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Spectral analysis of the anti-reflective algebra

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Abstract

Anti-reflective boundary conditions have been studied in connection with fast deblurring algorithms, in the case of d-dimensional objects (signals for $d = 1$, images for $d = 2$). Here we study how, under the assumption of strong symmetry of the point spread functions and under mild degree conditions, the associated matrices depend on a symbol and define an algebra homomorphism. Furthermore, the eigenvalues can be exhaustively described in terms of samplings of the symbol and other related functions, and appropriate $O(n^d \log(n))$ arithmetic operations algorithms can be derived for the related computations. These results, in connection with the use of the anti-reflective transform, are of interest when employing filtering type procedures for the reconstruction of noisy and blurred objects.

Keywords: Boundary conditions; Fast trigonometric transforms; Matrix algebras

1. Introduction

We consider the deblurring problem of blurred and noisy d-dimensional objects with space invariant point spread functions (PSFs) (see e.g. [1]) and with anti-reflective (AR) boundary conditions (BCs), see [2] for the original proposal. More in detail, the discrete blurring and noising model in d dimensions is formulated by the following set of equations:

$$g_i = \sum_{j \in \mathbb{Z}^d} f_j h_{i-j} + \nu_i, \quad i \in \mathbb{Z}^d. \tag{1}$$

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In the above relations, for $s \in \mathbb{Z}^d$, the tensor $\mathbf{h} = (h_s)$ represents the (discrete) blurring operator (the discrete PSF), $\boldsymbol{\nu} = (\nu_s)$ is the noise contribution, $\mathbf{g} = (g_s)$ is the blurred and noisy observed object, and $\mathbf{f} = (f_s)$ represents the true object to be reconstructed. Given $\mathbf{h}$ and some statistical knowledge of $\boldsymbol{\nu}$, the problem is to recover the unknown "true" object $\mathbf{f}$, in a fixed field of view (FOV) described by $s \in \{1, \ldots, n\}^d$, from the knowledge of $\mathbf{g}$ in the same FOV. Assuming that the support of $\mathbf{h}$ is a d-dimensional cube of volume $q^d > 1$, it follows that system (1) is under-determined, because we have to solve n^d equations and $(n + q - 1)^d$ unknowns are involved. In order to cope with this problem while maintaining the quality of the restored object, very recently AR-BCs have been proposed [2]. Indeed for $d = 2$ the AR-BCs preserve at the boundary both the continuity of the image and the continuity of its normal derivative (C^1 continuity when dealing with signals). Therefore typical artifacts called ringing effects are negligible with respect to the other BCs [2,3], at least for piece-wise smooth images: on the other hand, when fine textures appear close to the boundary, then we should be aware that none of these extrapolation based BCs provide a good model for the values outside the scene. Furthermore, unlike periodization and reflection, the anti-reflection model does not guarantee that the obtained values fall inside the natural range (typically $\{0, \ldots, 255\}$). In practice the latter does not represent a problem since these outlying values appear in general only outside the scene i.e. outside the FOV, and therefore are simply ignored. More recently, in [4], we extended the proposal and the analysis to an arbitrary number $d > 1$ of dimensions: from the viewpoint of the modeler the quality of the proposal is similar to that of the 2-dimensional case and, from an algebraic/computational point of view, the resulting matrix structure is d-level Toeplitz + Hankel plus a d-level structured low rank matrix. Despite the apparent involved expression of the resulting matrix, the matrix–vector product is again possible by d-level fast Fourier transforms (FFTs) by using embedding arguments (refer to Section 3 of [5] for the explicit treatment of the case $d = 2$), while the solution of a linear system can be obtained in $O(n^d \log(n))$ operations (ops) by j-level fast sine transforms (FSTs), $j \leqslant d$, if the PSF is strongly symmetric and the associated symbol satisfies a mild degree condition [4]. We define a d-dimensional PSF $\mathbf{h}$ to be *strongly symmetric* if it is symmetric with respect to each index i.e. if $h_s = h_{|s|}$, $\forall s \in \mathbb{Z}^d$, with $s = (s_1, \ldots, s_d)$ being a d-index and $|s| = (|s_1|, \ldots, |s_d|)$, see [1]. Moreover, given a PSF $\mathbf{h}$ represented by a d-dimensional tensor or mask, we associate the symbol $h(y)$ defined as

$$h(y) = \sum_{j \in \mathbb{Z}^d} h_j \exp(\mathrm{i}\langle j, y\rangle), \quad \langle j, y\rangle = \sum_{i=1}^{d} y_i j_i, \qquad \mathrm{i}^2 = -1 \tag{2}$$

for all $y \in \mathbb{R}^d$. For *mild degree condition* we mean that $j_i \geqslant n - 2$ for some $i \in \{1, \ldots, d\}$ necessarily implies $h_j = 0$. Moreover, in real applications, a PSF is usually *normalized*, i.e., $\sum_{s_1,\ldots,s_d=-m}^{m} h_s = 1$. In the rest of the paper, when it is not differently specified, we suppose that the PSF $\mathbf{h}$ is strongly symmetric, normalized, and satisfies the above mild degree condition.

We recall that many concrete applications satisfy by nature the above hypotheses (Gaussian blur, some out of focus etc., see e.g. [1]).

The main focus of this paper concerns the spectral properties of the AR-BC matrices, in connection with the symbol h in (2) and in analogy with the well-studied (multilevel) Toeplitz/circulant case (see [6] and references therein). More in detail, we prove an algebra homomorphism between the space of multivariate cosine polynomials, associated with strongly symmetric PSFs, and the d-level AR-BC algebra: to be precise, the set of d-variate cosine polynomials is not viewed as an infinite dimensional ring, but as a finite dimensional algebra of dimension not exceeding that of the d-level AR-BC algebra, by considering a special definition of the polynomial product/inversion based on interpolation (see (22) and the discussion in Section 3.1, just before Theorem 5). In that case we are able to indicate an algorithm of $O(n^d \log(n))$ complexity, based on FFTs or suitable FSTs, for the determination of all the eigenvalues of any AR-BC matrix: in addition those eigenvalues can be described analytically in terms of some special symbols associated with the main symbol h. The study of the corresponding eigenvectors or, in other words, the definition of the anti-reflective transform, and the design of related fast procedures deserve a specific attention and are fully treated in [7].

The paper is organized as follows: in Section 2 we review the matrix structures arising from d-dimensional AR-BCs, we show that they define a matrix algebra such that the resolution of a linear system and the matrix–vector product reduce to the computation of few j-dimensional discrete sine transforms of type I (DST-I), with $j \leqslant d$ and with a total cost of $O(n^d \log(n))$ ops. In Section 3, we discuss the spectral features of the involved matrices and their relations with the symbol induced from the PSF mask. Finally, in Section 4 we briefly mention the applicability of the previous computational proposals in connection with filtering techniques and the AR transform introduced in [7].

2. The algebra of matrices induced by AR-BCs

In Section 2.1, for $d \geqslant 1$, we first introduce the $\tau^{(d)}$ class, related to the multilevel DST-I matrices [8]. In a second step, we define the classes of matrices $\mathscr{S}^{(d)}$, which are inherently related to the algebras $\tau^{(d)}$. In particular we recall the structure of algebra of the space $\mathscr{S}^{(d)}$, which is related to AR-BC matrices. Moreover, it is known [4] that any operation in $\mathscr{S}^{(d)}$ such as matrix–vector product, matrix-matrix product, eigenvalue computation, and linear system solution can be carried out within $O(n^d \log(n))$ ops. Section 2.2 is devoted to briefly describe the AR-BCs in the case of d-dimensional objects, providing the structure of the matrix A which represents the corresponding blurring operator: the nice fact is that all matrices arising from the imposition of d-level AR-BCs in our hypothesis (PSF strongly symmetric, normalized and satisfying a mild degree condition) are elements of $\mathscr{S}^{(d)}$. In particular, it will be shown in Section 3 that they form a commutative subalgebra of $\mathscr{S}^{(d)}$.

2.1. *The $\mathscr{S}^{(d)}$ algebras and their computational features*

Let Q be the DST-I matrix of order n with entries $[Q]_{i,j} = \sqrt{\frac{2}{n+1}} \sin(\frac{ij\pi}{n+1})$, $i, j = 1, \ldots, n$ (see [8]). It is known that the Q is orthogonal and symmetric ($Q^{-1} = Q^{\mathrm{T}} = Q$). For any $\mathbf{v} \in \mathbb{R}^n$, the matrix–vector product $Q\mathbf{v}$ can be computed in $O(n \log(n))$ ops by using the FST. In the multidimensional case, setting $Q^{(d)} = Q \otimes \cdots \otimes Q$ (d times, $\otimes$ being the Kronecker product) and for $\mathbf{v} \in \mathbb{R}^{n^d}$, $Q^{(d)}\mathbf{v}$ can be computed in $O(n^d \log(n))$ ops by the FST of level d. Let $\tau^{(d)} = \{Q^{(d)} D Q^{(d)} : D = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n^d}), \lambda_i \in \mathbb{R}\}$. Let $X \in \tau^{(d)}$, then $Q^{(d)} X \mathbf{e}_1 = D Q^{(d)} \mathbf{e}_1$, with $\mathbf{e}_j$ denoting the jth vector of the canonical basis of $\mathbb{R}^{n^d}$, that is, the eigenvalues λ_i of X are given by $\lambda_i = \frac{[Q^{(d)}(X\mathbf{e}_1)]_i}{[Q^{(d)}\mathbf{e}_1]_i}$, $i = 1, \ldots, n^d$. Hence for all $X \in \tau^{(d)}$ its eigenvalues can be obtained by means of one d-level DST-I of its first column and X is uniquely determined by its first column.

