A weak energy identity and the length of necks for a sequence of Sacks–Uhlenbeck $\alpha$-harmonic maps

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Abstract
In this paper we discuss the convergence behavior of a sequence of $\alpha$-harmonic maps $u_\alpha$ with $E_\alpha(u_\alpha) < C$ from a compact surface $(M, g)$ into a compact Riemannian manifold $(N, h)$ without boundary. Generally, such a sequence converges weakly to a harmonic map in $W^{1,2}(M, N)$ and strongly in $C^\infty$ away from a finite set of points in $M$. If energy concentration phenomena appears, we show a generalized energy identity and discover a direct convergence relation between the blow-up radius and the parameter $\alpha$ which ensures the energy identity and no-neck property. We show that the necks converge to some geodesics. Moreover, in the case there is only one bubble, a length formula for the neck is given. In addition, we also give an example which shows that the necks contain at least a geodesic of infinite length.

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1. Introduction

Let \((M, g)\) be a smooth compact Riemann surface without boundary, and \((N, h)\) be an \(n\)-dimensional smooth compact Riemannian manifold without boundary. By Nash’s isometric embedding theorem, we assume that \(N\) is isometrically embedded in some Euclidean space \(N \hookrightarrow \mathbb{R}^K\) for the sequel.

We define the Sobolev space of \(W^{1,p}\)-maps from \(M\) into \(N\), denoted by \(W^{1,p}(M, N)\), as
\[
W^{1,p}(M, N) = \{ u \in W^{1,p}(M, \mathbb{R}^K) : u(x) \in N \text{ for a.e. } x \in M \}.
\]

If \(u \in W^{1,2}(M, N)\), we define the energy density \(e(u)\) of \(u\) by
\[
e(u) = \text{Trace}_g u^* h = |\nabla_g u|^2,
\]
where \(u^* h\) is the pull-back of the metric tensor \(h\). In a local coordinate system of \(x \in M\), the energy density can be expressed as
\[
e(u)(x) = g^{ij}(x) h_{\alpha\beta}(u(x)) \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}.
\]

The energy \(E(u)\) of \(u\) is defined by
\[
E(u) = \int_M e(u) \, dV_g,
\]
and the critical points of \(E(u)\) are called harmonic maps from \(M\) into \(N\). Hence, a harmonic map \(u\) satisfies the corresponding Euler–Lagrange equation in the sense of distribution
\[
\tau(u) = \Delta u + A(u)(\nabla u, \nabla u) = 0,
\]
where \(A(\cdot, \cdot)\) is the second fundamental form of \(N\) in \(\mathbb{R}^K\).

Eells and Sampson first employed the heat flow method to approach the existence problems of harmonic maps and deformed successfully a map from a closed manifold into a manifold with non-positive sectional curvature into a homotopic harmonic map. Concretely, they considered the heat flow of harmonic maps (or the negative gradient flow of the energy functional \(E(u)\)):
\[
\frac{\partial u}{\partial t} = \tau(u).
\]

If one can establish the global existence of the above flow with respect to the time variable \(t\), or roughly speaking, the flow flows to infinity smoothly, then we are able to find a sequence \(u_k = u(x, t_k)\) such that \(u_k\) converges to a harmonic map as \(t_k \to +\infty\) (see [9]). Generally, such a flow does not exist globally (see [6] and [2]).

From the viewpoint of calculus of variations, it is also not easy to find a harmonic map, since \(E\) does not satisfy the well-known Palais–Smale condition when the dimensions of domain manifold \(\dim(M) \geq 2\). In particular, when \(\dim(M) = 2\), it is well known that the energy functional \(E\) is of conformal invariance and the corresponding variational problem possesses a non-compact invariance group and represents limiting cases where the Palais–Smale condition
just fails. Therefore, harmonic maps from a surface are of special importance and interest. In fact, mathematicians have paid more attention to this case. To prove the existence of harmonic maps from a closed surface Sacks and Uhlenbeck in their pioneering paper [17] introduced a perturbed energy functional which satisfies the Palais–Smale condition, hence obtained so called $\alpha$-harmonic maps, as critical points of perturbed functional, to approximate harmonic maps. More precisely, for every $u \in W^{1,2\alpha}(M,N)$ Sacks and Uhlenbeck defined the so called $\alpha$-energy functional $E_{\alpha}$ as

