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Taylor series associated with a differential-difference operator on the real line

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Abstract

We extend the classical theory of Taylor series to a first-order differential-difference operator Λ on the real line which includes as a particular case the Dunkl operator associated with the reflection group $\mathbf{Z}_2$ on $\mathbf{R}$. More precisely, we establish first a generalized Taylor formula with integral remainder, and then specify sufficient conditions for a function on $\mathbf{R}$ to be expanded as a generalized Taylor series. Moreover, we provide a criterion of analyticity for functions on $\mathbf{R}$ involving the differential-difference operator Λ.

Keywords: Differential-difference operator; Generalized Taylor series; Analyticity criterion

0. Introduction

In this paper we consider the first-order differential-difference operator on $\mathbf{R}$

$$\Lambda f = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right), \tag{1}$$

where

$$A(x) = |x|^{2\alpha+1}B(x), \quad \alpha \geqslant -\tfrac{1}{2}, \tag{2}$$

B being a positive C^∞ even function on $\mathbf{R}$. In the case $A(x) = |x|^{2\alpha+1}$, $\alpha \geqslant -\frac{1}{2}$, we regain the differential-difference operator

$$D_\alpha f = \frac{\mathrm{d}f}{\mathrm{d}x} + \left(\alpha + \frac{1}{2}\right)\frac{f(x)-f(-x)}{x}, \tag{3}$$

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which is known as the Dunkl operator of index $\alpha + \frac{1}{2}$ associated with the reflection group $\mathbf{Z}_2$ on $\mathbf{R}$. Dunkl operators are parameterized first-order differential-difference operators on some Euclidean space that are related to finite reflection groups. They are introduced in [4,5] in connection with a generalization of the classical theory of spherical harmonics. For the mathematical and physical applications of such operators we refer to the literature cited in [11]. For instance, the one-dimensional Dunkl operator D_α plays a major role in the study of quantum harmonic oscillators governed by Wigner's commutation rules [6,9].

A quite new commutative harmonic analysis on the real line related to the differential-difference operator Λ was initiated in [7,8] in which several analytic structures on $\mathbf{R}$ were generalized. Through this paper, the classical Taylor series theory on $\mathbf{R}$ is extended to the differential-difference operator Λ. More explicitly, we establish in Section 1 the following generalized Taylor formula with integral remainder:

$$T^x f(y) = \sum_{p=0}^{n} b_p(y)\Lambda^p f(x) + \int_{-|y|}^{|y|} W_n(y,z)\, T^x \Lambda^{n+1} f(z) A(z)\, \mathrm{d}z, \tag{4}$$

where T^x, $x \in \mathbf{R}$, stand for the generalized translation operators tied to the differential-difference operator Λ; $\{W_p\}$ and $\{b_p\}$, $p=0,1,2,\ldots$, being two sequences of functions constructed inductively from the function A. In analogy to the classical setting, we determine sufficient conditions under which a C^∞ function f on $\mathbf{R}$ may be expanded as a generalized Taylor series in a neighborhood of an arbitrary point $x \in \mathbf{R}$; that is, conditions which ensure that for $|y|$ small enough, the integral remainder in (4) tends to 0 as $n \to \infty$. Moreover, it turns out that, except for the Dunkl operator case, the generalized Taylor series as discussed here are not power series. In other words, the b_p, $p = 1,2,3,\ldots$, are in general not polynomials. Nevertheless, we provide in Section 2 a criterion of analyticity on $\mathbf{R}$ involving the differential-difference operator Λ; that is, a criterion characterizing an analytic function f on $\mathbf{R}$ by means of the sequence $\{\Lambda^p f\}$, $p=0,1,2,\ldots$. The chief device in the proof of this criterion will be results from the theory of hypo-analytic operators (see [1]).

The notion of Taylor series was first extended in [2] to the Bessel differential operator $L_\alpha = \mathrm{d}^2/\mathrm{d}x^2 + ((2\alpha+1)/x)\,\mathrm{d}/\mathrm{d}x$, $\alpha \geqslant -\frac{1}{2}$. Such an extension was essentially aimed to allow a formal introduction of a generalized translation operation on the half line tied to the Bessel operator L_α. Later, Trimèche [12] extended the theory of Delsarte to more general second-order differential operators of the form

$$L = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \alpha \geqslant -\tfrac{1}{2}. \tag{5}$$

It is pointed out that all the results obtained in [12] emerge as easy consequences of those stated in the present article.

1. Generalized Taylor series

In the first part of this section we look for a Taylor formula with integral remainder, in which the differential-difference operator Λ replaces the usual first derivative operator $\mathrm{d}/\mathrm{d}x$. Naturally, the construction of such a formula will require a number of preliminary results.

