Incremental stochastic model for the temporal distribution of peak traffic demand*

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This paper presents the incremental stochastic user equilibrium (ISUE) model for predicting how travellers select their departure times from an origin in a single origin-destination pair system if they have desired times of arrival at the destination. The temporal distribution of the peak traffic is the result of commuters’ selections of departure times. The stochastic user equilibrium (SUE) model is one of the techniques for estimating this distribution, but is computationally cumbersome to apply. In addition, existence and uniqueness of the solution to the SUE formulation and its approximations have not been proven. The ISUE is another approximation to the SUE, and it is based on an approach for which existence and uniqueness of the solution have been established.

Keywords: peak traffic demand, stochastic model

Introduction

Several techniques have been proposed for reducing traffic congestion during the peak period. One such technique is the variable work start time, also known as flexible working hours. Even though many people believe that this approach will reduce congestion, there are no satisfactory quantitative methods for estimating the magnitude of reduction in traffic congestion that would result given a specific range of flexibility in work start times. Where variable work start times have been implemented, there have been no convincing reported empirical data that can isolate the effect of these measures on traffic congestion. Even though the intuitive belief is very convincing, there is a need to develop a quantitative model capable of being used for estimating congestion reductions resulting from variable work start times. However, one needs to understand (1) how a commuter selects a departure time from home and (2) the travel route as a function of desired arrival time at work. The commuter’s decisions about departure times result in the temporal distribution and route assignment of traffic during the peak period.

The need of a model for estimating the temporal distribution and route assignment of peak traffic demand has been outlined by Alfa. He pointed out that existing models are limited to a simple network with only one origin-destination (O-D) pair and one travel corridor between the O-D pair. These models cannot handle realistic practical networks that involve multiple O-D pairs and complex networks with several nodes and interconnecting links, mainly because there are no simple models for estimating travel times in such complex networks when traffic flow is time dependent. Also, the existing models for the simple network with one O-D pair are computationally cumbersome in the first place, or they are based on an unrealistic premise. The model presented in this paper is intended to alleviate the computational problem associated with some of the models for the simple network with one O-D pair.

Two of the existing types of models have been advocated as having the potential for adaptation to a realistic network. They are the deterministic user equilibrium

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approach (DUE) and the stochastic user equilibrium approach (SUE). The SUE is more realistic than the DUE in representing the traveller’s true behaviour, but at the same time it is computationally more cumbersome to apply. In addition, existence and uniqueness of the solution for the approximate approaches developed for the SUE by Alfa and Minh and by De Palma et al., even though reasonable and realistic, have not been proven. An incremental stochastic user equilibrium (ISUE) approach for approximating the SUE is presented here. This approach is computationally less cumbersome than the SUE, and the existence and uniqueness of the solution for this approach has been established. One of the recommendations in Ref. 1 is that both the SUE and DUE be pursued further to develop realistic and simpler approaches that can be incorporated in larger network models. The aim of this paper is to develop a more computationally manageable approximation to the SUE approach for the single O-D system.

Model description

Concept and assumptions

It is believed that most travellers during the peak period are commuters who usually have predetermined times at which they wish to arrive at their destinations. Destination target time (DTT) is defined as the time at which a traveller desires to arrive at the destination. A traveller attaches some cost to arriving at the destination earlier or later than the DTT. He or she also attaches costs to delay in the system. A traveller then selects the departure time to minimize the total cost. The temporal distribution of traffic demand, observed on the road network during the peak period, is mainly the result of the interaction of all commuters’ selected departure times during this period.

Consider an O-D pair connected by one route with a bottleneck in between, as shown in Figure 1. The bottleneck is assumed to have a capacity of 1/S vehicles per unit time. It is assumed that delay is experienced only at the bottleneck. Travel times between O and B1 and between B2 and D are assumed to be constant. Let T0 be the sum of these two constant travel times, where T0 is the minimum (zero-flow) travel time between the O-D pair. Consider a traveller who departed from O at time t. If w(t) is the delay to this traveller at the bottleneck, then t2(t) is the arrival time at D, then

\[ t_2(t) = t + T_0 + w(t) \]  

Let t0 be the traveller’s DTT. Further, let C_u(w(t)), C_b(t_D - t_1(t))^+, and C_v(t_1(t) - t_2)^+ be the costs for delay in the system, and for early and late arrivals at the destination, respectively, where (e)^+ = max(0, e). The total cost C(t, w(t)) to this traveller, who began the trip at time t and was delayed w(t) units of time in the system, is

\[ C(t, w(t)) = C_u(w(t)) + C_b(t_D - t_1(t))^+ + C_v(t_1(t) - t_2)^+ \]  

The traveller selects the departure time t so that C(t, w(t)) is minimized.