Moreover, the eigenvalues of a $\tau^{(1)}$ matrix are given by the cosine function

$$v(y) = \sum_{|j| \leqslant n-1} v_j \exp(\mathrm{i} j y), \tag{3}$$

sampled at

$$y \in G_n^{(1)} \equiv \left\{ \frac{k\pi}{n+1},\ k = 1, \ldots, n \right\}, \tag{4}$$

where $v_s = v_{|s|}$ for $|s| \leqslant n-1$ (i.e., $\lambda_k = v(\frac{k\pi}{n+1})$, $k = 1, \ldots, n$). In such case the $\tau^{(1)}$ matrix is denoted by $\tau^{(1)}(v)$ and is called the $\tau^{(1)}$ matrix generated by the function or symbol $v(y)$, while $\mathbf{v} = (v_0, \ldots, v_{n-1})^{\mathrm{T}} \in \mathbb{R}^n$ denotes the coefficient mask of the function v as expressed in (3). For $d > 1$, as in the one-dimensional case, a spectral characterization of the matrices in $\tau^{(d)}$ is possible. The key information is contained in a d-dimensional mask. Indeed, let us consider a d-dimensional strongly symmetric tensor $\mathbf{v} = (v_s)$, where $v_s = v_{|s|}$, $|s| \leqslant (n-1)e$ with e being the vector of all ones. The related matrix has eigenvalues described by the d-variate cosine function

$$v(y) = \sum_{|j| \leqslant (n-1)e} v_j \exp(\mathrm{i}\langle j, y\rangle) \tag{5}$$

sampled at

$$y \in G_n^{(d)} \equiv \left\{ \left(\frac{j_1\pi}{n+1}, \ldots, \frac{j_d\pi}{n+1} \right),\ j_k = 1, \ldots, n,\ k = 1, \ldots, d \right\}, \tag{6}$$

and is denoted by $\tau^{(d)}(v)$. The class $\tau^{(d)}$ will be indicated explicitly by $\tau_k^{(d)}$ when the corresponding matrix size k^d is not clear from the context.

The mask $\mathbf{v}$ of the Fourier coefficients of the function v in (5) defines not only the spectrum of $\tau^{(d)}(v)$ but also its entries. Such characterization of the $\tau^{(d)}$ class is important for analyzing the structure of the AR-BC matrices. We start with the case of $d = 1$. Let us define the shift of any vector $\mathbf{x} = (x_0, \ldots, x_{n-1})^{\mathrm{T}}$ as $\sigma(\mathbf{x}) = (x_1, x_2, \ldots, x_{n-1}, 0)^{\mathrm{T}}$. We define $T(\mathbf{x})$ as the n-by-n symmetric Toeplitz matrix whose first column is $\mathbf{x}$ and $H(\mathbf{x}, \mathbf{y})$ as the n-by-n Hankel matrix whose first and last column are $\mathbf{x}$ and $\mathbf{y}$, respectively. Then

$$\tau^{(1)}(v) = T(\mathbf{v}) - H(\sigma^2(\mathbf{v}), J\sigma^2(\mathbf{v})) \tag{7}$$

where J is the flip matrix, i.e., $J_{s,t} = 1$ if $s + t = n + 1$ and zero otherwise. This means that scalar $\tau^{(1)}$ structures are special instances of Toeplitz plus Hankel matrices. Now for $d > 1$, the description of $\tau^{(d)}$ can be given recursively. Briefly, every $\tau^{(d)}$ matrix is represented as (7), where v_j is the $\tau^{(d-1)}$ matrix associated with the $(d-1)$-dimensional mask $\mathbf{v}_{(j,s_2,\ldots,s_d)}$ and where the matrix J is replaced by $J \otimes I_{n^{d-1}}$.

The algebra $\tau^{(d)}$ of proper dimension represents a building block for defining the algebra $\mathscr{S}^{(d)}$. More specifically, for $d \geqslant 1$, we define the classes of $n^d \times n^d$ matrices $\mathscr{S}^{(d)}$ as follows. For $d = 1$, $M \in \mathscr{S}^{(1)}$ if

$$M = \begin{pmatrix} \alpha & & \\ \mathbf{v} & \hat{M} & \mathbf{w} \\ & & \beta \end{pmatrix}, \tag{8}$$

with $\alpha, \beta \in \mathbb{R}$, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{n-2}$, and $\hat{M} \in \tau_{n-2}^{(1)}$. For $d > 1$, $M \in \mathscr{S}^{(d)}$ if

$$M = \begin{pmatrix} \alpha & & \\ \mathbf{v} & M^* & \mathbf{w} \\ & & \beta \end{pmatrix}, \tag{9}$$

with $\alpha, \beta \in \mathscr{S}^{(d-1)}$, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{(n-2)n^{d-1}\times n^{d-1}}$, $\mathbf{v} = (v_j)_{j=1}^{n-2}$, $\mathbf{w} = (w_j)_{j=1}^{n-2}$, $v_j, w_j \in \mathscr{S}^{(d-1)}$ and $M^* = (M^*_{i,j})_{i,j=1}^{n-2}$ having an external $\tau^{(1)}_{n-2}$ structure, with $M^*_{i,j} \in \mathscr{S}^{(d-1)}$. Blanks in (8) and (9) denote null blocks. As for the $\tau^{(d)}$ algebra, the class $\mathscr{S}^{(d)}$ will be indicated explicitly by $\mathscr{S}^{(d)}_k$ when the corresponding matrix size k^d is not clear from the context (for more details on links and relation between classes $\mathscr{S}^{(d)}$ and $\tau^{(d)}$ see [4]).

In the rest of this subsection we recall some relevant computational findings, whose derivations and details can be found in [4]. Let $M \in \mathscr{S}^{(d)}$ and $\mathbf{f}, \mathbf{g} \in \mathbb{R}^{n^d}$, then the following computations can be carried out with $O(n^d \log(n))$ ops:

(1) solve the system $M\mathbf{f} = \mathbf{g}$ with $\det(M) \neq 0$;
(2) compute the matrix–vector product $\mathbf{g} = M\mathbf{f}$;
(3) compute the eigenvalues of M.

Furthermore, the space $\mathscr{S}^{(d)}$ is an algebra of dimension $(3n-4)^d$.

As an application of the above results consider the linear system arising from the Tikhonov regularization with reblurring strategy proposed in [9,5]. Such a linear system has coefficient matrix $M^2 + \mu\mathscr{T}^2$ or $M^2 + \mu\mathscr{L}$, where $\mathscr{T}$, $\mathscr{L}$ are any even-order differential-like operators with AR-BCs, M is an AR-BC matrix, and μ is a positive Tikhonov-like regularization parameter. Therefore, by the algebra structure of the class $\mathscr{S}^{(d)}$, we deduce that $M^2 + \mu\mathscr{T}^2, M^2 + \mu\mathscr{L} \in \mathscr{S}^{(d)}$ so that the recursive procedure contained in the proof of the previous item (1) can be employed [4]. Furthermore, it is worth mentioning that a direct (non-recursive) procedure for the 1D, 2D, and 3D case is described in [3,4]: it shows an interesting geometric interpretation in terms of lower dimensional faces of the unit d-dimensional cube, $d \leqslant 3$, and the same geometric-combinatorial flavor will appear again in the spectral results of Section 3.

2.2. The d-dimensional AR-BCs and the related matrices

The AR-BCs impose a global symmetry around the boundary points: for $d = 1$ the latter choice corresponds to a central symmetry around the considered boundary point while for $d \geqslant 2$ we have a symmetry around each $(d-1)$-dimensional affine space (straight line for $d = 2$) supporting the considered $(d-1)$-dimensional face (segment for $d = 2$) of the boundaries. More specifically, in the one-dimensional case, if f_1 is the left boundary point and the f_n is the right one, then the external points f_{1-j} and f_{n+j}, $j \geqslant 1$, are computed as function of the internal points according to the rules $f_{1-j} - f_1 = -(f_{j+1} - f_1)$ and $f_{n+j} - f_n = -(f_{n-j} - f_n)$. Therefore, if the support of the centered blurring function is $q = 2m + 1 \leqslant n$, we have $f_{1-j} = 2f_1 - f_{j+1}$ and $f_{n+j} = 2f_n - f_{n-j}$, for $j = 1, \ldots, m$. Following the analysis given in [2], the structure of the 1D AR-BC matrix A is

$$A = \left(\begin{array}{c|c|c} z_1 + a_0 & \mathbf{0}^{\mathrm{T}} & \mathbf{0} \\ \vdots & & a_m \\ z_m + a_{m-1} & \hat{A} & z_m + a_{m-1} \\ a_m & & \vdots \\ \mathbf{0} & \mathbf{0}^{\mathrm{T}} & z_1 + a_0 \end{array}\right), \tag{10}$$

where $a_j = h_j$, $z_j = 2\sum_{k=j}^{m} h_k$, hence $A_{1,1} = A_{n,n} = z_1 + a_0 = 1$ thanks to the normalization condition, and $\hat{A} \in \mathbb{R}^{(n-2)\times(n-2)}$ is

$$\hat{A} = T(\mathbf{u}) - H(\sigma^2(\mathbf{u}), J\sigma^2(\mathbf{u})), \tag{11}$$

with $\mathbf{u} = (h_0, h_1, \ldots, h_m, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^{n-2}$. According to the brief discussion of Section 2.1, relation (11) implies that $\hat{A} = \tau^{(1)}_{n-2}(h)$. Therefore the AR-BC matrix A is a special case of (8), i.e., $A \in \mathscr{S}^{(1)}$, with $\alpha = \beta = 1$, $v_j = w_{n-1-j} = h_j + 2\sum_{k=j+1}^{m} h_k$, and with $\hat{M} = \hat{A} \in \tau^{(1)}_{n-2}$.

For introducing the AR-BCs in a d-dimensional setting, it is enough to apply anti-reflection with respect to every axis, separately [4,3]. The resulting matrix is described as follows.