$$E_{\alpha}(u) = \int_{M} (1 + |\nabla u|^{2})^{\alpha} \, dV_{g},$$

which can be regarded as a perturbation of energy functional $E$, and considered $\alpha$-harmonic maps, i.e. the critical points of $E_{\alpha}$ in $W^{1,2\alpha}(M,N)$, which satisfy the following equation:

$$\Delta_{\alpha}u_{\alpha} + (\alpha - 1) \frac{\nabla_{g}|\nabla_{g}u_{\alpha}|^{2}}{1 + |\nabla_{g}u_{\alpha}|^{2}} \nabla_{g}u_{\alpha} + A(u_{\alpha})(du_{\alpha}, du_{\alpha}) = 0. \quad (1.1)$$

We call such an $\alpha$-harmonic map as Sacks–Uhlenbeck $\alpha$-harmonic map. If there is a subsequence $u_{\alpha_{k}}$ which converges smoothly as $\alpha_{k} \to 1$, then $u_{\alpha_{k}}$ will converge to a harmonic map. Such smooth convergence fails generally, thus, Sacks and Uhlenbeck developed a powerful method and some techniques to investigate the blow-up phenomena for such a variational problem.

Later, Struwe [18] used the heat flow method of Eells and Sampson to approach the existence problems for harmonic maps from a closed surface and he obtained almost the same results as in [17]. Chang showed the same results as in [18] for the case where the domain manifold is a compact surface with smooth boundary (see [1]).

No matter which method, Sacks–Uhlenbeck perturbation or heat flow approach, is adopted to produce a sequence of two-dimensional approximate harmonic maps, generally we cannot exclude the bubbling phenomena when we consider the convergence of such a sequence. That is to say, it is possible that the limiting map (a smooth harmonic map $u_{0}$) is just a trivial map, although such a sequence converges smoothly away from finitely many points (which are called blow-up points). On the other hand, around every blow-up point $p$, the energy concentrates and there are some non-trivial harmonic maps $w_{i}$ from $S^{2}$ to $N$, i.e. some bubbles for the sequence of approximate harmonic maps.

For a sequence of approximate harmonic maps from a compact Riemann surface $M$ into $N$ with bounded energy, naturally one pays attention to the following two problems: One is whether the energy identity holds true or not, i.e.

$$\lim_{k \to +\infty} \int_{M} |\nabla u_{k}|^{2} \, dV_{g} = \int_{M} |\nabla u_{0}|^{2} \, dV_{g} + \sum_{i=1}^{m} E(w_{i})?$$

where $m$ is the number of all bubbles. The other is what we can say about necks joining bubbles if they exist?

When $u_{k} = u(x, t_{k})$ is a subsequence of a heat flow for two-dimensional harmonic maps, the above two problems were deeply studied. The energy identities have been proved by Qing [15] (for the case $N = S^{n}$), Ding and Tian [7] for the general case. Lin and Wang provided another proof of the energy identity in [13]. Furthermore, Qing and Tian [16] proved that there is no
neck if blow-up phenomena happens at infinite time and Ding [5] proved a more general result. It is worthy to point out that Topping [19] provided a surprising example of heat flow $u(x, t)$ for which the blow-up phenomena appears at a finite time $T$ and the weak limit $\lim_{t \to T} u(x, t)$ is not continuous. Also, one has known that for some special sequences of $\alpha$-harmonic maps the energy identity is true. But it was obtained by some methods which are completely different from the one of [7]. Now we would like to mention the following cases.

If $\{u_\alpha\}$ is a sequence of minimizing $\alpha$-harmonic maps (every $u_\alpha$ is the minimizer of $E_\alpha$), each $u_\alpha$ of which belongs to the same homotopy class, Chen and Tian [3] proved that the necks consist of some geodesics of finite length, and moreover this implies no loss of energy in necks for the sequence, i.e. the energy identity holds true.

