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On the support of solutions to the Zakharov–Kuznetsov equation ☆

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ABSTRACT

In this article we prove that sufficiently smooth solutions of the Zakharov–Kuznetsov equation:

$$\partial_t u + \partial_x^3 u + \partial_x \partial_y^2 u + u \partial_x u = 0,$$

that have compact support for two different times are identically zero.

1. Introduction

In this article we consider the Zakharov–Kuznetsov equation

$$\partial_t u + \partial_x^3 u + \partial_x \partial_y^2 u + u \partial_x u = 0, \quad (x, y) \in \mathbb{R}^2, \ t \in [0, 1]. \tag{1.1}$$

Eq. (1.1) is a bidimensional generalization of the Korteweg–de Vries (KdV) equation which is a mathematical model to describe the propagation of nonlinear ion-acoustic waves in magnetized plasma (see [12]).

Our goal in this article is to prove that a sufficiently smooth solution $u = u(x, y, t)$ of (1.1) which has compact support at two different times must vanish identically. Results concerning local and global well-posedness for the Cauchy problem associated to Eq. (1.1) can be found in the articles [5,1,7,9,8].

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In [11], Saut and Scheurer proved a result concerning a general class of dispersive-dissipative equations, including the KdV equation, which affirms that if a sufficiently smooth solution $u = u(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, of this type of equation, vanishes in a nonempty open set of $\mathbb{R}^n \times \mathbb{R}$, then it is identically zero.

Kenig, Ponce and Vega in [6] proved that if a sufficiently smooth solution u of the KdV equation is such that for some $B \in \mathbb{R}$, and two different times $t = 0$ and $t = 1$,

$$\operatorname{supp} u(\cdot,0), \operatorname{supp} u(\cdot,1) \subset (-\infty, B], \tag{1.2}$$

then $u \equiv 0$. First of all, they observed that with this condition on the support at time $t = 0$, the solution presents exponential decay to the right ($x > 0$) for every $t > 0$, which enables the use of a Carleman type estimate in order to show that the solution is zero in a half-strip $[R, +\infty) \times [0,1]$. In particular, u vanishes in a nonempty open set of $\mathbb{R} \times [0,1]$, which permits to apply Saut–Scheurer's result to conclude that $u \equiv 0$.

Using refinements of the method in [6], unique continuation principles have been successively improved for the KdV and Schrödinger equations (see for example [3] and [4]).

In [2], Bourgain introduced an approach, based on Complex Analysis methods, to prove that if sufficiently smooth solutions of certain dispersive equations, including the KdV equation, are compactly supported on a nontrivial time interval, then they are identically zero.

Although the result in [2] is weaker, in the KdV case, than that in [11], unlike Saut and Scheurer's result, Bourgain's result can be obtained for the Zakharov–Kuznetsov equation. In fact, Panthee in [10] proved the following result:

Theorem 1.1. *Let $u \in C([0,1]; H^4(\mathbb{R}^2))$ be a solution of Eq.* (1.1) *such that for some $B > 0$*

$$\operatorname{supp} u(t) \subset [-B,B] \times [-B,B] \quad \forall t \in [0,1]. \tag{1.3}$$

Then $u \equiv 0$.

In our work we will only require condition (1.3) for two different times. More precisely, we prove the following result.

Theorem 1.2. *Let $u \in C([0,1]; H^4(\mathbb{R}^2)) \cap C^1([0,1]; L^2(\mathbb{R}^2))$ be a solution of* (1.1) *such that, for some $B > 0$,*

$$\operatorname{supp} u(0), \operatorname{supp} u(1) \subseteq [-B,B] \times [-B,B].$$

Then, $u \equiv 0$.