Theorem 1 [4]. *Let* $\mathbf{h}$ *be the d-dimensional PSF, then the $n^d \times n^d$ blurring matrix A, with $n \geqslant 3$ and AR-BCs, has the form*

$$A = \left(\begin{array}{c|c|c} z_1 + a_0 & 0 & 0 \\ \vdots & & a_m \\ z_m + a_{m-1} & A^* & z_m + a_{m-1} \\ a_m & & \vdots \\ 0 & 0 & z_1 + a_0 \end{array}\right), \tag{12}$$

where zeros denote null matrices of proper order and especially:

- $a_j \in \mathbb{R}^{n^{d-1}\times n^{d-1}}$ *is the AR-BC matrix related to the $(d-1)$-dimensional PSF* $\mathbf{h}^{(d)}_{(d-1,\{1\},j)} = (h_{j,k})^{m}_{k_1,\ldots,k_{d-1}=-m}$ *and* $z_j = 2\sum_{k=j}^{m} a_k$,
- $A \in \mathscr{S}^{(d)}$.

Clearly, from the previous Theorem 1, it follows that $z_1 + a_0$ in (12) is the AR-BC matrix related to the mask $\mathbf{h}^{(d)}_{(d-1,\{1\})} = \sum_{j=-m}^{m} \mathbf{h}^{(d)}_{(d-1,\{1\},j)}$ which is normalized and strongly symmetric.

In conclusion, from Theorem 1 the AR-BC matrices belong to $\mathscr{S}^{(d)}$ and more specifically, in the next section, we will see that the set of AR-BC matrices constitute a commutative subalgebra of $\mathscr{S}^{(d)}$. As a final tool for the theoretical analysis of Section 3 we report the following corollary, which is also important for an efficient implementation of the truncated SVD as in [1]: its importance concerns the case of separable symbols, for which a natural Kronecker decomposition of the associated matrices can be observed.

Corollary 2 [4]. *Let* $\mathbf{h}$ *be such that* $h_s = b_{s_1} c_{(s_2,\ldots,s_d)}$, *then*:

- $\mathbf{b} = (b_{s_1})^{m}_{s_1=-m}$ *and* $\mathbf{c} = (c_{(s_2,\ldots,s_d)})^{m,\ldots,m}_{s_2=-m,\ldots,s_d=-m}$ *can be chosen as strongly symmetric and normalized PSFs of dimensions* 1 *and* $d-1$, *respectively*,
- *the $n^d \times n^d$ blurring matrix A with AR-BCs has the form $A = B \otimes C$ where B and C are the AR-BC matrices associated with* $\mathbf{b}$ *and* $\mathbf{c}$, *respectively*.

3. Structural and spectral analysis of AR-BC matrices by symbol

We study how the AR-BC matrix associated with a mask $\mathbf{h}$ depends on the symbol h defined in (2). We will consider structural properties and spectral properties by discussing both the general case and, in more detail, the case where h is a multivariate cosine polynomial that satisfies the mild degree condition.

3.1. The $AR_n(\cdot)$ operators

We define $\mathscr{C}_l^{(d)}$ the set of d-variate real-valued cosine polynomials of degree at most l in every variable (the degree l will be omitted when not necessary). Let $h \in \mathscr{C}_l^{(1)}$, then its Fourier coefficients are such that $h_i = h_{-i} \in \mathbb{R}$ with $h_j = 0$ if $|j| > l$, and we can define the one-level $AR_n(\cdot)$ operator

$$AR_n(h(y)) = \begin{pmatrix} h(0) & & \\ \mathbf{v}_{n-2}(h) & \tau_{n-2}^{(1)}(h) & \mathbf{v}'_{n-2}(h) \\ & & h(0) \end{pmatrix}, \tag{13}$$

where for $\mathbf{x} \in \mathbb{R}^n$ we define $\mathbf{x}' = J\mathbf{x}$ and

$$\mathbf{v}_{n-2}(h) = \tau_{n-2}^{(1)}(\phi(h))\mathbf{e}_1, \tag{14}$$

$$(\phi(h))(y) = \frac{h(y) - h(0)}{2(\cos(y) - 1)}. \tag{15}$$

It is interesting to observe that $h(y) - h(0)$ has a zero of order at least 2 at zero, hence $\phi(h) \in \mathscr{C}_{l-1}^{(1)}$ and $(\phi(h))(0) = -h''(0)/2$, in other words the function is well defined at zero (note that $0 \notin G_n^{(1)}$ for any n).

The multidimensional case is simply treated by tensor products. If $h(y) \in \mathscr{C}^{(d)}$, then its Fourier coefficients form a real d-dimensional tensor which is strongly symmetric. In addition, $h(y)$, with $y \in \mathbb{R}^d$, can be written as a linear combination of terms of the form

$$m(y) = \prod_{j=1}^{d} \cos(\alpha_j y_j) \tag{16}$$

where $\alpha_j \in \mathbb{N}^+$. Therefore, for $n \in \mathbb{N}^d$ we define

$$AR_n(m) \equiv AR_n(m(y)) = AR_{n_1}(\cos(\alpha_1 y_1)) \otimes \cdots \otimes AR_{n_d}(\cos(\alpha_d y_d)), \tag{17}$$

and we force

$$AR_n(\alpha h_1(y) + \beta h_2(y)) = \alpha AR_n(h_1(y)) + \beta AR_n(h_2(y)) \tag{18}$$

$\forall \alpha, \beta \in \mathbb{R}$ and $\forall h_1, h_2 \in \mathscr{C}^{(d)}$. The interesting fact is that the given operator describes in a functional way the AR-BC matrices introduced in Theorem 1. For showing the latter statement, which is fully contained in the subsequent Theorem 4, we need a simple lemma.

Lemma 3. *Let $k \in \mathbb{N}$ and $x \in \mathbb{R}$. Then*

$$\frac{k}{2} + \sum_{j=1}^{k-1}(k-j)\cos(jx) = \frac{1}{2}\frac{\cos(kx) - 1}{\cos(x) - 1} = (\phi(\cos(k\cdot))(x),$$

extending by continuity when $\frac{x}{2\pi} \in \mathbb{Z}$.

Proof. The proof consists in a direct check: denoting by $\bar{\alpha}$ the complex conjugate of α,

$$\frac{k}{2}+\sum_{j=1}^{k-1}(k-j)\cos(jx)=\frac{1}{2}\left(z_k(x)+\overline{z_k(x)}-k\right), \tag{19}$$

where

$$\begin{aligned}
z_k(x) &= \sum_{j=0}^{k-1}(k-j)\mathrm{e}^{\mathrm{i}jx}=\sum_{j=0}^{k-1}\sum_{q=0}^{j}\mathrm{e}^{\mathrm{i}qx}\\
&= \sum_{j=0}^{k-1}\frac{\mathrm{e}^{\mathrm{i}(j+1)x}-1}{\mathrm{e}^{\mathrm{i}x}-1}=\frac{\mathrm{e}^{\mathrm{i}x}\frac{\mathrm{e}^{\mathrm{i}kx}-1}{\mathrm{e}^{\mathrm{i}x}-1}-k}{\mathrm{e}^{\mathrm{i}x}-1}\\
&= \frac{\mathrm{e}^{\mathrm{i}(1+k)x}-(k+1)\mathrm{e}^{\mathrm{i}x}+k}{(\mathrm{e}^{\mathrm{i}x}-1)^2}=\frac{\sum_{j=1}^{k}\mathrm{e}^{\mathrm{i}jx}-k}{\mathrm{e}^{\mathrm{i}x}-1}\\
&= \frac{\left(\sum_{j=1}^{k}\mathrm{e}^{\mathrm{i}jx}-k\right)(\mathrm{e}^{-\mathrm{i}x}-1)}{(\mathrm{e}^{\mathrm{i}x}-1)(\mathrm{e}^{-\mathrm{i}x}-1)}=\frac{-\mathrm{e}^{\mathrm{i}kx}-k\mathrm{e}^{\mathrm{i}x}+k+1}{2-2\cos(x)}.
\end{aligned}$$

Therefore

$$z_k(x)+\overline{z_k(x)}=\frac{-\cos(kx)-k\cos(x)+k+1}{1-\cos(x)}=\frac{\cos(kx)-1}{\cos(x)-1}+k$$

and, by replacing this expression in (19), the statement is proven since, by (15), the latter coincides with $(\phi(\cos(k\cdot))(x)$. $\square$

In the following, for the sake of notational simplicity, we set $\mathscr{A}_m(\cos(ky))=\mathscr{A}_m[k]$ with $\mathscr{A}$ being any of the symbols $\tau^{(1)}$, AR and $\mathbf{v}$ introduced in (14). This because by linearity several properties on $h(y)\in\mathscr{C}_m^{(1)}$ can be proved in a simpler fashion on the basis $\cos(ky)$ of $\mathscr{C}_m^{(1)}$ for $k=0,\ldots,m$.

Theorem 4. *Let* $\mathbf{h}$ *be the* d*-dimensional PSF and let* h *be the related symbol as in* (2). *Then the* $n^d\times n^d$ *blurring matrix* A, *with* $n\geqslant 3$ *and AR-BCs, in* (12) *is such that* $A=AR_n(h)$.