Notation. Let $\mathscr{C}$ be the subset of $\mathbf{R}^2$ defined by $\mathscr{C} = \{(x,y) \in \mathbf{R}^2 : 0 < |y| \leqslant |x|\}$.

Define on $\mathscr{C}$ two sequences of functions $\{u_p(x,y)\},\{v_p(x,y)\}$, $p=0,1,2,\dots,$ via the following recursive integral formulae:

$$u_0(x,y)=\frac{\operatorname{sgn}(x)}{2A(x)},\quad v_0(x,y)=\frac{\operatorname{sgn}(y)}{2A(y)}, \tag{6}$$

$$u_{p+1}(x,y)=\int_{|y|}^{|x|} v_p(x,z)\,\mathrm{d}z,\qquad v_{p+1}(x,y)=\frac{\operatorname{sgn}(y)}{A(y)}\int_{|y|}^{|x|} u_p(x,z)A(z)\,\mathrm{d}z. \tag{7}$$

This pair of families of functions enjoys the following properties.

Lemma 1. (i) *$u_p, v_p\in C^1(\mathscr{C})$ for all $p=0,1,2,\dots,$ and satisfy for $0<|y|<|x|$ the relations*

$$\Lambda_x u_{p+1}(x,y)=u_p(x,y),\qquad \Lambda_y u_{p+1}(x,y)=-v_p(x,y), \tag{8}$$

$$\Lambda_x v_{p+1}(x,y)=v_p(x,y),\qquad \Lambda_y v_{p+1}(x,y)=-u_p(x,y). \tag{9}$$

(ii) *The sequences $\{u_p(x,y)\}$ and $\{v_p(x,y)\}$ may also be computed recursively for $p=2,3,4,\dots,$ by the formulae*

$$u_p(x,y)=2\int_{|y|}^{|x|} u_{p-2}(x,s)u_1(s,y)A(s)\,\mathrm{d}s,\qquad v_p(x,y)=2\int_{|y|}^{|x|} v_{p-2}(x,s)v_1(s,y)A(s)\,\mathrm{d}s. \tag{10}$$

Proof. An induction argument gives assertion (i). Assertion (ii) follows easily by combining identities (7). □

Remark 1. Appealing to (10), we show inductively that for any $p=0,1,2,\dots,u_{2p}(\cdot,y)$ is odd, $u_{2p+1}(\cdot,y)$ is even, $|u_{2p}(x,y)|=u_{2p}(|x|,y)$, and $u_{2p+1}(x,y)\geqslant 0$.

Notation. For $x\in\mathbf{R}$ put $\eta(x)=\inf_{|y|\leqslant|x|}B(y),\sigma(x)=\sup_{|y|\leqslant|x|}B(y)$, and $\omega(x)=\sigma(x)/\eta(x)$.

We shall need the following estimates.

Lemma 2. *We have*

$$0\leqslant u_1(x,y)\leqslant\begin{cases}\dfrac{|y|^{-2\alpha}-|x|^{-2\alpha}}{4\alpha\eta(x)} & \text{if } \alpha\neq 0,\\[2ex] \dfrac{\log(|x|/|y|)}{2\eta(x)} & \text{if } \alpha=0,\end{cases} \tag{11}$$

$$|u_{2p}(x,y)|\leqslant\left(\frac{x^2\omega(x)}{2\alpha+2}\right)^p\frac{u_1(x,y)}{|x|}\quad \text{for } p=1,2,3,\dots, \tag{12}$$

$$0\leqslant u_{2p+1}(x,y)\leqslant\left(\frac{x^2\omega(x)}{2\alpha+2}\right)^p u_1(x,y)\quad \text{for } p=0,1,2,\dots\ . \tag{13}$$

Proof. Inequalities (11) follow readily from (6) and (7). Let us check (12). By (10) we have

$$|u_2(x,y)| = 2\int_{|y|}^{|x|} |u_0(x,s)|u_1(s,y)A(s)\,\mathrm{d}s = \frac{1}{A(x)}\int_{|y|}^{|x|} u_1(s,y)A(s)\,\mathrm{d}s$$
$$\leqslant \frac{1}{A(x)}\int_0^{|x|} A(s)\,\mathrm{d}s\; u_1(x,y) \leqslant \frac{|x|\omega(x)}{2\alpha+2}u_1(x,y). \tag{14}$$

This implies that (12) holds for $p=1$. Moreover, using (10) and (14) we find

$$|u_4(x,y)| = 2\int_{|y|}^{|x|} |u_2(x,s)|u_1(s,y)A(s)\,\mathrm{d}s \leqslant 2u_1(x,y)\int_{|y|}^{|x|} |u_2(x,s)|A(s)\,\mathrm{d}s$$
$$\leqslant u_1(x,y)\frac{|x|\omega(x)}{\alpha+1}\int_0^{|x|} u_1(x,s)A(s)\,\mathrm{d}s.$$