The proof of Theorem 1.2 follows the ideas of Kenig, Ponce and Vega in [6]. In first place, we observe that if the solutions of the Zakharov–Kuznetsov equation have exponential decay for $x > 0$ and $y \in \mathbb{R}$ at time $t = 0$, and exponential decay for $x < 0$ and $y \in \mathbb{R}$ at time $t = 1$, then these solutions have exponential decay as $x^2 + y^2$ goes to infinity at all times $t \in [0,1]$. This fact allows us to use a Carleman estimate of $L^2 - L^2$ type, in order to establish that for the function u in Theorem 1.2 there exists $B > 0$ such that $\operatorname{supp} u(t) \subset [-B,B] \times [-B,B]$ for all $t \in [0,1]$. In this manner, by Theorem 1.1, $u \equiv 0$.

From now on, we will say that $f \in H^k(e^{2\beta x}\,dx\,dy)$ if $\partial^\alpha f \in L^2(e^{2\beta x}\,dx\,dy)$ for all multi-index $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| \leqslant k$. In a similar way we define $H^k(e^{2\beta x} e^{2\beta y}\,dx\,dy)$.

The decay property of the solutions of the Zakharov–Kuznetsov equation, mentioned before, plays a central role in this article and it is proved in the following theorem:

Theorem 1.3. *Let $u \in C([0,1]; H^4(\mathbb{R}^2)) \cap C^1([0,1]; L^2(\mathbb{R}^2))$ be a solution of* (1.1)*.*

(i) *If for all $\beta > 0$, $u(0) \in L^2(e^{2\beta x}e^{2\beta|y|}\,dx\,dy)$, then u is a bounded function from $[0,1]$ with values in $H^3(e^{2\beta x}e^{2\beta|y|}\,dx\,dy)$ for all $\beta > 0$.*
(ii) *If for all $\beta > 0$, $u(1) \in L^2(e^{-2\beta x}e^{2\beta|y|}\,dx\,dy)$, then u is a bounded function from $[0,1]$ with values in $H^3(e^{-2\beta x}e^{2\beta|y|}\,dx\,dy)$ for all $\beta > 0$.*

In particular, if the conditions for $u(0)$ and $u(1)$ given in (i) *and* (ii)*, respectively, are satisfied, then u is bounded from $[0,1]$ to $H^3(e^{2\beta|x|}e^{2\beta|y|}\,dx\,dy)$ for all $\beta > 0$.*

The Carleman type estimates are proved in the following theorem:

Theorem 1.4. *Let $w \in C([0,1]; H^3(\mathbb{R}^2)) \cap C^1([0,1]; L^2(\mathbb{R}^2))$, be a function such that for all $\beta > 0$:*

(i) *w is bounded from $[0,1]$ with values in $H^3(e^{2\beta|x|}e^{2\beta|y|}\,dx\,dy)$, and*
(ii) *$w' \in L^1([0,1]; L^2(e^{2\beta|x|}e^{2\beta|y|}\,dx\,dy))$.*

Then, for all $\lambda \neq 0$,

$$\left\|e^{\lambda x}w\right\| \leqslant \left\|e^{\lambda x}w(0)\right\|_{L^2(\mathbb{R}^2)} + \left\|e^{\lambda x}w(1)\right\|_{L^2(\mathbb{R}^2)} + \left\|e^{\lambda x}\left(w' + \partial_x^3 w + \partial_x\partial_y^2 w\right)\right\|, \tag{1.4}$$

where $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^2\times[0,1])}$.

A similar estimate also holds with y instead of x in the exponents.

Our proof of (1.4) relies only on the Fourier transform in the space variables and on the elementary properties of absolutely continuous functions in the time variable.

The paper is organized as follows: in Section 2 we prove Theorem 1.3 and in Section 3 we prove Theorem 1.4. Finally, in Section 4, using Theorems 1.3, 1.4 and 1.1, we establish Theorem 1.2.

Throughout this article the letter C will denote diverse positive constants which may change from line to line and depend on parameters which are clearly established in each case.

