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An Initial-Value Method for Fredholm Integral Equations with Generalized Degenerate Kernels

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An initial-value method is derived for integral equations with generalized degenerate kernels. Both the necessity and sufficiency of the Cauchy system are demonstrated.

1. INTRODUCTION

Initial-value methods for Fredholm integral equations have been under investigation for several years [1, 2] (see [4] for an extensive bibliography on initial value methods). There are several reasons for desiring an initial-value formulation. First of all, modern digital computers are ideally suited for solving large systems of initial-valued differential equations, and the solution can be easily studied as a function of some parameter. In this paper, our solutions will be given as functions of the interval length for generalized degenerate kernels having the form

$$k(t, y) = \int_0^1 \alpha(t, z)\, \beta(y, z)\, dz. \tag{1.1}$$

The derivation is shorter and more general than that in [1].

In Sec. 2, we derive the Cauchy system for the solution, and in Sec. 3, a proof of sufficiency is given.

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2. Necessity of the Cauchy System

Consider the Fredholm integral equation of the second kind

$$U(t, x) = g(t) + \int_0^x k(t, y)\, U(y, x)\, dy, \qquad 0 \leqslant x \leqslant X, \tag{2.1}$$

where the forcing term $g(t)$ is assumed to be an L_2 function and $k(t, y)$ is a generalized degenerate L_2 kernel of the form

$$k(t, y) = \int_0^1 \alpha(t, z)\, \beta(y, z)\, dz. \tag{2.2}$$

Under the above assumptions, (2.1) possesses a unique solution for sufficiently small x, $0 \leqslant x \leqslant X$.

If we introduce the function $e(z, x)$ by the relation

$$e(z, x) = \int_0^x \beta(y, z)\, U(y, x)\, dy, \tag{2.3}$$

then the solution $U(t, x)$ of (2.1) is given by

$$U(t, x) = g(t) + \int_0^1 e(z, x)\, \alpha(t, z)\, dz. \tag{2.4}$$

Our purpose in this paper is to show that $e(z, x)$ may be obtained by solving an initial-value problem. Substituting (2.4) into (2.3), we find that $e(z, x)$ satisfies the Fredholm integral equation

$$e(z, x) = \int_0^x \beta(y, z)\, g(y)\, dy + \int_0^1 \left\{ \int_0^x \beta(y, z)\, \alpha(y, z')\, dy \right\} e(z', x)\, dz', \qquad 0 \leqslant z \leqslant 1. \tag{2.5}$$

If we differentiate in (2.5) with respect to x, we find that $e_x(z, x)$ satisfies a Fredholm equation of the form

$$e_x(z, x) = \beta(x, z) \left\{ g(x) + \int_0^1 \alpha(x, z')\, e(z', x)\, dz' \right\} + \int_0^1 \left\{ \int_0^x \beta(y, z)\, \alpha(y, z')\, dy \right\} e_x(z', x)\, dz'. \tag{2.6}$$

The resolvent $R(z, z', x)$ of the kernel

$$\int_0^x \beta(y, z)\, \alpha(y, z')\, dy$$

satisfies the integral equation

$$R(z, z', x) = \int_0^x \beta(y', z)\, \alpha(y', z')\, dy' + \int_0^1 \left\{\int_0^x \beta(y, z)\, \alpha(y, z'')\, dy\right\} R(z'', z', x)\, dz''. \tag{2.7}$$

In terms of the resolvent, the solution for $e_x(z, x)$ is given by

$$e_x(z, x) = \left[g(x) + \int_0^1 \alpha(x, z')\, e(z', x)\, dz'\right] \times \left[\beta(x, z) + \int_0^1 R(z, z', x)\, \beta(x, z')\, dz'\right], \qquad 0 \leqslant x \leqslant X. \tag{2.8}$$