This concept is also applicable to the evening peak period. During this period a traveller, instead of having a DTT, will have an origin ‘takeoff’ time, i.e., the desired departure time from the origin. Late and early departures and delays will have costs attached to them. The focus of this paper is the morning peak problem.

The model

It was shown in the previous section how a traveller perceives costs, given the departure time from home and delay, or travel time, in the system. Define at as the probability that a traveller chooses to depart from home at time t. The SUE model was described in Ref. 1 as stating that

\[ a(t) = Pr\{C(t, w(t)) = \min[C(s, w(s))], \forall s \in [t_1, t_2], t_1 \leq t \leq t_2\} \]  

where t1 and t2 are the earliest and latest times a traveller would start a journey from home.

The delay w(t) depends on all a(s) from time s = t1 up to time s = t. Hence, evaluating equation (3) involves an iterative process for which convergence cannot be assured. Existence and uniqueness of the solution to this problem have not been established. Alfa and Minh have proposed simpler models than equation (3), attempting to emulate the SUE. There have not been any proofs of existence or uniqueness of solutions to either one of their models.

This paper offers a much simpler approximation to the SUE, called the ISUE model. A discrete time approach is adopted. Departures and arrivals are assumed to occur at equally spaced time epochs sequentially numbered 0, 1, 2, ….

Incremental stochastic user equilibrium approach

This approach assumes that at some stage, the departure process of travellers in the system is known. What it then predicts is how a new traveller would select a departure time, given that all other travellers have settled for specific choices of departure times. The addition of this new traveller to the system leads to a small adjustment in the overall departure process, which depends on the number of travellers already using the system. The process is initiated by the first traveller. It is similar in principle to the incremental assignment technique.

Consider a situation in which there are I travellers who use the system regularly, and all have chosen their departure times from home. Let a_n(t) be the probability that a traveller, out of these I travellers, departs from home at epoch n. Let A_n(t) be the corresponding number of departures at epoch n. For a departure process a_n(t), n = 0, 1, 2, …, there are corresponding delays w_n(t). Let W_n(t) be the probability that a traveller who departs from home at epoch n is delayed i units of time. W_n(t) can be evaluated as in Alfa and Alfa and Minh.

Suppose a new traveller, not among the I travellers, who wishes to use this system, has to decide on a depar-
ture time from home. Let \( a'_d(I) \) be the probability that the departure occurs at epoch \( n \). If it is assumed that this new traveller wishes to select a departure time such that the total cost is minimum, then \( a'_d(I) \) can be estimated by

\[
a'(l) = \text{Pr}\{C_n < C_u; \forall u \in L_n\}
\]

\[
= \sum_{v=1}^{N-1} \frac{1}{v+1} \text{Pr}\{C_n < C_u; \forall u \in L_{n,v}; \forall \varphi \in L_{n,v}\} \forall n \in L
\]

where \( C_n = \text{total cost to the traveller if departure from home occurs at epoch } n \), \( C_n \) can assume values in the set \( \{C(n, i) \mid i = 1, 2, \ldots, (I + 1) \times S\} \).

\[
N = t_2 - t_1
\]

\[
L \equiv \{i \mid i = 1, 2, 3, \ldots, N\}
\]

\[
L_n \equiv \{i \mid i = n, n+1, \ldots, n+l\}
\]

\[
L_{n,v} \equiv \{i \mid i = n, n+1, \ldots, n+l\}
\]

such that \( C_n = C_n = C_{n-1} = \cdots = C_{n+l} \).

The arguments leading to equation (4) are as follows:

This traveller will choose to depart from the origin at time epoch \( n \) if it will lead to a cost \( C_n \) that is less than the cost \( C_u \) for departing at any other time epoch \( u \), \( u \in L_n \). This gives the first term on the right side of equation (4). In addition, if there are \( v \) other time epochs other than \( n \) at which the traveller can depart and have the same minimum cost as departing at epoch \( n \), then he or she would choose to depart at any of these \( v \) epochs or epoch \( n \) with the same probability, provided that the total costs to be incurred at these epochs, including epoch \( n \), are less than at any other epoch. This leads to the second term on the right side of equation (4).

Equation (4) is a special case of equation (3) in Ref. 4, where the existence of a positive and unique solution for equation (3) was proven. Equation (4) is the key equation in the ISUE approach.