2. A priori estimates (Proof of Theorem 1.3)

The proof of Theorem 1.3 is based on the following lemmas.

The first lemma is an interpolation result which can be proved using the Three-line Theorem:

Lemma 1. *For $s > 0$ and $\beta > 0$ let $f \in H^s(\mathbb{R}^2) \cap L^2(e^{2\beta x}\,dx\,dy)$. Then, for $\theta \in [0,1]$:*

$$\left\|J^{\theta s}\left(e^{(1-\theta)\beta x}f\right)\right\|_{L^2} \leqslant C\left\|J^s f\right\|_{L^2}^{\theta}\left\|e^{\beta x}f\right\|_{L^2}^{1-\theta}, \tag{2.5}$$

where $[J^s f]^\wedge(\xi) := (1+|\xi|^2)^{s/2}\widehat{f}(\xi)$ and $C = C(s,\beta)$.

(*Here, $\widehat{\ }$ denotes the spatial Fourier transform in $\mathbb{R}^2$, and $\xi = (\xi_1, \xi_2)$, where (ξ_1, ξ_2) are the variables in the frequency space corresponding to the space variables (x, y).*)

Similarly, if $f \in H^s(\mathbb{R}^2) \cap L^2(e^{2(\beta x+\beta y)}\,dx\,dy)$. Then, for $\theta \in [0,1]$:

$$\left\|J^{\theta s}\left(e^{(1-\theta)(\beta x+\beta y)}f\right)\right\|_{L^2} \leqslant C\left\|J^s f\right\|_{L^2}^{\theta}\left\|e^{\beta x+\beta y}f\right\|_{L^2}^{1-\theta}. \tag{2.6}$$

The exponential decay in Theorem 1.3 is obtained in two steps. In the first step we establish the boundedness of $u(t)$ in the space $H^3(e^{2\beta x}\,dx\,dy)$, and then, using this fact, we prove the boundedness of $u(t)$ in the space $H^3(e^{2\beta x+2\beta y}\,dx\,dy)$. The conclusion of the proof follows from the symmetry properties of the equation.

Lemma 2. *Let $u \in C([0,1]; H^4(\mathbb{R}^2)) \cap C^1([0,1]; L^2(\mathbb{R}^2))$ be a solution of (1.1) such that for all $\beta > 0$, $u(0) \in L^2(e^{2\beta x}\,dx\,dy)$. Then u is a bounded function from $[0,1]$ with values in $H^3(e^{2\beta x}\,dx\,dy)$ for all $\beta > 0$.*

Proof. We will first prove that $t \mapsto u(t)$ is bounded from $[0,1]$ with values in $L^2(e^{2\beta x}\,dx\,dy)$. Let $\varphi \in C^\infty(\mathbb{R})$ be a decreasing function with $\varphi(x) = 1$ if $x < 1$ and $\varphi(x) = 0$ if $x > 10$. For $n \in \mathbb{N}$ we define

$$\phi_n(x) := e^{2\beta\theta_n(x)},$$

where $\theta_n(x) := \int_0^x \varphi(\frac{x'}{n})\,dx'$.

It is easily seen that for every n, ϕ_n is an increasing function, $\phi_n(x) = e^{2\beta x}$ if $x \leqslant n$, and $\phi_n(x) \equiv d_n \leqslant e^{20\beta n}$ if $x > 10n$. Also, $\phi_n \leqslant \phi_{n+1}$ for every n and

$$\left|\phi_n^{(j)}(x)\right| \leqslant C_{j,\beta}\phi_n(x) \quad \forall j \in \mathbb{N},\ \forall x \in \mathbb{R}.$$

Multiplying Eq. (1.1) by $u\phi_n$ and integrating by parts in $\mathbb{R}^2_{xy}$ we obtain:

$$\frac{1}{2}\frac{d}{dt}\int u^2\phi_n + \frac{3}{2}\int (\partial_x u)^2\phi_n' - \frac{1}{2}\int u^2\phi_n''' + \frac{1}{2}\int (\partial_y u)^2\phi_n' - \frac{1}{3}\int u^3\phi_n' = 0.$$