Proof. For $d=1$, the matrix A in (10) is linearly depending on the function h generated by the PSF $\mathbf{h}$, since it is linearly depending on the Fourier coefficients h_i, $i\in\mathbb{Z}$. By the degree condition, $h\in\mathscr{C}_{n-3}^{(1)}$ and in particular

$$h(y)=h_0+2\sum_{k=1}^{m}h_k\cos(ky),\quad m\leqslant n-3.$$

Therefore the identity $A=AR_n(h)$ follows from the same identity on the polynomial basis $\cos(ky)$, i.e., it is enough to study the case of $h(y)=\cos(ky)$, for $k=1,\ldots,n-3$ (the case $k=0$ is trivial). The Fourier coefficients of $\cos(ky)$ can be described by $h_s=\frac{1}{2}\delta_{k,|s|}$, $s\in\mathbb{Z}$, where $\delta_{k,|s|}$ is the Kronecker delta centered at k, for $k=1,\ldots,n-3$. Hence, by the Toeplitz-minus-Hankel representation in (7), the 1-level AR-BC matrix A in (10) associated with $\cos(ky)$ can be written as

$$
\begin{array}{r}
0 \;\to \\
1 \;\to \\
\vdots \\
k-1 \;\to \\
k \;\to \\
k+1 \;\to \\
\vdots \\
\vdots \\
n-2 \;\to \\
n-1 \;\to
\end{array}
\left(\begin{array}{c|ccccccc|c}
1 & \multicolumn{7}{c|}{0} & \\
\hline
1 & & & -\frac{1}{2} & 0 & \frac{1}{2} & & & \\
\vdots & & \ddots & & & & \ddots & & \\
1 & -\frac{1}{2} & & & & & & \ddots & \\
\frac{1}{2} & 0 & & & & & & \frac{1}{2} & \\
 & \frac{1}{2} & & & & & & 0 & \frac{1}{2} \\
 & & \ddots & & & & & -\frac{1}{2} & 1 \\
 & & & \ddots & & & \ddots & & \vdots \\
 & & & & \frac{1}{2} & 0 & -\frac{1}{2} & & 1 \\
\hline
 & \multicolumn{7}{c|}{0} & 1
\end{array}\right)
$$

$$
= \begin{pmatrix} 1 & & \\ \mathbf{v}_{n-2,k} & \tau_{n-2}^{(1)}[k] & \mathbf{v}'_{n-2,k} \\ & & 1 \end{pmatrix}
$$

with

$$
\mathbf{v}_{m,k} = [\,\underbrace{1, \ldots, 1, \tfrac{1}{2}}_{k}, \underbrace{0, \ldots, 0}_{m-k}\,]^{\mathrm{T}}.
$$

We notice that $\mathbf{v}_{m,0} = \mathbf{0}$ while for negative k it is evident that $\mathbf{v}_{m,k} \equiv \mathbf{v}_{m,|k|}$. Therefore, since $\cos(ky)$ evaluated at zero is equal to 1, by (13) it follows that the above expression of the AR-BC matrix A coincides with $AR_n[k]$ if we prove that $\mathbf{v}_{n-2,k} = \mathbf{v}_{n-2}[k] \equiv \tau_{n-2}^{(1)}((\phi(\cos(k\cdot)))(y))\mathbf{e}_1$. A direct algebraic check based again upon (7) clearly shows that the latter is true: in fact, by the Fourier expansion in Lemma 3, the nonzero Fourier coefficients of $\varphi_k(y) \equiv (\phi(\cos(k\cdot)))(y)$ are $a_i = (k-i)/2$, $i = 0, \ldots, k-1$. Hence the first column of $\tau_{n-2}^{(1)}(\varphi_k)$ is exactly $\mathbf{v}_{n-2,k}$ for $k \leqslant n-3$, since, by the Toeplitz-Hankel representation in (7), such a first column vector can be expressed as the first column of the Toeplitz part $[a_0, \ldots, a_{k-1}, 0, \ldots, 0]^{\mathrm{T}} \in \mathbb{R}^n$, minus the first column of the Hankel part $[a_2, \ldots, a_{k-1}, 0, \ldots, 0]^{\mathrm{T}} \in \mathbb{R}^n$. In conclusion we can state that

$$
A = AR_n[k] \equiv AR_n(\cos(ky)) \quad \text{for } |k| \leqslant n-3. \tag{20}
$$

For $d > 1$, the claimed thesis follows from the tensor definition of the linear operator $AR_n(\cdot)$ in (13), (16)–(18), and from the analogous tensor decomposition of the AR-BC matrix A, which is contained in Corollary 2. $\square$

For the sake of completeness we have to observe that $AR_n(h)$, for h violating the degree condition, is not the AR-BC matrix A as defined in Section 2.2. Indeed, in such a case, the PSF has infinite support and the central block of size $n-2$ (and entries of size n^{d-1}) in A may fail to have a block $\tau^{(1)}$ structure. This should be not surprising since the same trouble is observed, with slightly different degree conditions, also when dealing with periodic and reflective BCs. From the viewpoint of the modeler, the problem is not substantial at all, since the degree condition does not hold only for Fourier coefficients of the PSF which are infinitesimal (since at least one index i_j is of the order of n_j and because of the normalization condition). Therefore, we can state that the extension represented by the $AR_n(\cdot)$ operator, as defined in (13), (16)–(18) is canonical in the sense that it does not differ so much from the real AR-BC matrix and at the same time, as we will see in the next Theorem 5, it preserves important theoretical features also when the degree condition is violated. More specifically, $AR_n(\cdot)$ can be viewed as an algebra homomorphism between $\mathscr{C}_{n-2}^{(d)}$ and $\mathscr{S}^{(d)}$ that is

$$AR_n(h_1 \odot h_2) = AR_n(h_1) \odot AR_n(h_2), \quad \odot \in \{+, \cdot\}, \tag{21}$$

In this respect, we should observe that the addition $+$ is the standard addition both for matrices and polynomials, while the product $\cdot$ in the right hand-side is the usual matrix-matrix product and the product in the left hand-side is a special internal product in the space $\mathscr{C}_{n-2}^{(d)}$. More precisely, for $d = 1$ and $h_1, h_2 \in \mathscr{C}_{n-2}^{(d)}$ the product $h_1 \cdot h_2$ is the unique polynomial $h \in \mathscr{C}_{n-2}^{(d)}$ that satisfies the following interpolation condition

$$h(y) = z_y, \quad z_y \equiv h_1(y)h_2(y), \quad \forall y \in G_{n-2}^{(1)}. \tag{22}$$

Notice that if the degree of h_1 plus the degree of h_2 does not exceed $n - 2$, then, by the uniqueness of the interpolation, the polynomial h coincides with the product in the usual sense of the standard ring of polynomials: in other words, the coefficients of h are obtained by convolution among those of h_1 and those of h_2. The very same idea applies when considering the inversion. Moreover, in the multidimensional case, i.e., $d \geqslant 2$, the definition is formally identical. There exists only a delicate issue since, in general, the interpolation problem in many dimensions is not necessarily associated with an invertible Vandermonde matrix. In fact, the pairwise distinction of the nodes is not sufficient for guaranteeing such an invertibility. However, in our setting this is not a problem since the grid $G_{n-2}^{(d)}$ is in tensor form and therefore it is simple to check the unique solvability of the corresponding interpolation problem in $\mathscr{C}_{n-2}^{(d)}$. In conclusion, with this careful definition of the product/inversion and with the standard definition of the addition, $\mathscr{C}_{n-2}^{(d)}$ has become an algebra of vector-space dimension equal to $(n-2)^d < (3n-4)^d = \dim(\mathscr{S}^{(d)})$ with $n \geqslant 3$.

Similar canonical extensions can be defined for completing the periodic and reflective BC matrices, when the related degree condition of the PSF is not fulfilled.

Theorem 5. *With the above definition of the operator $AR_n(\cdot)$ we have*

(1) $\alpha AR_n(h_1) + \beta AR_n(h_2) = AR_n(\alpha h_1 + \beta h_2)$,
(2) $AR_n(h_1)AR_n(h_2) = AR_n(h_1 h_2)$,

for $\alpha, \beta \in \mathbb{R}$ and for $h_1, h_2 \in \mathscr{C}_{n-2}^{(d)}$.

Proof. We start giving a proof of statement (1). The linearity follows directly from the linearity of all the involved operators and, in particular, of the Fourier coefficients with respect to the symbol. Therefore $h(0) = \alpha h_1(0) + \beta h_2(0)$, $[\mathbf{v}_{n-2}(h)]_j = \alpha[\mathbf{v}_{n-2}(h_1)]_j + \beta[\mathbf{v}_{n-2}(h_2)]_j$, $j = 1, \ldots, n-2$ and $\tau_{n-2}^{(1)}(h) = \alpha\tau_{n-2}^{(1)}(h_1) + \beta\tau_{n-2}^{(1)}(h_2)$. This shows the desired property for $d = 1$. The case of $d > 1$ follows from this and the very definition of the $AR_n(\cdot)$ operator in the multilevel case, where just in (18), the linearity is forced.

Statement (2) is a bit more difficult to prove: it suffices to treat the case $d = 1$, since by (17) and (18), the d-level case follows from this. By defining $h \equiv h_1 h_2$, we find

$$AR_n(h_1(y))AR_n(h_2(y)) = \begin{pmatrix} h_1(0)h_2(0) & & \\ \mathbf{v} & \tau_{n-2}^{(1)}((h_1h_2)(y)) & \mathbf{w} \\ & & h_1(0)h_2(0) \end{pmatrix},$$

with

$$\mathbf{v} = h_1(0)\mathbf{v}_{n-2}(h_2) + \tau_{n-2}^{(1)}(h_1)\mathbf{v}_{n-2}(h_2),$$
$$\mathbf{w} = \tau_{n-2}^{(1)}(h_1)J_{n-2}\mathbf{v}_{n-2}(h_2) + h_2(0)J_{n-2}\mathbf{v}_{n-2}(h_1).$$

Since $\tau_{n-2}^{(1)}(h_1h_2) = \tau_{n-2}^{(1)}(h)$, $h_1(0)h_2(0) = h(0)$, and $\tau_{n-2}^{(1)}(g)J_{n-2} = J_{n-2}\tau_{n-2}^{(1)}(g)\ \forall g \in \mathscr{C}_{n-2}^{(d)}$ (every $\tau^{(d)}$ is centro-symmetric), all we have to prove is that

$$\mathbf{v}_{n-2}(h) = h_2(0)\mathbf{v}_{n-2}(h_1) + \tau_{n-2}^{(1)}(h_1)\mathbf{v}_{n-2}(h_2), \tag{23}$$

since $\mathbf{w} = J_{n-2}\mathbf{v} \equiv \mathbf{v}'$. Of course, by linearity, it is enough to consider the case of the product with symbols $h_1(y) = \cos(sy)$ and $h_2(y) = \cos(ty)$ for $s, t \in \mathbb{N}^+$: this more specific case is easier to handle thanks to the sparsity of their Fourier expansion. Therefore, for proving (23), it is enough to show that, for the product $AR_n[s]AR_n[t]$, it holds

$$\mathbf{v}_{n-2}(\cos(sy)\cos(ty)) = \mathbf{v}_{n-2}[s] + \tau_{n-2}^{(1)}[s]\mathbf{v}_{n-2}[t]. \tag{24}$$

