

$$\exp(mc\phi) = \frac{P_0 + mc}{P}, \qquad \exp(-mc\phi) = \frac{P_0 - mc}{P}, \qquad \exp(2mc\phi) = \frac{q_2^2}{q_1^2} = \frac{P_0 + mc}{P_0 - mc}. \tag{4.13}$$

When $q_2^2 = q_1^2$, $m = 0$, whereas $\phi \neq 0$. At the other limit, when q_2^2 and q_1^2 tend to infinity, $m \neq 0$, whereas ϕ will tend to zero.

Consider the limit when $q_1^2 \to q_2^2$. Evidently, this limit corresponds to the *massless particle*, because $2mc = (q_2^2 - q_1^2) = 0$. At this limit formulae for the momenta (4.12) admit regular behavior

$$P(m=0) = \frac{1}{\phi}, \qquad P_0(m=0) = \frac{1}{\phi}.$$

Denote the momenta at the point $m = 0$ by $P(m=0) = P_0(m=0) = \pi_0$. Then hyperbolic argument ϕ obtains the following interpretation:

$$\phi(m=0) = \frac{1}{P(m=0)} = \frac{1}{\pi_0}. \tag{4.14}$$

Use this interpretation of ϕ in formulae (4.12). In this way we come to the mapping from momentum of the massless particle onto the momentum of the massive one

$$\frac{mc}{P} = \sinh\left(\frac{mc}{\pi_0}\right), \qquad \frac{mc}{P_0} = \tanh\left(\frac{mc}{\pi_0}\right). \tag{4.15}$$

Formula for the velocity is obtained by using relationship $cP = vP_0$, which follows from the last formula of Eq. (4.2). Then,

$$\frac{v}{c} = \frac{1}{\cosh(\frac{mc}{\pi_0})}.$$

Formula (4.13) now can be written as follows:

$$\exp\left(2\frac{mc}{\pi_0}\right) = \frac{1+\sqrt{1-\frac{v^2}{c^2}}}{1-\sqrt{1-\frac{v^2}{c^2}}}.$$

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