Therefore, discarding positive terms and applying Sobolev embeddings,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int u^2\phi_n &\leqslant \frac{1}{2}C_{3,\beta}\int u^2\phi_n + \frac{1}{3}\left\|u(t)\right\|_{L^\infty(\mathbb{R}^2)}C_{1,\beta}\int u^2\phi_n \\ &\leqslant \left(C_{3,\beta} + C\|u\|_{C([0,1];H^2)}\right)\int u^2\phi_n \equiv C_{\beta,u}\int u^2\phi_n,\end{aligned}$$

and applying Gronwall's lemma and the Monotone Convergence Theorem with $n \to \infty$ we conclude that

$$\int u(t)^2 e^{2\beta x}\,dx\,dy \leqslant C\int u(0)^2 e^{2\beta x}\,dx\,dy \quad \forall t \in [0,1], \tag{2.7}$$

which proves that $t \mapsto u(t)$ is bounded from $[0,1]$ with values in $L^2(e^{2\beta x}\,dx\,dy)$.

Since this boundedness holds for each $\beta > 0$, and, on the other hand, $u \in C([0,1]; H^4)$, we can apply the interpolation inequality (2.5) with $s = 4$, $\theta = \frac{3}{4}$, to conclude that $t \mapsto u(t)$ is bounded from $[0,1]$ with values in $H^3(e^{2\beta x}\,dx\,dy)$, which completes the proof of Lemma 2. □

Proof of Theorem 1.3. *Proof of* (i). Our first step will be to prove that the u is bounded from $[0,1]$ to $L^2(e^{2\beta x}e^{2\beta y}\,dx\,dy)$.

Since $u(0) \in L^2(e^{2\beta x}e^{2\beta|y|}\,dx\,dy)$, then $u(0) \in L^2(e^{2\beta x}\,dx\,dy)$, and in consequence, by Lemma 2, u is bounded from $[0,1]$ with values in $H^3(e^{2\beta x}\,dx\,dy)$ for all $\beta > 0$.

Let $w(t) := e^{\beta x}u(t)$. Since u is a solution of (1.1), it follows that w satisfies the equation

$$e^{\beta x}u' - \beta^3 w + 3\beta^2\partial_x w - 3\beta\partial_x^2 w + \partial_x^3 w - \beta\partial_y^2 w + \partial_x\partial_y^2 w - \beta u w + u\partial_x w = 0. \tag{2.8}$$

Let us notice that, since $u(t) \in H^3(e^{2\beta x}\,dx\,dy)$, and u satisfies Eq. (1.1), all terms in the former equation belong to $L^2(\mathbb{R}^2)$.

For $n \in \mathbb{N}$ let us define $\phi_n(y) := e^{2\beta\theta_n(y)}$, where the function θ_n is the same function defined in the proof of Lemma 2.

Multiplying Eq. (2.8) by $w\phi_n(y)$ and integrating by parts in $\mathbb{R}^2_{xy}$ we obtain:

$$\int e^{\beta x} u' w\phi_n - \beta^3 \int w^2\phi_n + 3\beta \int (\partial_x w)^2\phi_n + \beta \int (\partial_y w)^2\phi_n - \frac{1}{2}\beta \int w^2\phi_n''$$
$$+ \int (\partial_y w)(\partial_x w)\phi_n' - \beta \int u w^2\phi_n - \frac{1}{2}\int w^2(\partial_x u)\phi_n = 0. \tag{2.9}$$

For the first term we will see that

$$t \mapsto \int_{\mathbb{R}^2} e^{\beta x} u(t) w(t)\phi_n(y)\,dx\,dy = \int w^2\phi_n$$

is absolutely continuous in $[0, 1]$ and that

$$\frac{1}{2}\frac{d}{dt}\int w^2\phi_n = \int e^{\beta x} u' w\phi_n \quad \text{a.e. } t \in [0,1]. \tag{2.10}$$