From the identity $\cos(sy)\cos(ty) = \frac{1}{2}[\cos((s+t)y) + \cos((s-t)x)]$, we infer $AR_n(\cos(sy)\cos(ty)) = \frac{1}{2}(AR_n[s+t] + AR_n[s-t])$ and then

$$\mathbf{v}_{n-2}(\cos(sy)\cos(ty)) = \frac{1}{2}(\mathbf{v}_{n-2}[s+t] + \mathbf{v}_{n-2}[s-t]). \tag{25}$$

Therefore, for proving $AR_n[s]AR_n[t] = AR_n(\cos(sy)\cos(ty))$, from (24) and (25) we are reduced to show that

$$\mathbf{v}_{n-2}[s] + \tau_{n-2}^{(1)}[s]\mathbf{v}_{n-2}[t] = \frac{1}{2}(\mathbf{v}_{n-2}[s+t] + \mathbf{v}_{n-2}[s-t]). \tag{26}$$

Recalling that $\mathbf{v}_{n-2}[k] = \tau_{n-2}(\varphi_k)\mathbf{e}_1$, $\varphi_k(y) \equiv (\phi(\cos(k\cdot)))(y)$, by the definition of $\mathbf{v}_{n-2}(\cdot)$ in (14), the (26) is reduced to an identity between $\tau_{n-2}^{(1)}$ matrices. Finally, from (15), it is clear that (26) follows from the trigonometric identity

$$\frac{\cos(sy)-1}{2(\cos(y)-1)} + \cos(sy)\frac{\cos(ty)-1}{2(\cos(y)-1)} = \frac{1}{2}\left(\frac{\cos((s+t)y)-1}{2(\cos(y)-1)} + \frac{\cos((s-t)y)-1}{2(\cos(y)-1)}\right)$$

that is equal to $\cos(sy)\cos(ty) = \frac{1}{2}(\cos((s+t)y) + \cos(s-t)y)$ which is known to be true. $\square$

Remark 6. It is interesting to observe that, as a consequence of the above Theorem 5, the $AR_n(\cdot)$ operator restricted to symbols related to strongly symmetric normalized PSFs behaves as a group homomorphism with respect to the product. Indeed if $\mathbf{h_1}$ and $\mathbf{h_2}$ are d-dimensional strongly symmetric normalized PSFs, then the convolution $\mathbf{h} = \mathbf{h_1} * \mathbf{h_2}$ is clearly still a d-dimensional strongly symmetric normalized PSF. In terms of symbols, this means that $h \in \mathscr{C}^{(d)}$ having non-negative Fourier coefficients, is such that $h(0) = 1$ (analogously to h_j, $j = 1, 2$), and furthermore $AR_n(h) = AR_n(h_1)AR_n(h_2)$ since h coincides with h_1h_2.

Remark 7. Theorems 4 and 5 implicitly state that the matrices $AR_n(h)$, where $h \in \mathscr{C}^{(d)}$, form a commutative subalgebra of $\mathscr{S}^{(d)}$. Consequently, if there exists some matrix $AR_n(h)$ with pairwise distinct eigenvalues (such an example is easy to construct) then all the matrices in the subalgebra are diagonalizable by the same similarity, which means the existence of an anti-reflective transform. This issue is studied in [7].

Remark 8 (*Practical use of our functional characterization*). The results shown in this section so far, especially Theorem 4 and Theorem 5, are of practical relevance in the Tikhonov regularization with reblurring (see the end of Section 2.1). In fact let $M = AR_n(h)$ be the AR-BC blurring matrix, let $\mathscr{Z} = AR_n(z)$ be an even-order AR-BC differential-like operator (the regularizing operator), and let $\mu > 0$ be the Tikhonov-like regularization parameter: then we have to solve linear systems with coefficient matrix $B = M^2 + \mu\mathscr{Z}$. With the algorithm proposed in [4], we work in the algebra $S_n^{(d)}$ for dealing with the matrix B. Conversely, thanks to our new functional characterization we know that $B = AR_n(h^2 + \mu z)$ and therefore we just work on the PSFs by making simple convolutions, with a subsequent great saving of computational cost especially when h and z have local Fourier support. Analogous considerations can be repeated verbatim for any regularizing procedure which employs the reblurring idea.

In this last part of the section we show that the general algebras $\mathscr{S}^{(d)}$ are not as nice as their subalgebras induced by the $AR_n(\cdot)$ operators. Precise statements are contained in the following proposition.

Proposition 9. *For every algebra $\mathscr{S}_n^{(d)}$ with $d \geqslant 1$ and $n \geqslant 3$,*

(1) $\exists M \in \mathscr{S}_n^{(d)}$ *such that M is non-diagonalizable*;
(2) $\exists M, N \in \mathscr{S}_n^{(d)}$ *such that $MN \neq NM$.*

Proof. Both the claims can be easily proved for $d = 1, n \geqslant 3$: the general case follows from this.

For the first claim we consider the matrix $M \in \mathscr{S}_n^{(1)}$ in (8) where $\beta \neq \alpha$, α belongs to the spectrum of $\hat{M}$, and $\mathbf{v}$ is a vector not belonging to the space spanned by the columns of $\hat{M} - \alpha I_{n-2}$, $n \geqslant 3$. We impose $M\mathbf{x} = \alpha\mathbf{x}$ and we show that the geometric multiplicity of α is different from the algebraic one, which in turn implies that M does not possess a basis of eigenvectors. Let us write $\mathbf{x} \in \mathbb{R}^n$ as $\mathbf{x} = (x_1, \tilde{\mathbf{x}}^{\mathrm{T}}, x_n)^{\mathrm{T}}$. Then $M\mathbf{x} = \alpha\mathbf{x}$ if and only if

$$\alpha x_1 = \alpha x_1, \quad x_1\mathbf{v} + \hat{M}\tilde{\mathbf{x}} + x_n\mathbf{w} = \alpha\tilde{\mathbf{x}}, \quad \beta x_n = \alpha x_n$$

and, by the assumptions, we deduce $x_n = x_1 = 0$ and $\hat{M}\tilde{\mathbf{x}} = \alpha\tilde{\mathbf{x}}$. Consequently, if ν is the algebraic and the geometric multiplicity of α as eigenvalue of $\hat{M} \in \tau_{n-2}^{(1)}$, then $\nu + 1$ is the algebraic multiplicity of α as eigenvalue of M, while its geometric multiplicity is still ν.

For the proof of the second claim is enough to choose $M, N \in \mathscr{S}_n^{(1)}$, $n \geqslant 3$,

$$M = \begin{pmatrix} \alpha & & \\ \mathbf{v} & \hat{M} & \mathbf{w} \\ & & \beta \end{pmatrix}, \qquad N = \begin{pmatrix} \alpha_{(1)} & & \\ \mathbf{v}_{(1)} & \hat{M}_{(1)} & \mathbf{w}_{(1)} \\ & & \beta_{(1)} \end{pmatrix},$$

such that $\alpha_{(1)}\mathbf{v} + \hat{M}\mathbf{v}_{(1)} \neq \alpha\mathbf{v}_{(1)} + \hat{M}_{(1)}\mathbf{v}$: the latter clearly implies that $MN \neq NM$ even if MN and NM are both $\mathscr{S}_n^{(1)}$ matrices. $\square$

We now motivate our practical interest in the algebras $\mathscr{S}_n^{(d)}$. The first point comes from [4], where the algorithms for eigenvalue computation, matrix–vector multiplication, and linear system solution of AR-BC matrices were described in the larger $\mathscr{S}_n^{(d)}$ framework. The second is more technical and, at the same time, more substantial: when dealing with a multigrid procedure as e.g. the V-cycle, we have observed that, starting at the highest level with a matrix $AR_n(h)$ in the commutative subalgebra with $d = 1$, the structure at the lower levels is maintained only in a weak sense. More precisely, the first projected matrix A_k is of size k with $n = 2k - 1$ and it cannot be viewed in general as $AR_k(h_{\text{new}})$ for some symbol h_{new}, but it still belongs to $\mathscr{S}_k^{(1)}$. More generally, if at size n the original coefficient matrix lies in $\mathscr{S}_n^{(1)}$, then all the projected matrices at the different levels i and size n_i will continue to belong to $\mathscr{S}_{n_i}^{(1)}$, $i = 1, \ldots, L$, $L = O(\log n)$. The reasoning plainly extends to the case of a general $d \geqslant 1$.

3.2. *The eigenvalues of the AR-BC matrices*

The spectral structure of any d-dimensional matrix $AR_n(h)$ is concisely described in the following results, whose proof is heavily based upon Theorem 1 and upon the following remark.