By adding the resulting departure process of this additional traveller to the existing departure process of the \( I \) travellers, we obtain the new departure process involving all \( I + 1 \) travellers. With this additional traveller in the system, the departure time probability for all \( I + 1 \) travellers \( a'(l+1) \) is thus

\[
a'(l+1) = \rho a'_d(l) + (1 - \rho)a'_d(l)
\]

\[
0 \leq \rho \leq 1, \quad 1 \leq n \leq N, \quad I \geq 0
\]

Thus, the new departure time probability is a sum of two components. It is weighted sum of the probability distribution of departure times of all the initial \( I \) travellers and of the additional traveller. By assuming that the weight \( \rho \) depends on the magnitude of \( I \), we can approximate \( \rho \) by \( I/(I+1) \). Hence,

\[
a'_d(I+1) \approx \frac{a'_d(l) + \rho a'_d(l)}{I+1}
\]

\[
I \leq n \leq N, \quad I \geq 0
\]

and

\[
A'_d(I+1) = A'_d(l) + a'_d(l)
\]

\[
I \geq 0, \quad 1 \leq n \leq N
\]

Equation (5a) is the recursive procedure for the incremental stochastic model. It is initiated by

\[
a'_d(1) = a'_d(0)
\]

where \( a'_d(0) \) is the departure process of the first traveller when he or she is the only traveller using the system, and it is easily evaluated. For example, if service at the bottleneck is \( I/S \) vehicles per unit epoch \( (S \geq 1) \), and \( t_0 \) is the traveller's destination target time, then

\[
a'_d(0) = \begin{cases} 1 & \text{if } n = t_0 - S, \\ 0 & \text{otherwise.} \end{cases}
\]

Our major task is evaluating equation (4). We do it using equation (9). Let

\[
U^{i,j}_{k,m} = \begin{cases} 1 & \text{if } C(n, i) < C(m, j), \\ 0 & \text{otherwise.} \end{cases}
\]

and

\[
V^{j,k}_{i,m,w} = \begin{cases} 1 & \text{if } C(n, i) = C(m, j), \\ 0 & \text{otherwise.} \end{cases}
\]

where \( C(n, i) = C_n \) for \( n^{*}(\tau) = i \) - i.e., the cost if the traveller departed from origin \( O \) at epoch \( n \) and was delayed \( \tau \) units of time in the system.

Illustrative example. Consider a simple network of one O-D pair with one direct link between the origin and destination and a bottleneck between them. Let the capacity of the bottleneck be \( 20 \) vehicles per minute. Assume there are \( 1500 \) travellers going from this origin to the destination. Each traveller uses his or her own vehicle; hence, vehicle occupancy is assumed to be \( 1.0 \). Assume that each traveller wishes to arrive at the destination at \( 8:30 \) A.M. and that each attaches costs per unit time to earliness, lateness, and delay of \( 1.0, 2.0, \) and \( 4.0 \), respectively. The problem is to determine how these travellers select their departure times from the origin.

![Figure 2 Departure rate for all 1500 travellers](image-url)
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For practical purposes, consider the system of equally spaced time epochs sequentially numbered 0, 1, 2, ... with epoch spacings of 5 minutes. Consider the system between 7:15 A.M. and 8:55 A.M. Then there are 21 epochs 0, 1, 2, ..., 20, with the destination target time of 8:30 being epoch 15. Further, consider travel through the system to be in groups (cf. super commuter in Ref. 2) of 100 travellers. Hence, the capacity of the bottleneck is 1 group per epoch, i.e., 100 travellers per 5 minutes. Total demand is thus 15 groups. The resulting departure process (rate) distribution for all these travellers is shown in Figure 2. Selective departure rate distributions as the total demand increased from 500 travellers (5 groups) up to 1500 travellers (15 groups) are shown in Figure 3. The departure process (probability) distributions for the additional 100 travellers (1 group) when the number already using the system varies from 5 groups to 15 groups is shown in Figure 4.

Discussion
It is believed that stochastic models are more realistic than deterministic models for this type of problem. The main problem with a 'true' stochastic model in this case is that it is computationally cumbersome to implement and also that existence and uniqueness of the solution has not been proved for it or its approximations, developed by Alfa and Minh and De Palma et al. The major advantages of the ISUE presented in this paper are that it is computationally more feasible than the 'true' stochastic model, and the existence and uniqueness of its solution has been proven. The computational complexity associated with evaluating equation (9) is $O(3S_3N^2)$; for evaluating $W_k$ for use in equation (9) the complexity is $O(SN)$. Attempts should now be made to see how it can be incorporated into a route assignment model for extension to a realistic network.

A noticeable disadvantage with the ISUE is in the selection of incremental size. The most appropriate increment is one (unit) group. But how large a group should be depends on the spacing of the epochs, or vice versa. As group size increases, precision of results decreases. When it comes to using the ISUE in traffic assignment, this problem becomes critical. The question then is not just the size of the increment, but the zones from which loading should begin. This problem is the same as that encountered in the incremental assignment technique.

To use this or any other model for estimating the temporal distribution of peak traffic demand, the cost parameters must first be determined. Studies to estimate these parameters were reported by Abkowitz, Small, Alfa, and Hendrickson and Plank.

Conclusion
This paper showed that it is possible to obtain a theoretically sound and computationally feasible approximation to the stochastic model for the temporal distribution of peak traffic demand. The next problem to address is the development of an approximate procedure for estimating delays in a realistic complex network when flow is time dependent.

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