Jost considered the energy identity for a minimax sequence of $\alpha$-harmonic maps. Suppose that $M$ is a compact Riemann surface and $A$ is a parameter manifold. Let $h_0 : M \times A \to N$ be a continuous map, and $H$ be the set of such continuous maps $h : M \times A \to N$ that $h$ are homotopic to $h_0$ and satisfy $h(t) \in W^{1,2\alpha}(M, N)$ for any fixed $t \in A$. Set

$$\beta_\alpha(H) = \inf_{h \in H} \sup_{t \in A} E_\alpha(h(\cdot, t)).$$

We can deduce from Jost’s result [10] that there is at least a sequence $\{u_{\alpha_k}\}$, which attained $\beta_{\alpha_k}(H)$, satisfies the energy identity as $\alpha_k \to 1$. Recently, T. Lamm gave a simple proof of this energy identity [11] (also see [4] and [14]).

The energy identity for an $\alpha$-harmonic map sequence with bounded energy is still open up to now. It seems that the methods used to show the identity for heat flow, or a sequence of approximate maps with tension fields $\tau$ bounded in $L^2$, are not powerful enough to prove the energy identity for such an $\alpha$-harmonic map sequence. We think that the key difficulty lies on the identity (2.5) in this paper. Indeed, for a sequence of maps with tension fields $\tau$ bounded in $L^2$, then from (2.5) we can see easily

$$\int_{\partial B_r} \left| \frac{\partial u_k}{\partial r} \right|^2 dS_0 - \frac{1}{2} \int_{\partial B_r} |\nabla_0 u_k|^2 dS_0 = O \left( \int_{B_r} |\tau(u_k)||\nabla u_k| dV_g \right) + O(1),$$

then the right-hand side of the above identity is bounded, therefore we can derive the energy identity for heat flow or a sequence of maps with tension fields $\tau$ bounded in $L^2$. However, for a sequence of $\alpha$-harmonic maps a very “bad” term

$$\frac{\alpha - 1}{r} \int_{\partial B_r} (1 + |\nabla u_\alpha|^2)^{\alpha - 1} |\nabla u_\alpha|^2 dV_g$$

appears in (2.5). We will see that it is not so easy to control such a term.

In this paper, we attempt to establish a generalized energy identity and study the asymptotic behaviors of neck domains. When we encounter the situation at a blow-up point there are several bubbles of a sequence of $\alpha$-harmonic maps, we will be concerned with more general $\alpha$-energy functionals which are of the following forms:

$$E_{\alpha, \epsilon_\alpha}(u) = \int_M \left( \epsilon_\alpha + |\nabla_g u_\alpha|^2 \right)^\alpha dV_{g_\alpha},$$
where $\alpha > 1$ and $0 < \epsilon_{\alpha} < 1$. In the beginning of Section 2, furthermore it will be explained why one needs to consider such a kind of functionals. On the other hand, also it is of interest to consider whether a sequence of critical points corresponding such a family of functionals converges or not.

In consideration of the above motivations and reasons, more precisely we will focus on the following problems: “Given a sequence of maps $u_\alpha$ from $(B_\sigma, g_\alpha)$ to $(N, h)$, each of which is a critical point of the corresponding $E_{\alpha, \epsilon_\alpha}(u)$, with $E_{\alpha, \epsilon_\alpha}(u_\alpha) \leq \Theta$ and

$$0 < \beta_0 < \lim_{\alpha \to 1} \epsilon_\alpha^{\alpha-1} \leq 1.$$ 

Here $\Theta$ and $\beta_0$ are positive constants, $B_\sigma$ is a ball in $\mathbb{R}^2$ centered at the origin, and $g_\alpha = e^{\phi_\alpha}((dx_1)^2 + (dx_2)^2)$ with $\phi_\alpha(0) = 0$ and $\phi_\alpha \to \phi \in C^\infty(B_\sigma)$ smoothly. Moreover, without loss of generality we assume that $u_\alpha \to u_0$ in $C^k_{\text{loc}}(B_\sigma \setminus \{0\})$. For such a sequence $\{u_\alpha\}$, does the energy identity hold true? If the necks joining bubbles exist, do the necks consist of some geodesics? And if so, can we give the length formula of such geodesics?”