In fact, since $u \in C^1([0,1]; L^2(\mathbb{R}^2))$ and for $m \in \mathbb{N}$, $\phi_m(x)\phi_n(y) \in L^\infty(\mathbb{R}^2)$

$$\frac{d}{dt}\big\langle u(t), \phi_m(\cdot_x)\phi_n(\cdot_y)u(t)\big\rangle = 2\int u'(t)\phi_m(x)\phi_n(y)u(t).$$

Thus the fundamental theorem of Integral Calculus implies that

$$\int u(t)\phi_m(x)\phi_n(y)u(t) - \int u(0)\phi_m(x)\phi_n(y)u(0) = 2\int_0^t \left[\int u'(\tau)\phi_m(x)\phi_n(y)u(\tau)\,dx\,dy\right] d\tau.$$

An easy application of Dominated Convergence Theorem in the former equality gives, when m goes to ∞, that

$$\int u(t)e^{2\beta x}\phi_n(y)u(t) - \int u(0)e^{2\beta x}\phi_n(y)u(0) = 2\int_0^t \left[\int u'(\tau)e^{2\beta x}\phi_n(y)u(\tau)\,dx\,dy\right] d\tau,$$

which implies (2.10).

Taking into account that $|\phi_n'(y)| = |2\beta\varphi(\frac{y}{n})\phi_n(y)| \leqslant 2\beta\phi_n(y)$, from (2.9) and (2.10), it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int w^2\phi_n &\leqslant \beta^3\int w^2\phi_n - \beta\int\big((\partial_x w)^2 - 2|\partial_x w||\partial_y w| + (\partial_y w)^2\big)\phi_n + \frac{1}{2}\beta C_{2,\beta}\int w^2\phi_n \\ &\quad + \beta C\|u\|_{C([0,1];H^2(\mathbb{R}^2))}\int w^2\phi_n + C\|\partial_x u\|_{C([0,1];H^2(\mathbb{R}^2))}\int w^2\phi_n \\ &\equiv C_{\beta,u}\int w^2\phi_n - \beta\int\big(|\partial_x w| - |\partial_y w|\big)^2\phi_n \\ &\leqslant C_{\beta,u}\int w^2\phi_n \quad \text{a.e. } t\in[0,1],\end{aligned} \tag{2.11}$$

which, as in Lemma 2, implies that u is bounded from $[0,1]$ to $L^2(e^{2\beta x}e^{2\beta y}\,dx\,dy)$. This, together with the fact that $u \in C([0,1];H^4)$ and the interpolation inequality (2.6) with $s=4$ and $\theta=\frac{3}{4}$, shows that u is bounded from $[0,1]$ with values in $H^3(e^{2\beta x}e^{2\beta y}\,dx\,dy)$ for all $\beta>0$.

Finally, if we define $\widetilde{u}(x,y,t) := u(x,-y,t)$, then $\widetilde{u}$ is also a solution of (1.1), with $\widetilde{u}(0) \in L^2(e^{2\beta x}e^{2\beta|y|}\,dx\,dy)$ and therefore $\widetilde{u}$ is bounded from $[0,1]$ with values in $H^3(e^{2\beta x}e^{2\beta y}\,dx\,dy)$ for all $\beta>0$, i.e. u is bounded from $[0,1]$ with values in $H^3(e^{2\beta x}e^{-2\beta y}\,dx\,dy)$; which proves (i).