Remark 10. As already observed in [4], if we exchange the first and the t-th variable, $t = 2, \ldots, d$, both in the ordering of the equations and of the unknowns, then the structure in (12) is exactly the same, but the partial PSFs will change and, in particular, a_j is the $(d-1)$-dimensional AR-BC matrix related to the PSF $\mathbf{h}^{(d)}_{(d-1,\{t\},j)}$ and $a_0 + z_1$ is the $(d-1)$-dimensional AR-BC matrix related to the PSF $\mathbf{h}^{(d)}_{(d-1,\{t\})} = \sum_{j=-m}^{m} \mathbf{h}^{(d)}_{(d-1,\{t\},j)}$. For giving a precise definition of the above PSFs, we observe that the FOV $\{1, \ldots, n\}^d$ can be seen as multiindices of a uniform gridding in the unit cube $C_d = [0, 1]^d$ and we describe the various partial PSFs in terms of C_d and of its faces. We say that X is a k-dimensional face (indeed a k-dimensional unit cube) of C_d, $0 \leqslant k \leqslant d$, if $X = \partial C_d \cap \mathscr{V}$ with $\mathscr{V}$ being a k-dimensional affine space of $\mathbb{R}^d$. We want to define a k-dimensional PSF starting from a given d-dimensional strongly symmetric PSF with $0 \leqslant k \leqslant d$. There will be a unique set of parallel k-dimensional faces uniquely determined by $d - k$ directions to which these faces are orthogonal. For instance, $\mathbf{h}^{(2)}_{(1,\{2\})}$ represents the PSF of the two 1-dimensional faces orthogonal to the second axis x_2; analogously $\mathbf{h}^{(d)}_{(d-1,\{t\})}$ is the PSF of the two $(d-1)$-dimensional faces orthogonal to the axis x_t. Now we generalize this idea. Let $d \geqslant 1$, $0 \leqslant k \leqslant d$, let $\mathscr{F} \subset \{1, \ldots, d\}$ with $\#\mathscr{F} = d - k$, and let $z = (z_j)_{j \in \mathscr{F}}$. We define

$$\mathbf{h}^{(d)}_{(k,\mathscr{F},z)} = (h_s)_{\substack{s_j=-m,\ldots,m,\, j \notin \mathscr{F} \\ s_j = z_j,\, j \in \mathscr{F}}} \tag{27}$$

and

$$\mathbf{h}^{(d)}_{(k,\mathscr{F})} = \sum_{j \in \mathscr{F}} \sum_{z_j=-m}^{m} \mathbf{h}^{(d)}_{(k,\mathscr{F},z)}. \tag{28}$$

As already anticipated $\mathbf{h}^{(d)}_{(k,\mathscr{F})}$ is the k-dimensional PSF (still strongly symmetric and normalized if $\mathbf{h}$ is) associated with the set of parallel k-dimensional faces orthogonal to every axis x_j with $j \in \mathscr{F}$. It is clear that for $k = 0$, $\mathscr{F} = \{1, \ldots, d\}$ and $\mathbf{h}^{(d)}_{(k,\mathscr{F})} \equiv 1$, owing to the normalization condition, while, for $k = d$, $\mathscr{F} = \emptyset$ and $\mathbf{h}^{(d)}_{(k,\mathscr{F})} \equiv \mathbf{h}$.

More in detail, we first consider the simplest case of permutationally symmetric PSFs. We define a PSF $\mathbf{h}$ to be *permutationally symmetric* if $h_s = h_{|s_\sigma|}$, $s_\sigma = (s_{\sigma(1)}, \ldots, s_{\sigma(d)})$, $\forall s \in \mathbb{Z}^d$, for every permutation $\sigma(\cdot)$ of $\{1, \ldots, d\}$. Of course the strong symmetry follows from the permutational symmetry. Moreover, in such case, since the specific variables do not have any specific importance, from (27) and (28), we deduce that $\mathbf{h}^{(d)}_{(k,\mathscr{F},z)} = \mathbf{h}^{(d)}_{(k,\mathscr{F}',z)} \equiv \mathbf{h}^{(d)}_{(k,z)}$, $z \in \{-m, \ldots, m\}^{d-k}$ and $\mathbf{h}^{(d)}_{(k,\mathscr{F})} = \mathbf{h}^{(d)}_{(k,\mathscr{F}')} = \mathbf{h}^{(d)}_{(k)}$, whenever $\#\mathscr{F} = \#\mathscr{F}' = d - k$.

Theorem 11. *Let the d-dimensional PSF* $\mathbf{h}$ *be permutationally symmetric. Let $\alpha_k^{(d)}$, $0 \leqslant k \leqslant d$, be the cardinality of k-dimensional faces of the unit d-dimensional cube C_d. Then the eigenvalues of $A = AR_n(h)$ are given by*

$$\bigcup_{k=0}^{d} \{h_{(k)}^{(d)}(y)|_{y \in G_{n-2}^{(k)}}\}^{\alpha_k^{(d)}}, \tag{29}$$

where $\alpha_k^{(d)}$ counts the multiplicity (algebraic and geometric) of the eigenvalues listed in the related set, the functions $h_{(k)}^{(d)}$ are the symbols associated with $\mathbf{h}_{(k)}^{(d)}$, where the notation $h_{(k)}^{(d)}(y)|_{y \in G_{n-2}^{(k)}}$ indicates the set of numbers obtained as a sampling of $h_{(k)}^{(d)}(y)$ for $y \in G_{n-2}^{(k)}$ defined in (6), and where for $k = 0$ the set $G_{n-2}^{(k)}$ is empty and we set $\{h_{(0)}^{(d)}(y)|_{y \in G_{n-2}^{(0)}}\} = \{1\}$.

Proof. We provide a nice combinatorial interpretation of (29) which is useful to better understand the proof. First, through a proper bijection, we identify the FOV $\{1, \ldots, n\}^d$ with a uniform grid D_d of the unit cube $C_d = [0, 1]^d$; D_d is a *discrete* d-dimensional cube, which has its own faces and facets (defined as intersection of D_d with the faces and facets of C_d). As example, D_d has 2^d vertices described by s, with $s_i \in \{1, n\}$, $i = 1, \ldots, d$, and $2d$ facets. We observe that $\alpha_k^{(d)} = \binom{d}{k} 2^{d-k}$: indeed, for any fixed set I of $d - k$ directions, there is a well-identified set of slices of D_d which have dimension k and are orthogonal to the directions in I; these slices are n^{d-k} since they have $d - k$ fixed parameters and each parameter can be chosen in $\{1, \ldots, n\}$, but a face is obtained only if each parameter belongs to $\{1, n\}$.

The combinatorial interpretation is the following: if we consider the algorithm proposed in Section 2.2.1 of [4], we see that it is possible to compute all the unknowns if we solve first the unknowns which belong to the lowest affine dimension faces of D_d, and in particular the solution of the (remaining) unknowns which belong to a common face of D_d, which has dimension k, is done at once by solving a $\tau^{(k)}$ system. It follows that there exists a common permutation of unknowns and equations such that the matrix A is block upper-triangular, and the diagonal blocks are the $\tau^{(k)}$ matrices which are solved in the cited algorithm of [4], and this means that we have $\alpha_k^{(d)}$ matrices in $\tau^{(k)}$, for all $k = 0, \ldots, d$. The last statement is the thesis of the present theorem.

A formal proof of what above stated can be given by recursion on d. It is an exercise to prove that there exists a common permutation of unknowns and equations such that the matrix A is block upper-triangular and it has several $\tau^{(k)}$ blocks on the diagonal (generated by $h_{(k)}^{(d)}$); here the difficult point is to clarify *how many* blocks of *what size* appear on the diagonal of $\hat{A}$, so this will be done by recursion. We state the property $\mathscr{P}(d)$ as: "there exists a permutation matrix P such that PAP^{T} is block upper triangular and the diagonal blocks are $\tau^{(k)}(h_{(k)}^{(d)})$ (with multiplicity $\alpha_k^{(d)}$) for $k = 0, \ldots, d$". It follows from $\mathscr{P}(d)$ that the eigenvalues of A are those in (29). $\mathscr{P}(1)$ is straightforward since we have twice the block 1, which is 1-by-1, and once the block $\tau_{n-2}^{(1)}(h_{(1)}^{(1)})$. For the inductive step we use (9) and $\hat{P}^{(d)} = \begin{pmatrix} & I_{n-2} & \\ 1 & & \\ & & 1 \end{pmatrix} \otimes I_{n^{d-1}}$ to prove

$$\hat{P}^{(d)} A (\hat{P}^{(d)})^{\mathrm{T}} = \begin{pmatrix} M^* & \times & \times \\ & \alpha & \\ & & \beta \end{pmatrix}. \tag{30}$$

Now, by induction, the eigenvalues in the second and third diagonal block are

$$\bigcup_{k=0}^{d-1}\left\{h^{(d)}_{(k)}(y)|_{y\in G^{(k)}_{n-2}}\right\}^{2\alpha_k^{(d-1)}},\tag{31}$$

since in (30) both α and β are the $(d-1)$-level AR-BC matrix $AR_n(h^{(d)}_{d-1})$. The first diagonal block M^* has external $\tau^{(1)}$ structure and internal $(d-1)$-level entries which belong to $\mathscr{S}^{(d)}$, so $\mathscr{P}(d-1)$ can still be applied to its internal blocks of M^*. It follows that, if we permute the indices (in M^* only) according to $(1, 2, \ldots, n) \to (2, \ldots, n, 1)$, we get a structure which is external $(d-1)$-level $\mathscr{S}^{(d-1)}$ and internal $\tau^{(1)}$: by the property $\mathscr{P}(d-1)$ applied to the external $d-1$ levels of the block we obtain that there exists a permutation matrix P such that PM^*P^{T} is block upper triangular and the diagonal blocks are $\tau^{(k)}(h^{(d)}_{(k)})$ (with multiplicity $\alpha_k^{(d)}$) for $k = 0, \ldots, d-1$ *and such blocks have inner* $\tau^{(1)}$ *structure*. The eigenvalues are

$$\bigcup_{k=0}^{d-1}\left\{h^{(d)}_{(k+1)}(y)|_{y\in G^{(k+1)}_{n-2}}\right\}^{\alpha_k^{(d-1)}}=\bigcup_{k=1}^{d}\left\{h^{(d)}_{(k)}(y)|_{y\in G^{(k)}_{n-2}}\right\}^{\alpha_{k-1}^{(d-1)}}.\tag{32}$$

To complete the proof, we just have to observe that (29) is the union of (31) and (32). Indeed $\alpha_0^{(d)} = 2\alpha_0^{(d-1)}$, and $\alpha_k^{(d)} = 2\alpha_k^{(d-1)} + \alpha_{k-1}^{(d-1)}$ for $k = 1, \ldots, d-1$, and $\alpha_d^{(d)} = 1 = \alpha_{d-1}^{(d-1)}$. □

We can now deal with the more general case, when the PSF is strongly symmetric but not necessarily permutationally symmetric.