We will adopt some methods and techniques in [7] and [5] to discuss the above problems. In order to make detailed analysis on the energy on neck domains, we will employ some suitable variational formulas to establish some useful integral identities, and make use of $\epsilon$-regularity theory and blow-up analysis to derive some a priori estimates by which the “bad” term mentioned in the above can be controlled effectively. We only show a weak energy identity for such a sequence. On the other hand, we also study the convergence behaviors of the neck domains. Precisely, we provide a new method to show that the necks converge to geodesics and obtain a formula on the length of the geodesics. In particular, we discover the relation between the blow-up radius and the parameter $\alpha$ which ensures the energy identity and no-neck property.

In order to state our main results, here we introduce some basic facts about harmonic maps and some related notions which are needed for the sequel. For a more detailed discussion of the facts reviewed here, we refer the reader to various articles cited below.

It is well known that there always exists an isothermal coordinate system in a small neighborhood of every $p \in M$ since $\dim(M) = 2$, i.e. there is a real function $\varphi$ with $\varphi(p) = 0$ such that the metric can be locally expressed as

$$ds^2_M = e^\varphi((dx_1)^2 + (dx_2)^2).$$

As the energy functional $E$ is invariant under conformal transformations, so, essentially we only need to consider the blow-up phenomena in a small ball $B_r$ in Euclidean plane, centered at the origin, with a metric given by $g = e^\varphi((dx_1)^2 + (dx_2)^2)$ where $\varphi$ is a smooth real function on $B_r$ satisfying $\varphi(0) = 0$.

Now, we assume that $\{u_\alpha\}$ ($\alpha \to 1$) is a sequence of $\alpha$-harmonic maps from $(M, g)$ to $(N, h)$ with bounded $\alpha$-energy, i.e.

$$E_{\alpha, \epsilon_\alpha}(u_\alpha) < \Theta.$$ 

Then, by the theory of Sacks and Uhlenbeck, there exists a sequence of $\alpha_k$-harmonic maps $u_{\alpha_k}$ which converges to a harmonic map $u_0 : M \to N$ smoothly away from finitely many points $\{x_i\}$ as $\alpha_k \to 1$. Furthermore, we suppose that there are $n_0$ bubbles at a singular point $x_1$. Then, for every $j \in \{1, \ldots, n_0\}$ there are a sequence of points $\{x_{\alpha_k}^j \in M : k \in \mathbb{Z}\}$ with $x_{\alpha_k}^j \to x_1$ as $\alpha_k \to 1$, ...
and a sequence of positive real numbers \( \{ \lambda_{\alpha_k} \in \mathbb{Z} \} \) with \( \lambda_{\alpha_k} \to 0 \) as \( \alpha_k \to 1 \), such that the sequence of maps given by

\[
v_{\alpha_k}^j = u_{\alpha_k}(x_{\alpha_k}^j + \lambda_{\alpha_k}^j x)
\]

converges in \( C^k_{\text{loc}}(\mathbb{R}^2 \setminus \{ p_1^j, p_2^j, \ldots, p_n^j \}) \) to a non-trivial harmonic map, denoted by

\[
v^j : S^2 \to N, \quad j = 1, 2, \ldots, n_0.
\]

Moreover, one of the following always holds true:

(H1) For any fixed \( R > 0 \), \( B_{R \lambda_i^\alpha}(x_i^\alpha) \cap B_{R \lambda_j^\alpha}(x_j^\alpha) = \emptyset \) whenever \( (\alpha - 1) \) are sufficiently small.

(H2) \( \frac{\lambda_i^\alpha}{\lambda_j^\alpha} + \frac{\lambda_j^\alpha}{\lambda_i^\alpha} \to +\infty \) as \( \alpha \to 1 \).