Proof of (ii). Property (ii) follows immediately from (i) by taking into account that the function defined by

$$(x,y,t) \mapsto u(-x,y,1-t)$$

is also a solution of Eq. (1.1) satisfying the hypothesis of (i). □

3. Estimates of the Carleman type (Proof of Theorem 1.4)

In the proof of Carleman's estimate of Theorem 1.4 we will use the following lemma:

Lemma 3. *Let $w \in C^1([0,1];L^2(\mathbb{R}^2))$ be a function such that for all $\beta>0$, w is bounded from $[0,1]$ with values in $L^2(e^{2\beta|x|}e^{2\beta|y|}\,dx\,dy)$ and $w' \in L^1([0,1];L^2(e^{2\beta|x|}e^{2\beta|y|}\,dx\,dy))$. Then, for all $\lambda \in \mathbb{R}$ and all $\xi=(\xi_1,\xi_2) \in \mathbb{R}^2$, the functions $t \mapsto \widehat{e^{\lambda x}w(t)}(\xi)$ and $t \mapsto \widehat{e^{\lambda y}w(t)}(\xi)$ are absolutely continuous in $[0,1]$ with derivatives $\widehat{e^{\lambda x}w'(t)}(\xi)$ and $\widehat{e^{\lambda y}w'(t)}(\xi)$ a.e. $t \in [0,1]$, respectively.*

Proof. By symmetry, it is sufficient to prove the lemma only for the weight $e^{\lambda x}$. Using Cauchy–Schwarz inequality, it is easy to see that for all $t \in [0,1]$ and $\lambda \in \mathbb{R}$, $e^{\lambda x}w(t) \in L^1(\mathbb{R}^2)$, and also that $e^{\lambda x}w' \in L^1(\mathbb{R}^2 \times [0,1])$ for all $\lambda \in \mathbb{R}$.

For $R>0$, let χ_R be the characteristic function of the square $[-R,R]\times[-R,R]$. Since $w \in C^1([0,1];L^2(\mathbb{R}^2))$, the function

$$t \mapsto \int_{\mathbb{R}^2} e^{-ix\xi_1}e^{-iy\xi_2}e^{\lambda x}\chi_R(x,y)w(t)(x,y)\,dx\,dy = \langle w(t), e^{ix\xi_1}e^{iy\xi_2}e^{\lambda x}\chi_R\rangle_{L^2(\mathbb{R}^2)} \tag{3.12}$$

defines a C^1 function of the variable t with derivative given by

$$t \mapsto \langle w'(t), e^{ix\xi_1}e^{iy\xi_2}e^{\lambda x}\chi_R\rangle_{L^2(\mathbb{R}^2)},$$

and in consequence

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-ix\xi_1}e^{-iy\xi_2}e^{\lambda x}\chi_R(x,y)w(t)(x,y)\,dx\,dy &= \int_0^t\int_{\mathbb{R}^2} e^{-ix\xi_1}e^{-iy\xi_2}e^{\lambda x}\chi_R(x,y)w'(\tau)(x,y)\,dx\,dy\,d\tau \\ &\quad + \int_{\mathbb{R}^2} e^{-ix\xi_1}e^{-iy\xi_2}e^{\lambda x}\chi_R(x,y)w(0)(x,y)\,dx\,dy.\end{aligned}$$

The lemma follows from the former equality by an application of the Lebesgue Dominated Convergence Theorem. □

Proof of Theorem 1.4. Let us define $g(t) := e^{\lambda x} w(t)$ and $h(t) := e^{\lambda x}(w'(t) + \partial_x^3 w(t) + \partial_x \partial_y^2 w(t))$. Then

$$h(t) = e^{\lambda x} w'(t) - \lambda^3 g(t) + 3\lambda^2 \partial_x g(t) - 3\lambda \partial_x^2 g(t) + \partial_x^3 g(t) - \lambda \partial_y^2 g(t) + \partial_x \partial_y^2 g(t). \tag{3.13}$$