Theorem 12. *Let the d-dimensional PSF* $\mathbf{h}$, *the eigenvalues of* $AR_n(h)$, $n \geqslant 3$, *are given by*

$$\bigcup_{k=0}^{d}\ \bigcup_{\substack{\mathscr{F}\subset\{1,\ldots,d\},\\ \#\mathscr{F}=d-k}}\left\{h^{(d)}_{(k,\mathscr{F})}(y)|_{y\in G^{(k)}_{n-2}}\right\}^{\beta^{(d)}(\mathscr{F})},\tag{33}$$

where the relevant notations are taken from Theorem 11, except for $\beta^{(d)}(\mathscr{F}) \in \mathbb{N}^+$ *which represents the number of k-dimensional faces orthogonal to the set of axes associated with the variables in* $\mathscr{F}$ *(notice that for $k = d$ we have* $\#\mathscr{F} = 0$, *i.e.*, $\mathscr{F} = \emptyset$ *and therefore* $\beta^{(d)}(\mathscr{F}) = 1$, *since the only d-dimensional face of C_d is the whole cube C_d itself).*

Proof. We use induction on the number of levels d. For $d = 1$, the basis is the same as in Theorem 11 because the sets in (29) and (33) coincide. On the other hand, for $d > 1$ the structure of the reasoning is identical. The notation is a bit different and more involved, since it is no longer true that the blocks $\tau^{(k)}$ share a common generator: now the generators of the $\tau^{(k)}$ blocks are $h^{(d)}_{(k,\mathscr{F})}$, for all $\mathscr{F} \subseteq \{1, \ldots, n\}$ such that $\#\mathscr{F} = d-k$, and each $\tau^{(k)}(h^{(d)}_{(k,\mathscr{F})})$ is repeated $\beta_k^{(d)}(\mathscr{F}) = 2^{d-k}$ times). Therefore we will not repeat the details here. □

It is clear that, under the assumptions of Theorem 12, the eigenvalues of any AR-BC matrix A are the values that several cosine polynomials take at the grid points of j-dimensional uniform meshes for $0 \leqslant j \leqslant d$. Therefore computing the spectrum can be carried out by means of $\sum_{j=0}^{d}\alpha_j^{(d)} = 3^d$ FFTs with a total cost of $O\left(\sum_{j=0}^{d}\alpha_j^{(d)}(n-2)^j\log(n-2)\right) = O(n^d\log(n))$ ops. Observe also that the eigenvalues of A are those of suitable $\tau^{(j)}_{n-2}$ matrices, for $0 \leqslant j \leqslant$

d, explicitly described in Theorem 1 and Remark 10, which can be computed by means of j-dimensional DSTs, $0 \leqslant j \leqslant d$, at the same asymptotic cost.

3.2.1. The computation of the eigenvalues: a fast spectral algorithm

Theorems 11 and 12 suggest a fast way to compute the eigenvalues of AR-BC matrices, which we describe in the case of permutationally symmetric PSF.

Define $\Lambda = \emptyset$. Then for $k = d$ down to 0 do:

(1) compute $\mathbf{h}^{(d)}_{(k)}$: nothing to do if $k = d$ since $\mathbf{h}^{(d)}_{(d)} = \mathbf{h}$, else if $k < d$ use $\mathbf{h}^{(d)}_{(k)} = \sum_{(k+1)\text{-th index}} \mathbf{h}^{(d)}_{(k+1)}$ where the sum is done point-wise with respect to the last index, i.e., all the k-dimensional slices of $\mathbf{h}^{(d)}_{(k+1)}$ which are orthogonal to the index $k+1$ are summed;
(2) compute the first column $\mathbf{a}^{(k)}$ of $\tau^{(k)}(h^{(d)}_{(k)})$ by use of $\mathbf{h}^{(d)}_{(k)}$;
(3) diagonalize $\tau^{(k)}(h^{(d)}_{(k)})$ by k-dimensional DST-I of size $(n-2)^k$ on $\mathbf{a}^{(k)}$, and define $\Lambda^{(k)}$ the set of the computed eigenvalues;
(4) compute $\alpha^{(d)}_k = \binom{d}{k} 2^{d-k}$;
(5) update $\Lambda := \Lambda \cup (\Lambda^{(k)})^{\alpha^{(d)}_k}$.

We remark that the eigenvalues in Λ can be rearranged in a one-to-one correspondence with the FOV: the $\alpha^{(d)}_0 = 2^d$ eigenvalues which are computed at the step $k = 0$ of the algorithm are all equal to 1, and correspond to the 2^d vertices of the FOV; all the eigenvalues which are computed at step $k > 0$ (they are $(n-2)^k$ with multiplicity $\alpha^{(d)}_k$) are associated with the *inner part* of the $\alpha^{(d)}_k$ k-dimensional faces of the FOV. In particular, the eigenvalues computed at step $k = d$ are $(n-2)^d$ (repeated just once) and correspond to the subset $\{2, \ldots, n-1\}^d$ of the FOV. We also remark that for $h \in \mathscr{C}^{(d)}_m$, the cost of computing all the $\mathbf{h}^{(d)}_{(k)}$ is $\sum_{k=1}^{d}(2m)(2m+1)^{d-k} = (2m+1)^d - 1$, which is less than $O(n^d \log(n))$ ops under the mild degree condition, and the cost of computing $\mathbf{a}^{(k)}$ is comparable with the one of $\mathbf{h}^{(d)}_{(k)}$.

The algorithm is a bit more involved in the case of strongly (but not permutationally) symmetric PSF, because the $\alpha^{(d)}_k$ matrices in $\tau^{(k)}$ do not have a common generator: for any k there are (potentially up to) $\binom{d}{k}$ different matrices $\tau^{(k)}(h^{(d)}_{(k,\mathscr{F})})$, each one with multiplicity 2^{d-k}. The extra task of the algorithm is that all the $\mathbf{h}^{(d)}_{(k,\mathscr{F})}$, $\mathscr{F} \subseteq \{1, \ldots, d\}$, $\#\mathscr{F} = d-k$ have to be computed, but the cost of determining all such $\mathbf{h}^{(d)}_{(k,\mathscr{F})}$, as well as the first columns $\mathbf{a}^{(k)}_{\mathscr{F}}$ of $\tau^{(k)}(h^{(d)}_{(k,\mathscr{F})})$, is still comparable with $(2m+1)^d - 1$.

3.3. The spectrum of AR-BC matrices with general PSFs

When the strongly symmetry condition considered so far is violated and when considering no assumptions on the support of the PSF, the AR-BC matrices can become arbitrarily non-normal and hence their spectral behavior becomes less regular: take in consideration a motion blur, where the zero Dirichlet matrix is $\epsilon I + (1-\epsilon)J$ with J Jordan block and small $\epsilon > 0$; in that case the zero Dirichlet, the reflective, and the anti-reflective matrices are severely non-normal, while the periodic matrix still is. However, if Tikhonov-like techniques are applied, then it is of interest to understand the spectral features of $A^{\mathrm{T}}A$ and of $A'A$, $X \mapsto X'$ being the correlation operation according to the reblurring strategy [5], X being a square blurring matrix with some imposed BCs. In both cases, formulae in close form (as those in Theorems 11 and 12) do not exist in general, but we can furnish asymptotical distribution results in the sense of [10] and localization, extremal results on the singular values of A and A'.

3.3.1. *Distribution results for sequences of AR-BC matrices*

When dealing with $A^{\mathrm{T}}A$ and by using proper tools [11,10], it is easily verified that the sequences $\{A^{\mathrm{T}}A\}$ and $\{T_n(|h|^2)\}$ are equally distributed, if the complex-valued function h, having arbitrary support, is smooth enough. In our case, thanks to the normalization condition and to the nonnegativity of the Fourier coefficients, the symbol h belongs to the Wiener algebra [12] and therefore h is at least continuous. As a consequence, the eigenvalues of $A^{\mathrm{T}}A$ behave asymptotically (that is for n large enough) as a uniform sampling of $|h|^2$ over $G_n^{(d)}$. This follows from standard tools: in fact, if $\{A_n\}$ and $\{B_n\}$ are equally distributed and $\{B_n\}$ is distributed as a function f, then the same is true for $\{A_n\}$, see [10]. In our setting, we know that $\{A^{\mathrm{T}}A\}$ and $\{T_n(|h|^2)\}$ are equally distributed since the involved Hankel matrix sequences are distributed as the zero function (see [11]); furthermore, by Szegö (see [10] and references therein), $\{T_n(f)\}$ is distributed as f over $[-\pi, \pi]^d$ so that the symbol $|h|^2$ is the distribution function for $\{A^{\mathrm{T}}A\}$ too.

The case of $\{A'A\}$ is a bit more involved and requires more recent tools, essentially because $A'A$ may fail to be symmetric. However, under the hypothesis that h is smooth enough (e.g. $h \in C^2$), it is plain to prove that $\{A'A\}$ is uniformly bounded in spectral norm (maximal singular value, [13]) and the trace norm (sum of all singular values, [13]) of $A'A - A^{\mathrm{T}}A$ divided by n^d, is infinitesimal as n tends to infinity. A recent perturbation result (see [14, Theorem 3.4]) implies that $\{A'A\}$ shows the same spectral distribution as $\{A^{\mathrm{T}}A\}$. In our case the sequence $\{A^{\mathrm{T}}A\}$ is distributed as $|h|^2$ over $[0, \pi]^d$ and hence the same is true for $\{A'A\}$, even if the related eigenvalues may have (generally infinitesimal) nonzero imaginary part. Since the symbol $|h|^2$ is nonnegative, it follows that $A'A$, for n large enough, can be regarded as a perturbation of a positive definite matrix. The latter statement leads naturally to the question if this is sufficient for applying safely a conjugate gradient algorithm. Of course this and related issues will be the subject of future researches.

3.3.2. *Localization and extremal spectral results for AR-BC matrices*

Here we are interested in giving uniform bounds and asymptotics on the extreme singular values and eigenvalues of AR-BC matrices. Since $\sigma_{\min} \leqslant |\lambda| \leqslant \sigma_{\max}$ for every square matrix A, for every λ eigenvalue of A, and with $\sigma_{\min}, \sigma_{\max}$ denoting the extreme singular values of A, we focus our attention on localization and asymptotic behavior of the extreme singular values. The main tool is general: in fact it applies to any choice of boundary conditions based on affine relationships between internal and external values. Therefore this invites us to derive results valid for every BCs and to briefly discuss the possible differences.