The initial condition for (2.8) is obtained by evaluating (2.5) at $x = 0$. Thus

$$e(z, 0) = 0. \tag{2.9}$$

At this point the resolvent $R(z, z', x)$ in (2.8) is still unknown. Hence, we shall proceed to derive an initial-value problem satisfied by $R(z, z', x)$. We can do this by differentiating in (2.7) with respect to x. Thus,

$$R_x(z, z', x) = \beta(x, z)\left[\alpha(x, z') + \int_0^1 \alpha(x, z'')\, R(z'', z', x)\, dz''\right] + \int_0^1 \left\{\int_0^x \beta(y, z)\, \alpha(y, z'')\, dy\right\} R_x(z'', z', x)\, dz''. \tag{2.10}$$

Notice that the kernel in (2.10) is the same as that given in (2.6). That is, the solution $R_x(z, z', x)$ of (2.10) may be expressed in terms of the resolvent $R(z, z', x)$ as

$$R_x(z, z', x) = \left[\alpha(x, z') + \int_0^1 \alpha(x, z'')\, R(z'', z', x)\, dz''\right] \times \left[\beta(x, z) + \int_0^1 R(z, z'', x)\, \beta(x, z'')\, dz''\right], \quad 0 \leqslant x \leqslant X. \tag{2.11}$$

The initial condition for the above equation is obtained by evaluating (2.7) at $x = 0$. Whence,

$$R(z, z', 0) = 0. \tag{2.12}$$

This completes the derivation of the initial value procedure. We solve (2.8), (2.9) and (2.11), (2.12) and then use (2.4) to obtain the solution of our original integral equation (2.1).

3. Sufficiency of the Cauchy System

In this section we shall prove that the initial-value system derived in the previous section is also sufficient for the original integral equation. Thus we wish to prove that given $e(z, x)$ and $R(z, z', x)$ as solutions of (2.8) and (2.11), $U(t, x)$, as given by (2.4), is the unique solution of (2.1).

Define the function $Q(z, z', x)$ by the relation

$$Q(z, z', x) = \int_0^x \beta(y, z)\, \alpha(y, z')\, dy + \int_0^1 \left[\int_0^x \beta(y, z)\, \alpha(y, z'')\, dy\right] R(z'', z', x)\, dz''. \tag{3.1}$$

We wish to prove that $Q(z, z', x) = R(z, z', x)$. We shall do this by showing that $Q(z, z', x)$ and $R(z, z', x)$ both satisfy the same differential equation with the same initial conditions. Differentiate in (3.1) with respect to x. We obtain

$$Q_x(z, z', x) = \beta(x, z)\left[\alpha(x, z') + \int_0^1 \alpha(x, z'')\, R(z'', z', x)\, dz''\right] + \int_0^1 \left[\int_0^x \beta(y, z)\, \alpha(y, z'')\, dy\right] R_x(z'', z', x)\, dz''. \tag{3.2}$$

Substituting (2.11) into (3.2) and simplifying, we get

$$Q_x(z, z', x) = \left[\alpha(x, z') + \int_0^1 \alpha(x, z'')\, R(z'', z', x)\, dz''\right] \times \left[\beta(x, z) + \int_0^1 \left(\int_0^x \beta(y, z)\, \alpha(y, z'')\, dy\right) \times \left(\beta(x, z'') + \int_0^1 \beta(x, z^{iv})\, R(z'', z^{iv}, x)\, dz^{iv}\right) dz''\right]. \tag{3.3}$$

Recalling the definition of Q, we find that (3.3) becomes the linear functional equation for Q,

$$Q_x(z, z', x) = \left[\alpha(x, z') + \int_0^1 \alpha(x, z'')\, R(z'', z', x)\, dz''\right] \times \left[\beta(x, z) + \int_0^1 \beta(x, z'')\, Q(z, z'', x)\, dz''\right]. \tag{3.4}$$

A solution of (3.4) is $Q = R$. By evaluating (3.1) at $x = 0$ we have that the initial condition for (3.4) is

$$Q(z, z', 0) = 0. \tag{3.5}$$

Comparing (2.12) and (3.5) and assuming uniqueness, we conclude that $Q(z, z', x) = R(z, z', x)$, which establishes (2.7).