Remark 1.1. One is easy to check that if \( (\lambda_i^\alpha, x_i^\alpha) \) and \( (\lambda_j^\alpha, x_j^\alpha) \) do not satisfy (H1) and (H2), then one is able to find subsequences of \( \lambda_i^\alpha, x_i^\alpha \) and \( \lambda_j^\alpha, x_j^\alpha \) such that

\[
\frac{\lambda_i^\alpha}{\lambda_j^\alpha} \to \lambda \in (0, \infty) \quad \text{and} \quad \frac{x_i^\alpha - x_j^\alpha}{\lambda_j^\alpha} \to a \in \mathbb{R}^2.
\]

Since

\[
u_a(x_i^\alpha + \lambda_j^\alpha x) = u_a\left(x_j^\alpha + \lambda_j^\alpha \left(\frac{x_i^\alpha - x_j^\alpha}{\lambda_j^\alpha} + \lambda_i^\alpha x\right)\right).
\]

we have \( v^j(x) = u^j(a + \lambda x) \), and then \( u^j \) and \( v^j \) are in fact the same bubbles.

In order to reveal the relation between the blow-up radius and the parameter \( \alpha \) which ensures the energy identity and no-neck property, we need to introduce the following quantities:

\[
u_j \equiv \liminf_{\alpha \to 1} (\lambda_j^\alpha)^{2-2\alpha}
\]

and

\[
u_j \equiv \liminf_{\alpha \to 1} (\lambda_j^\alpha)^{-\sqrt{\alpha-1}}.
\]

Obviously, \( \nu_j \in [1, \infty] \). It is also easy to see that \( \mu_j \in [1, \mu_{\text{max}}] \) for some positive number \( \mu_{\text{max}} \geq 1 \) (see Remark 1.2). We will see that the quantities \( \mu_j \) and \( \nu_j \) characterize completely the energy identity and the length of necks joining bubbles respectively.

Now, we are in the position to state our main results. The first task of this paper is to establish the following weak energy identity:

**Theorem 1.1.** Let \( B_\sigma = B_\sigma(0) \) be a ball in \( \mathbb{R}^2 \), \( g_\alpha = e^{\phi_\alpha(x)}((dx_1)^2 + (dx_2)^2) \) and \( g_0 = e^{\phi(x)}((dx_1)^2 + (dx_2)^2) \) be a family of metrics on \( B_\sigma \), where \( \phi_\alpha \in C^\infty(B_\sigma) \), \( \phi_\alpha(0) = 0 \), and
If \( \varphi \) converges smoothly to \( \varphi \in C^\infty(\overline{B}_\sigma) \). Assume \( \{u_\alpha,(B_\sigma,g_\alpha) \to N (\alpha \to 1) \) is a sequence of maps, each \( u_\alpha \) of which is a critical point of the corresponding functional \( E_{\alpha,\epsilon_\alpha}(u,B_\sigma) \), with \( E_{\alpha,\epsilon_\alpha}(u_\alpha,B_\sigma) < \Theta \) for any \( 0 < \alpha \leq 1 \) and

\[
0 < \beta_0 < \lim_{\alpha \to 1} \epsilon_\alpha^{\alpha^{-1}} \leq 1;
\]

moreover, \( u_\alpha \to u_0 \) in \( C^\infty_{\text{loc}}(B_\sigma \setminus \{0\},N) \) and there are \( n_0 \) bubbles \( v_j : S^2 \to N \) at point \( \{0\} \). Here \( \Theta \) and \( \beta_0 \) are positive constants. Then the following energy identity holds true

\[
\lim_{\delta \to 0} \lim_{\alpha \to 1} \int_{B_\delta(0)} (\epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2)^\alpha dV_{g_\alpha} = \sum_{j=1}^{n_0} \mu_j^2 E(v^j),
\]

where \( \mu_j \) is defined by (1.2).