We follow the very clean presentation of this type of boundary conditions given in Section 2 of [15] (see also Section 2 of [2] for the case of AR-BCs). Given any $\mathbf{x}$ vector of size n^d, and given A_{BC}, BC$\in$ {Dirichlet, Periodic, Reflective, Anti-Reflective}, there exists a unique d-level rectangular Toeplitz matrix T of size $n^d \times (n+m)^d$, with symbol $h(y)$ and independent of the chosen BCs, and there exists a unique vector $\mathbf{x}_{\mathrm{BC}}$ of proper size $(n+m)^d$, depending on the support of the PSF and such that its internal part of size n^d is $\mathbf{x}$, for which

$$A_{\mathrm{BC}}\mathbf{x} = T\mathbf{x}_{\mathrm{BC}}, \quad \mathbf{x}^{\mathrm{T}}\mathbf{x} < \mathbf{x}_{\mathrm{BC}}^{\mathrm{T}}\mathbf{x}_{\mathrm{BC}} < c_{\mathrm{BC}}\mathbf{x}^{\mathrm{T}}\mathbf{x}. \tag{34}$$

Here $c_{\mathrm{BC}} = 1$ for Dirichlet BCs and $c_{\mathrm{BC}} = 2^d$ for the remaining BCs with $m < n$. Therefore, taking into account (34), we find

$$\sigma^2_{\min}(A_{\mathrm{BC}}) = \min_{\mathbf{x}\in\mathbb{R}^{n^d}} \frac{\mathbf{x}^{\mathrm{T}} A^{\mathrm{T}}_{\mathrm{BC}} A_{\mathrm{BC}}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} > \min_{\mathbf{x}\in\mathbb{R}^{n^d}} \frac{\mathbf{x}^{\mathrm{T}}_{\mathrm{BC}} T^{\mathrm{T}} T \mathbf{x}_{\mathrm{BC}}}{\mathbf{x}^{\mathrm{T}}_{\mathrm{BC}}\mathbf{x}_{\mathrm{BC}}} \geqslant \sigma^2_{\min}(T);$$
$$\sigma^2_{\max}(A_{\mathrm{BC}}) = \max_{\mathbf{x}\in\mathbb{R}^{n^d}} \frac{\mathbf{x}^{\mathrm{T}} A^{\mathrm{T}}_{\mathrm{BC}} A_{\mathrm{BC}}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} > c_{\mathrm{BC}} \max_{\mathbf{x}\in\mathbb{R}^{n^d}} \frac{\mathbf{x}^{\mathrm{T}}_{\mathrm{BC}} T^{\mathrm{T}} T \mathbf{x}_{\mathrm{BC}}}{\mathbf{x}^{\mathrm{T}}_{\mathrm{BC}}\mathbf{x}_{\mathrm{BC}}} \leqslant c_{\mathrm{BC}}\sigma^2_{\max}(T).$$

By standard theory on Toeplitz matrices (specific work by Szegö, Widom; see [6] and references there reported), it is well known that

$$\sigma_{\min}(T) \geqslant c(h) \quad \text{and} \quad \sigma_{\max}(T) \leqslant \|h\|_\infty,$$

where by the normalization condition and by the nonnegativity of the Fourier coefficients, it holds $\|h\|_\infty = h(0) = 1$. Furthermore here $c(h)$ denotes the distance of convex hull of the range of h from the complex zero in the complex plane. As a first conclusion, independently of the considered BCs, we have

$$\sigma_{\min}(A_{\mathrm{BC}}) \geqslant c(h) \quad \text{and} \quad \sigma_{\max}(A_{\mathrm{BC}}) \leqslant \|h\|_\infty c_{\mathrm{BC}} = c_{\mathrm{BC}}. \tag{35}$$

Moreover, by the distributional results in Section 3.3.1 and by standard Lebesgue measure arguments, it follows that

$$\limsup_{n\to\infty} \sigma_{\min}(A_{\mathrm{BC}}) \leqslant \min_{y\in\mathbb{R}^d} |h(y)| \quad \text{and} \quad \liminf_{n\to\infty} \sigma_{\max}(A_{\mathrm{BC}}) \geqslant \|h\|_\infty = 1. \tag{36}$$

Therefore, putting together the second parts of (35) and (36), we deduce that the maximal singular value of A_{BC} has limit and this limit is $\|h\|_\infty = 1$, when $c_{\mathrm{BC}} = 1$, i.e., for Dirichlet BCs. On the other hand, we have $\min_{y\in\mathbb{R}^d} |h(y)| \geqslant c(h) \geqslant 0$ (notice that $|h(y)|$ has minimum thanks to the continuity of h and to its 2π-periodicity) and the limit of the minimal singular value of A_{BC} exists and is equal to $\min_{y\in\mathbb{R}^d} |h(y)|$ at least for the periodic BCs, and for the reflective BCs with a strongly symmetric PSF.

A similar analysis can be conducted with the very same arguments when dealing with the extreme singular values of the reblurring matrix $R_{\mathrm{BC}} = A'_{\mathrm{BC}} A_{\mathrm{BC}}$: in that case, the only novelty concerns the symbol of A'_{BC} which is easily identified as $\bar{h}(y)$, with $h(y)$ being the symbol of A_{BC}. By recalling that $h(y)$ and $\bar{h}(y)$ have the same modulus, it follows that

$$\sigma_{\min}(R_{\mathrm{BC}}) \geqslant c^2(h) \quad \text{and} \quad \sigma_{\max}(R_{\mathrm{BC}}) \leqslant \|h\|^2_\infty c^2_{\mathrm{BC}} = c^2_{\mathrm{BC}}, \tag{37}$$

$$\limsup_{n\to\infty} \sigma_{\min}(R_{\mathrm{BC}}) \leqslant \min_{y\in\mathbb{R}^d} |h(y)|^2 \quad \text{and} \quad \liminf_{n\to\infty} \sigma_{\max}(R_{\mathrm{BC}}) > \|h\|^2_\infty = 1, \tag{38}$$

with $\lim_{n\to\infty} \sigma_{\min}(R_{\mathrm{BC}}) = \min_{y\in\mathbb{R}^d} |h(y)|^2$ at least for the periodic BCs, and for the reflective BCs with a strongly symmetric PSF.

4. Concluding remarks

In this note we have analyzed the commutative algebra of the AR-BC matrices and its spectral structure, when strongly symmetric PSFs are involved. Every AR-BC matrix is associated with a proper d-variate cosine polynomial (like the sine or cosine algebras) and the eigenvalues are collectively uniform samplings of size $(n-2)^j$, $0 \leqslant j \leqslant d$, of such polynomial and other related partial polynomials. Therefore their computation can be done in a fast way, via the use of suitable FFT/FST-based procedures within $O(n^d \log(n))$ ops. Furthermore, any sum or product in the algebra, as required e.g. by the Tikhonov procedure with reblurring, can be performed in a straightforward and cheap way, by directly working on the involved symbols.

The definition of the AR-BC transform as studied in [7], joint with the results of this paper, is the basis for the definition of a new class of filtering-type methods for the regularization of the inverse problems arising in image/object reconstruction. In fact, by employing Tikhonov-like techniques and re-blurring [7], two computational tools are important: a fast algorithm for implementing the transform and a fast eigenvalue solver for the diagonalization. While the first is discussed in detail in [7], the second is provided in this note.

Finally, a further contribution of this note relies in the spectral analysis of the AR-BC matrices also in the case of general PSFs: of course, a full understanding of these properties could be of interest in the choice of the solution/regularization method in the nonsymmetric setting.

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References

[1] P.C. Hansen, J.G. Nagy, D.P. O'Leary, Deblurring Images: Matrices, Spectra, and Filtering, SIAM, Philadelphia, PA, 2006.

[2] S. Serra-Capizzano, A note on anti-reflective boundary conditions and fast deblurring models, SIAM J. Sci. Comput. 25 (3) (2003) 1307–1325.

[3] M. Donatelli, C. Estatico, J. Nagy, L. Perrone, S. Serra-Capizzano, Anti-reflective boundary conditions and fast 2d deblurring models, in: F.T. Luk (Ed.), Advanced Signal Processing Algorithms, Architectures, and Implementations VIII, vol. 5205, SPIE, 2004, pp. 380–389.

[4] A. Aricò, M. Donatelli, S. Serra-Capizzano, The anti-reflective algebra: structural and computational analysis, with application to image deblurring and denoising, Calcolo, submitted for publication.

[5] M. Donatelli, C. Estatico, A. Martinelli, S. Serra-Capizzano, Improved image deblurring with anti-reflective boundary conditions and re-blurring, Inverse Problems 22 (2006) 2035–2053.

[6] A. Böttcher, B. Silbermann, Introduction to Large Truncated Toeplitz Matrices, Springer-Verlag, New York, 1999.

[7] A. Aricò, M. Donatelli, J. Nagy, S. Serra-Capizzano, The anti-reflective transform and regularization by filtering, submitted for publication.

[8] D. Bini, M. Capovani, Spectral and computational properties of band symmetric Toeplitz matrices, Linear Algebra Appl. 52/53 (1983) 99–125.

[9] M. Donatelli, S. Serra-Capizzano, Anti-reflective boundary conditions and re-blurring, Inverse Problems 21 (2005) 169–182.

[10] E. Tyrtyshnikov, A unifying approach to some old and new theorems on distribution and clustering, Linear Algebra Appl. 232 (1996) 1–43.

[11] D. Fasino, P. Tilli, Spectral clustering properties of block multilevel Hankel matrices, Linear Algebra Appl. 306 (2000) 155–163.

[12] A. Zygmund, Trigonometric Series, Cambridge University Press, Cambridge, 1959.

[13] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.

[14] L. Golinskii, S. Serra-Capizzano, The asymptotic spectrum of non symmetrically perturbed symmetric Jacobi matrix sequences, J. Approx. Theory 144 (1) (2007) 84–102.

[15] M.K. Ng, R.H. Chan, W. Tang, A fast algorithm for deblurring models with Neumann boundary conditions, SIAM J. Sci. Comput. 21 (1999) 851–866.