But by (11),

$$0 \leqslant \int_0^{|x|} u_1(x,s)A(s)\,\mathrm{d}s \leqslant \frac{x^2\omega(x)}{8(\alpha+1)} \tag{15}$$

for any $\alpha \geqslant -\frac{1}{2}$. Therefore, (12) is true for $p=2$, and the full result follows by induction. Similarly, the majorization (13) is proved inductively by use of (10) and (15). □

Define on $\mathbf{R}$ the family of functions $\{b_p\}$ by setting $b_0(x)=1$, and for $p=1,2,3,\ldots,$

$$b_p(x) = \begin{cases} \displaystyle\int_{-|x|}^{|x|} u_{p-1}(x,y)A(y)\,\mathrm{d}y & \text{if } x \neq 0,\\ 0 & \text{if } x = 0. \end{cases} \tag{16}$$

Remark 2. By Remark 1 it follows that for any $p=0,1,2,\ldots,b_{2p}$ is even, b_{2p+1} is odd, and b_p is positive on $]0,\infty[$.

For our purpose of a Taylor formula involving the differential-difference operator Λ, a thorough investigation of the family $\{b_p\}$ seems unavoidable.

Proposition 1. *For any* $p=0,1,2,\ldots,$

$$b_p(x) = \mathcal{O}(x^p) \quad \textit{as } x \to 0. \tag{17}$$

Proof. By (16) we have for $x \neq 0$, $|b_1(x)| = 1/A(x)\int_0^{|x|} A(y)\,\mathrm{d}y \leqslant |x|\omega(x)/(2\alpha+2) = \mathcal{O}(x)$ as $x \to 0$. Moreover, from (12), (13) and (16) we deduce the estimates:

$$|b_{2p+1}(x)| \leqslant \left(\frac{x^2\omega(x)}{2\alpha+2}\right)^p \frac{b_2(x)}{|x|}, \qquad p=1,2,3,\ldots, \tag{18}$$

$$|b_{2p+2}(x)| \leqslant \left(\frac{x^2\omega(x)}{2\alpha+2}\right)^p b_2(x), \qquad p=0,1,2,\ldots\ . \tag{19}$$

As inequality (15) says that $0 \leqslant b_2(x) \leqslant x^2\omega(x)/4(\alpha+1)$, we see that for any $p=1,2,3,\dots, b_p(x) = \mathcal{O}(x^p)$ when $x \to 0$. $\square$

Proposition 2. *The functions b_p, $p=0,1,2,\dots$ are of class C^1 on $\mathbf{R}$ and satisfy the relation*

$$\Lambda b_{p+1} = b_p. \tag{20}$$

Proof. From its expression (16) it is clear that b_1 is differentiable on $]0,\infty[$, and for any $x>0$,

$$b_1'(x) = 1 - \frac{A'(x)}{A^2(x)} \int_0^x A(y)\,\mathrm{d}y = 1 - \left(\frac{2\alpha+1}{B(x)} + \frac{xB'(x)}{B^2(x)}\right) \int_0^1 B(tx)t^{2\alpha+1}\,\mathrm{d}t, \tag{21}$$

which tends to $1/(2\alpha+2)$ as $x \to 0^+$. This implies that b_1 is of class C^1 on $\mathbf{R}$ and $b_1'(0) = 1/(2\alpha+2)$. Further, it is immediate from (21) that $\Lambda b_1(x) = 1$ for all $x \in \mathbf{R}$. Now fix $p=2,3,4,\dots$. From (8) and (16) it is readily seen that $b_p \in C^1(\mathbf{R}\setminus\{0\})$ and

$$\Lambda b_p(x) = b_{p-1}(x) \quad \text{for all } x \neq 0. \tag{22}$$

But due to Remark 2, identity (22) becomes $b_p'(x) = b_{p-1}(x)$ for even p, and $b_p'(x) = b_{p-1}(x) - (A'/A)(x)b_p(x)$ for odd p. Therefore $b_p'(x) = \mathcal{O}(x^{p-1})$ as $x \to 0$, by virtue of (2) and (17). This immediately shows that $b_p \in C^1(\mathbf{R}), b_p'(0) = 0$, and that equality (22) also holds for $x=0$. $\square$

Starting from identity (20), we shall prove inductively that the b_p, $p=0,1,2,\dots$, are C^∞ functions on $\mathbf{R}$. We begin with the following technical lemma proved by a standard argument.

Lemma 3. *Let f be a function of class C^n on $[0,\infty[$, $n=0,1,2,\dots$. Then the function*

$$H_\alpha f(x) = \begin{cases} \dfrac{1}{x^\alpha}\displaystyle\int_0^x f(t)t^\alpha\,\mathrm{d}t & \text{if } x>0, \\ 0 & \text{if } x=0, \end{cases}$$

is of class C^{n+1} on $[0,\infty[$. Furthermore, $(H_\alpha f)^{(p)}(0) = p/(\alpha+p) f^{(p-1)}(0)$, for $p=1,2,\dots,n+1$.