From the hypotheses on w it can be seen that all terms in (3.13) are in $L^1(\mathbb{R}^2)$ for almost every $t \in [0,1]$. We take the spatial Fourier transform in (3.13) and apply Lemma 3 to obtain that

$$\frac{d}{dt}\widehat{g(t)}(\xi) + \big[-im(\xi) - a(\xi)\big]\widehat{g(t)}(\xi) = \widehat{h(t)}(\xi), \quad \text{a.e. } t \in [0,1], \tag{3.14}$$

where

$$m(\xi) := -3\lambda^2 \xi_1 + \xi_1^3 + \xi_1 \xi_2^2 \quad \text{and} \quad a(\xi) := \lambda^3 - 3\lambda \xi_1^2 - \lambda \xi_2^2.$$

Using (3.14), when $a(\xi) \leqslant 0$, we have

$$\widehat{g(t)}(\xi) = e^{im(\xi)t} e^{a(\xi)t} \widehat{g(0)}(\xi) + \int_0^t e^{im(\xi)(t-\tau)} e^{a(\xi)(t-\tau)} \widehat{h(\tau)}(\xi)\, d\tau, \quad \text{for all } t \in [0,1],$$

and when $a(\xi) > 0$, we choose to write

$$\widehat{g(t)}(\xi) = e^{-im(\xi)(1-t)} e^{-a(\xi)(1-t)} \widehat{g(1)}(\xi) - \int_t^1 e^{-im(\xi)(\tau-t)} e^{-a(\xi)(\tau-t)} \widehat{h(\tau)}(\xi)\, d\tau \quad \text{for all } t \in [0,1].$$

In any case, for all $t \in [0,1]$:

$$\big|\widehat{g(t)}(\xi)\big| \leqslant \big|\widehat{g(0)}(\xi)\big| + \big|\widehat{g(1)}(\xi)\big| + \int_0^1 \big|\widehat{h(\tau)}(\xi)\big|\, d\tau,$$

and estimate (1.4) follows from Plancherel's formula.

The proof of the estimate with the weight $e^{\lambda y}$ is similar. $\square$

4. Proof of Theorem 1.2

Proof. Let $\widetilde{\phi} \in C^\infty(\mathbb{R})$ be a non-decreasing function such that $\widetilde{\phi}(x) = 0$ for $x < 0$ and $\widetilde{\phi}(x) = 1$ for $x > 1$ and, for $R > B$, let $\phi(x) \equiv \phi_R(x) := \widetilde{\phi}(x - R)$. We define $w \equiv w_R := \phi(x)u$, and $v \equiv v_R := \phi(y)u$. Since $\operatorname{supp} u(0)$ and $\operatorname{supp} u(1)$ are compact, from Theorem 1.3 and Eq. (1.1), it follows that w and v satisfy the hypotheses of Theorem 1.4.

Taking into account that $w(0) = w(1) = 0$, from (1.4) we conclude that

$$\begin{aligned}
\big\|e^{\lambda x} w\big\| &\leqslant \big\|e^{\lambda x}\big(w' + \partial_x^3 w + \partial_x \partial_y^2 w\big)\big\| \\
&= \big\|e^{\lambda x}\big(\phi u' + \phi \partial_x^3 u + \phi \partial_x \partial_y^2 u + \phi''' u + 3\phi'' \partial_x u + 3\phi' \partial_x^2 u + \phi' \partial_y^2 u\big)\big\| \\
&\leqslant \big\|e^{\lambda x} \phi u \partial_x u\big\| + \big\|e^{\lambda x} F_{1\phi,u}\big\|,
\end{aligned} \tag{4.15}$$

where $\phi := \phi(x)$, $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^2 \times [0,1])}$ and

$$F_{1\phi,u} := \phi''' u + 3\phi'' \partial_x u + 3\phi' \partial_x^2 u + \phi' \partial_y^2 u.$$