Now define $f(z, x)$ by the relation

$$f(z, x) = \int_0^x \beta(y, z)\, g(y)\, dy + \int_0^1 \left(\int_0^x \beta(y, z)\, \alpha(y, z')\, dy \right) e(z', x)\, dz'. \tag{3.6}$$

We wish to show that $f(z, x) = e(z, x)$ by showing that both satisfy the same differential equation with the same initial conditions. We proceed by first differentiating in (3.6) with respect to x and obtain

$$\begin{aligned} f_x(z, x) &= \beta(x, z) \left[g(x) + \int_0^1 \alpha(x, z')\, e(z', x)\, dz' \right] \\ &\quad + \int_0^1 \left(\int_0^x \beta(y, z)\, \alpha(y, z')\, dy \right) e_x(z', x)\, dz'. \end{aligned} \tag{3.7}$$

Substituting (2.8) into (3.7) and simplifying yields

$$\begin{aligned} f_x(z, x) &= \left[g(x) + \int_0^1 \alpha(x, z')\, e(z', x)\, dz' \right] \\ &\quad \times \left[\beta(x, z) + \int_0^1 \left(\int_0^x \beta(y, z)\, \alpha(y, z')\, dy \right) \right. \\ &\quad \times \left. \left(\beta(x, z') + \int_0^1 \beta(x, z'')\, R(z', z'', x)\, dz'' \right) dz' \right]. \end{aligned} \tag{3.8}$$

Using (2.7), we find that (3.8) becomes

$$\begin{aligned} f_x(z, x) &= \left[g(x) + \int_0^1 \alpha(x, z')\, e(z', x)\, dz' \right] \\ &\quad \times \left[\beta(x, z) + \int_0^1 \beta(x, z')\, R(z, z', x)\, dz' \right], \qquad 0 \leqslant x \leqslant X. \end{aligned} \tag{3.9}$$

Evaluating (3.6) at $x = 0$, we find that the initial condition for (3.9) is

$$f(z, 0) = 0. \tag{3.10}$$

Comparing (2.8), (2.9) with (3.9), (3.10) we conclude that $e(z, x) = f(z, x)$.

Our next objective is to show that $U(t, x)$ as given by (2.4) does indeed provide the unique solution for (2.1).

Define the function $\tilde{U}(t, x)$ by the relation

$$\tilde{U}(t, x) = g(t) + \int_0^1 \alpha(t, z)\, e(z, x)\, dz. \tag{3.11}$$

We wish to prove that $\tilde{U}(t, x)$ satisfies (2.1), which then proves $U(t, x) = \tilde{U}(t, x)$. Substitute (2.5) into (3.11). We obtain

$$\tilde{U}(t, x) = g(t) + \int_0^1 \alpha(t, z) \left[\int_0^x \beta(y, z)\, g(y)\, dy + \int_0^1 \left(\int_0^x \beta(y, x)\, \alpha(y, z')\, dy\right) e(z', x)\, dz'\right] dz. \tag{3.12}$$

Interchanging the order of integration we may write (3.12) as

$$\tilde{U}(t, x) = g(t) + \int_0^x \left(\int_0^1 \alpha(t, z)\, \beta(y, z)\, dz\right)\left(g(y) + \int_0^1 \alpha(y, z')\, e(z', x)\, dz'\right) dy. \tag{3.13}$$

Recalling the relation for $\tilde{U}(t, x)$ as given by (3.11), we have

$$\tilde{U}(t, x) = g(t) + \int_0^x k(t, y)\, \tilde{U}(y, x)\, dy, \tag{3.14}$$

where

$$k(t, y) = \int_0^1 \alpha(t, z)\, \beta(y, z)\, dz. \tag{3.15}$$

Hence $\tilde{U}(t, x)$ and $U(t, x)$ satisfy the same integral equation and by uniqueness we have $\tilde{U}(t, x) = U(t, x)$. This completes the proof that our Cauchy system does indeed solve the original integral equation.

4. Discussion

We have shown both the necessity and the sufficiency of a Cauchy system for linear Fredholm integral equations with degenerate kernels. The eigenvalue problem will be treated in a forthcoming paper [3].

Examples showing the computational utility of Cauchy systems such as that given in Sec. 2 are contained in [1–3] and in many articles cited in [4].

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