Notation. For a function $f:\mathbf{R}\to\mathbf{C}$ denote by $f_{\mathrm{e}}(x) = (f(x)+f(-x))/2$, $f_{\mathrm{o}}(x) = (f(x)-f(-x))/2$ its even and odd part, respectively. Define $\mathscr{E}$ as the space of C^∞ complex-valued functions on $\mathbf{R}$, equipped with the topology of compact convergence of all derivatives. Let $\mathscr{E}_{\mathrm{e}}$ denote the subspace of $\mathscr{E}$ consisting of even functions. $\mathscr{E}_*$ stands for the subspace of $\mathscr{E}$ consisting functions f such that $f(0)=0$.

Lemma 4. *Let $m,n=0,1,2,\dots$. Let f be a function of class C^m on $\mathbf{R}$ such that $\Lambda^m f$ be of class C^n on $\mathbf{R}$. Then f is of class C^{m+n} on $\mathbf{R}$.*

Proof. It is enough to consider the case where $m=1$. For even f, the result is obvious since $\Lambda f = f'$ for such functions. For odd f, the result follows from Lemma 3, the relation $f(x) = 1/A(x)\int_0^x \Lambda f(t)A(t)\,\mathrm{d}t$, and expression (2) of A. For arbitrary f the lemma is a consequence of the relations $(\Lambda f)_{\mathrm{e}} = \Lambda(f_{\mathrm{o}})$, $(\Lambda f)_{\mathrm{o}} = \Lambda(f_{\mathrm{e}})$. $\square$

It is now possible to state the following proposition.

Proposition 3. *$b_p \in \mathscr{E}$ for each $p = 0, 1, 2, \ldots$.*

Proof. The result follows inductively by use of Proposition 2 and Lemma 4. □

The role of the b_p, $p = 0, 1, 2, \ldots$, in our generalized Taylor formula shall be analogous to that of the monomials $x^p/p!$, $p = 0, 1, 2, \ldots$, in the classical Taylor formula. To specify the connection between the families $\{b_p\}$ and $\{x^p/p!\}$, it is useful to recall from [7] the following result.

Theorem 1. *There exists a unique isomorphism V of $\mathscr{E}$ such that*

$$V\frac{\mathrm{d}}{\mathrm{d}x}f = \Lambda V f \quad \text{and} \quad Vf(0) = f(0) \text{ for all } f \in \mathscr{E}. \tag{23}$$

The operator V is said to be a transmutation operator between Λ and $\mathrm{d}/\mathrm{d}x$ on the space $\mathscr{E}$. For $A(x) = |x|^{2\alpha+1}$, $\alpha > -\frac{1}{2}$, this transmutation operator reads

$$Vf(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int_{-1}^{1}(1-t^2)^{\alpha-1/2}(1+t)f(xt)\,\mathrm{d}t, \tag{24}$$

and is referred to as the Dunkl intertwining operator of index $\alpha + \frac{1}{2}$ associated with the reflection group $\mathbf{Z}_2$ on $\mathbf{R}$ (see [5,11]).

We claim the following statement.

Proposition 4. *For any $n = 0, 1, 2, \ldots$,*

$$b_n(x) = V\left(\frac{y^n}{n!}\right)(x), \quad x \in \mathbf{R}. \tag{25}$$

In order to prove the proposition, we need the following simple lemma.

Lemma 5. *The mapping $f \to \Lambda f$ is one-to-one from $\mathscr{E}_*$ onto $\mathscr{E}$. The inverse mapping is given by*

$$\Lambda^{-1}f(x) = \int_0^x f_{\mathrm{o}}(y)\,\mathrm{d}y + \frac{1}{A(x)}\int_0^x f_{\mathrm{e}}(y)A(y)\,\mathrm{d}y.$$

Proof. If $f \in \mathscr{E}$ then (1) leads to $(\Lambda f)_{\mathrm{e}} = f'_{\mathrm{o}} + A'/A\, f_{\mathrm{o}}$, and $(\Lambda f)_{\mathrm{o}} = f'_{\mathrm{e}}$. That is,

$$f(x) = \int_0^x (\Lambda f)_{\mathrm{o}}(y)\,\mathrm{d}y + f(0) + \frac{1}{A(x)}\int_0^x (\Lambda f)_{\mathrm{e}}(y)A(y)\,\mathrm{d}y. \tag{26}$$