Since the derivatives of ϕ are supported in the interval $[R, R+1]$, it can be seen that

$$\left\|e^{\lambda x} F_{1\phi,u}\right\| \leqslant C e^{\lambda(R+1)}. \tag{4.16}$$

where $C = C(\|u\|_{C([0,1];H^2)})$, and is independent from λ and R. Therefore

$$\left\|e^{\lambda x}\phi u\right\| \leqslant \left\|e^{\lambda x}\phi u\right\| \|\partial_x u\|_{L^\infty([R,+\infty)\times\mathbb{R}\times[0,1])} + C e^{\lambda(R+1)}.$$

From Theorem 1.3, with $\beta = 1$ and Sobolev embeddings, there exists a constant C_1 such that

$$\left|\partial_x u(t)(x, y)\right| \leqslant C_1 e^{-x}.$$

Thus

$$\left\|e^{\lambda x}\phi u\right\| \leqslant C_1 e^{-R} \left\|e^{\lambda x}\phi u\right\| + C e^{\lambda(R+1)}. \tag{4.17}$$

Since, from Lemma 2 $\|e^{\lambda x}\phi u\| < \infty$, we can absorb the first term on the right-hand side of (4.17) by taking $R > B$ such that $C_1 e^{-R} < \frac{1}{2}$ to obtain that

$$\left\|e^{\lambda x}\phi u\right\| \leqslant C e^{\lambda(R+1)}.$$

And thus, since $\phi(x) = 1$ for $x \geqslant 2R$,

$$e^{2\lambda R}\left(\int_0^1 \int_{-\infty}^{\infty} \int_{2R}^{\infty} \left|u(t)(x, y)\right|^2 dx\,dy\,dt\right)^{1/2} \leqslant \left\|e^{\lambda x}\phi u\right\| \leqslant C e^{\lambda(R+1)}. \tag{4.18}$$

Since (4.18) is valid for all $\lambda > 0$, $2R > R+1$, and the constant C is independent from λ, by letting $\lambda \to +\infty$ it follows that

$$\left(\int_0^1 \int_{-\infty}^{\infty} \int_{2R}^{\infty} \left|u(t)(x, y)\right|^2 dx\,dy\,dt\right)^{1/2} = 0.$$

Thus $u \equiv 0$ in $[2R, \infty) \times \mathbb{R} \times [0, 1]$.

In a similar way, for $v := \phi(y)u$, taking into account that $v(0) = v(1) = 0$, an application of Carleman's estimate (1.4) with weight $e^{\lambda y}$ gives:

$$\begin{aligned}\left\|e^{\lambda y}\phi u\right\| = \left\|e^{\lambda y} v\right\| &\leqslant \left\|e^{\lambda y}\left(v' + \partial_x^3 v + \partial_x \partial_y^2 v\right)\right\| \\ &= \left\|e^{\lambda y}\left(\phi u' + \phi \partial_x^3 u + \phi \partial_x \partial_y^2 u + 2\phi' \partial_x \partial_y u + \phi'' \partial_x u\right)\right\| \\ &\leqslant \left\|e^{\lambda y}\phi u \partial_x u\right\| + \left\|e^{\lambda y} F_{2\phi,u}\right\|,\end{aligned}$$

where

$$F_{2\phi,u} := 2\phi' \partial_x \partial_y u + \phi'' \partial_x u.$$

Now we reason as above to conclude that $u \equiv 0$ in $\mathbb{R} \times [2R, \infty) \times [0, 1]$.

Finally, we notice that the function $(x, y, t) \mapsto u(-x, -y, 1-t)$ also satisfies the hypotheses of Theorem 1.2, which, by the former procedure, implies that $u \equiv 0$ in $(-\infty, -2R] \times \mathbb{R} \times [0,1] \cup \mathbb{R} \times (-\infty, -2R] \times [0,1]$.

In this manner, there exists $R > 0$ such that $\operatorname{supp} u(t) \subset [-2R, 2R] \times [-2R, 2R]$ for all $t \in [0,1]$. Then, by Theorem 1.1, $u \equiv 0$. □

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