This makes the result obvious. □

Proof of Proposition 4. Set $c_n(x) = V(y^n/n!)(x)$, $x \in \mathbf{R}$, $n = 0, 1, 2, \ldots$. Notice that $c_0 = b_0 = 1$ by virtue of (23) and (26). Moreover, (23) ensures that $c_n \in \mathscr{E}_*$ and $\Lambda c_n = c_{n-1}$ for all $n = 1, 2, 3, \ldots$. So using Propositions 1–3 and Lemma 5 we easily check by induction that $c_n = b_n$ for all $n = 0, 1, 2, \ldots$, which is the required result. □

Remark 4. If $A(x)=|x|^{2\alpha+1}$, $\alpha \geqslant -\frac{1}{2}$, then identities (24) and (25) entail

$$b_{2p}(x)=\frac{1}{(\alpha+1)_p\,p!}\left(\frac{x}{2}\right)^{2p}, \quad b_{2p+1}(x)=\frac{1}{(\alpha+1)_{p+1}\,p!}\left(\frac{x}{2}\right)^{2p+1}$$

for all $p=0,1,2,\ldots$. Moreover, it is easily shown that the $b_n, n=0,1,2,\ldots,$ are polynomials only in the Dunkl operator case.

Before we formulate our generalized Taylor formula, we need to introduce in the space $\mathscr{E}$ certain generalized translation operators $T^a, a\in\mathbf{R}$, tied to the differential-difference operator Λ . Such operators are defined in terms of the transmutation operator V via the formula

$$T^a f(x)=V_a V_x[V^{-1} f(a+x)], \quad x\in\mathbf{R}.$$

Clearly for $A(x)=1$, we regain the ordinary translation operators on $\mathbf{R}: f\to\tau^a f(x)=f(a+x)$. The T^a, $a\in\mathbf{R}$, are linear bounded operators from $\mathscr{E}$ into itself, and possess the following fundamental properties:

$$T^0=\text{identity}, \quad T^a f(x)=T^x f(a) \text{ and } \Lambda T^a=T^a\Lambda. \tag{27}$$

For more details about this generalized translation operation we refer to [7,8]. We can now state the first central result of this paper.

Theorem 2. *Let $f\in\mathscr{E}$ and $a\in\mathbf{R}$. Then for any $n=0,1,2,\ldots,$ we have the following generalized Taylor formula with integral remainder*:

$$T^a f(x)=\sum_{p=0}^{n} b_p(x)\Lambda^p f(a)+\int_{-|x|}^{|x|} W_n(x,y)T^a\Lambda^{n+1} f(y)A(y)\,\mathrm{d}y, \tag{28}$$

where

$$W_n(x,y)=u_n(x,y)+v_n(x,y). \tag{29}$$

In order to simplify the proof of Theorem 2, we first establish the following technical lemma.

Lemma 6. *Let f be a function of class C^{p+1} on $\mathbf{R}$, $p=1,2,3,\ldots$. Then*

$$\int_{-|x|}^{|x|} W_p(x,y)\Lambda^{p+1} f(y)A(y)\,\mathrm{d}y=\int_{-|x|}^{|x|} W_{p-1}(x,y)\Lambda^p f(y)A(y)\,\mathrm{d}y-b_p(x)\Lambda^p f(0)$$

Proof. First,

$$\int_{-|x|}^{|x|} u_p(x,y)\Lambda^{p+1} f(y)A(y)\,\mathrm{d}y=2\int_0^{|x|} u_p(x,y)(\Lambda^{p+1} f)_{\mathrm{e}}(y)A(y)\,\mathrm{d}y$$

$$=2\int_0^{|x|} u_p(x,y)\frac{\mathrm{d}}{\mathrm{d}y}[A(y)(\Lambda^p f)_{\mathrm{o}}(y)]\,\mathrm{d}y. \tag{30}$$

Observe that by Lemma 2, the function $y\to A(y)u_p(x,y)$ is bounded in a neighborhood of the origin. Further, the classical Taylor formula implies that $(\Lambda^p f)_{\mathrm{o}}(y)=cy+o(y)$ as $y\to 0$, for some constant c. Therefore, integrating (30) by parts and applying (8) we obtain

$$\int_{-|x|}^{|x|} u_p(x,y)\Lambda^{p+1}f(y)A(y)\,\mathrm{d}y = -2\int_0^{|x|} \Lambda_y u_p(x,y)(\Lambda^p f)_{\mathrm{o}}(y)A(y)\,\mathrm{d}y$$

$$= \int_{-|x|}^{|x|} v_{p-1}(x,y)\Lambda^p f(y)A(y)\,\mathrm{d}y. \tag{31}$$

Moreover,

$$\int_{-|x|}^{|x|} v_p(x,y)\Lambda^{p+1}f(y)A(y)\,\mathrm{d}y = 2\int_0^{|x|} v_p(x,y)(\Lambda^{p+1}f)_{\mathrm{o}}(y)A(y)\,\mathrm{d}y$$

$$= 2\int_0^{|x|} v_p(x,y)\frac{\mathrm{d}}{\mathrm{d}y}(\Lambda^p f)_{\mathrm{e}}(y)A(y)\,\mathrm{d}y. \tag{32}$$

As by (7) and (16), $\lim_{y\to 0^+} 2A(y)v_p(x,y) = b_p(x)$; an integration by parts in (32), as well as (9) yield

$$\int_{-|x|}^{|x|} v_p(x,y)\Lambda^{p+1}f(y)A(y)\,\mathrm{d}y = -2\int_0^{|x|} \Lambda_y v_p(x,y)(\Lambda^p f)_{\mathrm{e}}(y)A(y)\,\mathrm{d}y - b_p(x)\Lambda^p f(0)$$

$$= \int_{-|x|}^{|x|} u_{p-1}(x,y)\Lambda^p f(y)A(y)\,\mathrm{d}y - b_p(x)\Lambda^p f(0). \tag{33}$$

The result follows now by combining (31) and (33). □

Proof of Theorem 2. Because of (27) it is sufficient to consider the case where $a=0$. By (6) we have

$$\int_{-|x|}^{|x|} u_0(x,y)\Lambda f(y)A(y) = \frac{\mathrm{sgn}(x)}{A(x)}\int_0^{|x|} \frac{\mathrm{d}}{\mathrm{d}y}[A(y)f_{\mathrm{o}}(y)]\,\mathrm{d}y = f_{\mathrm{o}}(x),$$

$$\int_{-|x|}^{|x|} v_0(x,y)\Lambda f(y)A(y)\,\mathrm{d}y = \int_0^{|x|} f'_{\mathrm{e}}(y)\,\mathrm{d}y = f_{\mathrm{e}}(x) - f(0).$$

This yields the statement for $n=0$. For $n=1,2,3,\ldots,$ we get identity (28) inductively by use of Lemma 6. □

Remark 5. If p is a polynomial of degree k ($k=1,2,3,\ldots$), then (3) implies that $D_\alpha p$ is a polynomial of degree $k-1$. Therefore in the Dunkl operator case, the integral remainder in (28) will vanish whenever f is a polynomial of degree n.

Our next purpose is to determine sufficient conditions under which a function f in $\mathscr{E}$ may be expanded as a generalized Taylor series in the vicinity of an arbitrary point $a\in\mathbf{R}$. This will be achieved with some additional assumptions on the differential-difference operator Λ. For $\alpha > -\frac{1}{2}$, it was pointed out in [8] that the translation operators $T^a, a\in\mathbf{R}$, may be represented as

$$T^a f(x) = \int_{\mathbf{R}} f(y)\,\mathrm{d}\Omega_{a,x}(y), \quad f\in\mathscr{E}, \tag{34}$$

where for each $a, x \in \mathbf{R}, \Omega_{a,x}$ is a distribution on $\mathbf{R}$ with support in $[-|a|-|x|, -||a|-|x||] \cup [||a|-|x||, |a|+|x|]$. From now on we assume that the distributions $\Omega_{a,x}$ are uniformly norm-bounded measures, i.e, there is a constant $C > 0$ such that

$$\|\Omega_{a,x}\| \leqslant C \quad \text{for all } a, x \in \mathbf{R}. \tag{35}$$

With this additional assumption we readily check that

$$|T^a f(x)| \leqslant C \sup_{||y|-|a|| \leqslant |x|} |f(y)| \quad \text{for all } a, x \in \mathbf{R} \quad \text{and all } f \in \mathscr{E}. \tag{36}$$

Such an estimation will be the key tool in the proof of the next theorem.

Theorem 3. *Let $f \in \mathscr{E}$ and $a \in \mathbf{R}$. Suppose that there are $M, \rho > 0$ such that*

$$\sup_{||x|-|a|| \leqslant \rho} |\Lambda^n f(x)| \leqslant M^{n+1} n! \tag{37}$$

for all $n = 0, 1, 2, \ldots$. Then there exists an $r > 0$ such that

$$T^a f(x) = \sum_{p=0}^{\infty} b_p(x) \Lambda^p f(a) \tag{38}$$

for $|x| \leqslant r$. Furthermore, the series in (38) converges uniformly for $|x| \leqslant r$.

The following estimates for the functions b_p, $p = 0, 1, 2, \ldots$, sharpen those given by (18) and (19), and may be useful in the proof of Theorem 3.

Proposition 5. *For any $p = 0, 1, 2, \ldots$, and $x \geqslant 0$,*

$$0 \leqslant b_{2p}(x) \leqslant \frac{1}{(\alpha+1)_p \, p!} \left(\frac{x^2 \omega(x)}{4} \right)^p,$$

$$0 \leqslant b_{2p+1}(x) \leqslant \frac{1}{(\alpha+1)_{p+1} \, p!} \left(\frac{x}{2} \right)^{2p+1} (\omega(x))^{p+1}.$$

Proof. Let $\{\tilde{u}_p\}$ denote the family $\{u_p\}$ corresponding to the Dunkl operator. By (6),

$$|u_0(x,y)| = \frac{1}{2A(x)} \leqslant \frac{1}{2\eta(x)|x|^{2\alpha+1}} = \frac{|\tilde{u}_0(x,y)|}{\eta(x)}.$$

Further, by (6) and (7),

$$0 \leqslant u_1(x,y) = \int_{|y|}^{|x|} \frac{\mathrm{d}z}{2A(z)} \leqslant \frac{1}{\eta(x)} \int_{|y|}^{|x|} \frac{\mathrm{d}z}{2z^{2\alpha+1}} = \frac{\tilde{u}_1(x,y)}{\eta(x)}.$$

An induction argument shows that for all $p = 0, 1, 2, \ldots$,

$$|u_p(x,y)| \leqslant \frac{(\omega(x))^{[p/2]}}{\eta(x)} |\tilde{u}_p(x,y)|, \tag{39}$$

where $[p/2]$ is the integer part of $p/2$. The proposition follows now by combining (16), (39) and Remark 4. $\square$

Proof of Theorem 3. Set $R_n(a,x) = \int_{-|x|}^{|x|} W_n(x,y) T^a \Lambda^{n+1} f(y) A(y)\,\mathrm{d}y$. By (36) and (37) it follows that $|R_n(a,x)| \leqslant CM^{n+2}(n+1)! \int_{-|x|}^{|x|} |W_n(x,y)| A(y)\,\mathrm{d}y$ for $0 < |x| \leqslant \rho$. As by (29), $|W_n(x,y)| \leqslant |u_n(x,y)| + |v_n(x,y)| = u_n(|x|,y) + v_n(|x|,|y|)$, we deduce from (7) and (16) that

$$\int_{-|x|}^{|x|} |W_n(x,y)| A(y)\,\mathrm{d}y \leqslant b_{n+1}(|x|) + |x| b_n(|x|). \tag{40}$$

Therefore, $|R_n(a,x)| \leqslant C\,M^{n+2}(n+1)!\,(b_{n+1}(|x|) + |x| b_n(|x|))$ for $0 < |x| \leqslant \rho$. Now using Proposition 5 we get

$$|R_{2n}(a,x)| \leqslant CM^2 |x| (\omega(x) + 2(\alpha + n + 1))(M^2 x^2 \omega(x))^n \frac{(2n+1)!}{2^{2n+1}\,(\alpha+1)_{n+1} n!}$$

and

$$|R_{2n+1}(a,x)| \leqslant CM(M^2 x^2 \omega(x))^{n+1} \frac{(2n+3)!}{2^{2n+2}\,(\alpha+1)_{n+1}(n+1)!},$$

for $0 < |x| \leqslant \rho$. Applying Stirling's formula, we find

$$|R_{2n}(a,x)| = (M^2 x^2 \omega(x))^n \mathcal{O}(n^{-\alpha-1/2}) \text{ and } |R_{2n+1}(a,x)| = (M^2 x^2 \omega(x))^{n+1} \mathcal{O}(n^{-\alpha+1/2}), \tag{41}$$

for $0 < |x| \leqslant \rho$ and $n \to \infty$. Choose an $r \in]0,\rho[$ such that $0 < M^2 r^2 \omega(r) < 1$. As the function ω is increasing on $[0,\infty[$, we see by (41) that $\lim_{n\to\infty} R_n(a,x) = 0$ uniformly for $0 < |x| \leqslant r$. This ends the proof by virtue of Theorem 2. □

Remark 6. (i) According to Rösler [10], assumption (35) is satisfied in the Dunkl operator case, with $C = 4$.

(ii) Delsarte and Lions [3] have introduced in $\mathscr{E}_\mathrm{e}$ translation operators $S^a, a \in \mathbf{R}$, tied to the differential operator L. It was indicated in [7] that $S^a f(x) = [T^a f(x) + T^a f(-x)]/2$ for all $f \in \mathscr{E}_\mathrm{e}$. Accordingly, by taking f even in Theorems 2 and 3, we easily regain the corresponding results proved in [12] for the operator L.

2. Analyticity criterion

As already indicated in Remark 4, the generalized Taylor series (38) are power series only in the Dunkl operator case. Hence it may be useful to provide a criterion of analyticity on $\mathbf{R}$ involving the differential-difference operator Λ. Throughout this section we assume that the function B given by (2) is analytic on $\mathbf{R}$. Trimèche [12] has obtained the following analyticity criterion for even functions on $\mathbf{R}$ involving the differential operator L.

Theorem 4. *Let f be a function in $\mathscr{E}_\mathrm{e}$. Then f is analytic if and only if for any $\rho > 0$ there is an $M_\rho > 0$ such that*

$$\sup_{0 \leqslant x \leqslant \rho} |L^n f(x)| \leqslant (M_\rho)^{n+1} (2n)! \quad \textit{for all } n = 0, 1, 2, \ldots\,.$$

The purpose of this section is to establish an analogue of Theorem 4 for the differential-difference operator Λ. Namely, we claim the following result.

Theorem 5. *In order that a function f in $\mathscr{E}$ be analytic it is necessary and sufficient that for any $\rho > 0$ there be an $M_\rho > 0$ such that*

$$\sup_{|x|\leqslant\rho} |\Lambda^n f(x)| \leqslant (M_\rho)^{n+1} n! \quad \textit{for all } n = 0, 1, 2, \ldots . \tag{42}$$

Proof. Let f be an analytic function. By induction we check that for any $n = 0, 1, 2, \ldots,$

$$\Lambda^{2n} f = L^n(f_e) + \frac{d}{dx} L^n I(f_o) \quad \Lambda^{2n+1} f = L^{n+1} I(f_o) + \frac{d}{dx} L^n(f_e), \tag{43}$$

where I denotes the map defined on $\mathscr{E}$ by $Ig(x) = \int_0^x g(t)\,dt$, $x \in \mathbf{R}$. These relations when combined with Theorem 4, together with the identity $(d/dx)g(x) = 1/A(x) \int_0^x Lg(t)A(t)\,dt$ which is valid for any $g \in \mathscr{E}_e$, show that condition (42) is necessary. Conversely, suppose that an $f \in \mathscr{E}$ satisfy condition (42). Introduce the function $F(x, y) = \sum_{n=0}^{\infty} (i)^n (y^n/n!) \Lambda^n f(x)$. By hypothesis (42) and Lemma 4, we can find an $\varepsilon > 0$ such that $F(x, y)$ be of class C^∞ in the strip $\mathbf{R} \times]-\varepsilon, \varepsilon[$. Moreover $F(x, y)$ satisfies for $x \in \mathbf{R}$ and $|y| < \varepsilon$ the differential-difference equation

$$\Lambda_x^2 F(x, y) + \frac{\partial^2}{\partial y^2} F(x, y) = 0. \tag{44}$$

Write $F = p + q$ with $p(x, y) = (F(x, y) + F(-x, y))/2$ and $q(x, y) = (F(x, y) - F(-x, y))/2$. According to (44), the functions p and q satisfy on $\mathbf{R} \times]-\varepsilon, \varepsilon[$ the homogeneous differential equations

$$\Delta(xp(x, y)) + \left(x \frac{A'(x)}{A(x)} - 2\right) \frac{\partial p}{\partial x}(x, y) = 0, \tag{45}$$

$$\Delta(x^2 q(x, y)) + \left(x^2 \frac{A'(x)}{A(x)} - 4x\right) \frac{\partial q}{\partial x}(x, y) + \left(x^2 \left(\frac{A'(x)}{A(x)}\right)' - 2\right) q(x, y) = 0, \tag{46}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian on $\mathbf{R}^2$. As by (2) the functions $xA'(x)/A(x)$ and $x^2(A'(x)/A(x))'$ are analytic in $\mathbf{R}^2$, we deduce from [1, Corollary 1.2] that both the left-hand sides of (45) and (46) are hypo-analytic operators on $\mathbf{R}^2$. Therefore p and q are analytic in the strip $\mathbf{R} \times]-\varepsilon, \varepsilon[$, and so is F. To conclude the proof observe that $f(x) = F(x, 0)$ for all $x \in \mathbf{R}$. □

Remark 7. (i) According to identities (43), Theorem 5 immediately implies Theorem 4.

(ii) From (20) and Theorem 5 it follows that the b_n, $n = 0, 1, 2, \ldots,$ are analytic functions on $\mathbf{R}$.

Corollary 1. *$T^a f$ is analytic whenever f is analytic and $a \in \mathbf{R}$.*

Proof. Let f be analytic and $a \in \mathbf{R}$. By (27) and (36) we have for any $\rho > 0$ and $n = 0, 1, 2, \ldots,$

$$\sup_{|x|\leqslant\rho} |\Lambda^n T^a f(x)| = \sup_{|x|\leqslant\rho} |T^a \Lambda^n f(x)| \leqslant \sup_{||x|-|a||\leqslant\rho} |\Lambda^n f(x)| \leqslant \sup_{|x|\leqslant|a|+\rho} |\Lambda^n f(x)|.$$

From this and Theorem 5 we deduce that the function $T^a f$ is analytic on $\mathbf{R}$. □

Remark 8. A combination of Theorems 3, 5 and Corollary 1 shows that any analytic function on $\mathbf{R}$ may be expanded in a generalized Taylor series in a neighborhood of an arbitrary point $a \in \mathbf{R}$.

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