Remark 1.2. We claim that in Theorem 1.1 \( \mu_j \in [1,\mu_{\text{max}}] \) for some bounded positive number \( \mu_{\text{max}} \). Indeed, fixing an \( R > 0 \), we have

\[
\int_{B_R \setminus \bigcup_{i=1}^{n_0} B_{R_{\delta_j}}(x_j^0 + \lambda_{\alpha_k} p_i)} |\nabla_{g_{\alpha_k}} u_{\alpha_k}|^{2\alpha_k} dV_g = (\lambda_{\alpha_k}^j)^{2-2\alpha} \int_{B_{R} \setminus \bigcup_{i=1}^{n_0} B_{R_{\delta_j}}(x_j^0 + \lambda_{\alpha_k} p_i)} |\nabla_{g_{\alpha_k}} u_{\alpha_k}|^{2\alpha_k} dV_g(x_j^0 + \lambda_{\alpha_k} p_i), \tag{1.4}
\]

Since \( \lambda_{\alpha_k}^j < 1, x_j^0 \to 0 \) and

\[
\lim_{\alpha_k \to 1} \int_{B_R \setminus \bigcup_{i=1}^{n_0} B_{R_{\delta_j}}(x_j^0 + \lambda_{\alpha_k} p_i)} |\nabla_{g_{\alpha_k}} u_{\alpha_k}|^{2\alpha_k} dV_g \to \int_{B_R \setminus \bigcup_{i=1}^{n_0} B_{R_{\delta_j}}(p_i)} |\nabla_{0} v^j|^2 dx,
\]

then we have for large \( R \) and small \( \delta \)

\[
\mu_j \leq \lim_{\alpha_k \to 1} \frac{\int_{B_R \setminus \bigcup_{i=1}^{n_0} B_{R_{\delta_j}}(x_j^0 + \lambda_{\alpha_k} p_i)} |\nabla_{g_{\alpha_k}} u_{\alpha_k}|^{2\alpha_k} dV_g}{\int_{B_R \setminus \bigcup_{i=1}^{n_0} B_{R_{\delta_j}}(p_i)} |\nabla_{0} v^j|^2 dx}.
\]

Therefore, there holds

\[
\mu_j \leq \lim_{b \to 0} \lim_{R \to \infty} \lim_{\alpha_k \to 1} \frac{\int_{B_R \setminus \bigcup_{i=1}^{n_0} B_{R_{\delta_j}}(x_j^0 + \lambda_{\alpha_k} p_i)} |\nabla_{g_{\alpha_k}} u_{\alpha_k}|^{2\alpha_k} dV_g}{\int_{B_R \setminus \bigcup_{i=1}^{n_0} B_{R_{\delta_j}}(p_i)} |\nabla_{0} v^j|^2 dx} \leq \frac{1}{\Theta} \left( \lim_{\alpha_k \to 1} \epsilon_\alpha |B_\sigma| - E(u_0,B_\sigma) \right) = \mu_{\text{max}}, \tag{1.5}
\]

where $B_R = B_R(0)$ is a ball in $\mathbb{R}^2$, $|B_{\sigma}|$ denotes the area of $B_{\sigma}$ and
\[
\theta = \inf \{ E(u) : \text{all non-trivial harmonic maps } u \text{ from } S^2 \text{ into } N \}.
\]

This theorem tells us that the energy identity holds true if and only if $\mu_j = 1$. It provides a new route to approach the problem whether the energy on necks vanishes or not. Obviously, in the above theorem $u_0$ is a harmonic map from $(B_{\sigma}, g_0)$ into $N$. As a direct corollary, we obtain

**Theorem 1.2.** Let $M$ be a smooth closed Riemann surface and $N$ be a smooth compact Riemannian manifold without boundary. Assume that $u_{\alpha_k} \in C^\infty(M, N)$ $(\alpha_k \to 1)$ is a sequence of $\alpha_k$-harmonic maps with uniformly bounded $\alpha$-energy, i.e. $E_{\alpha_k}(u_{\alpha_k}) \leq \Theta$, and $x_1$ be the only blow-up point of the sequence $\{u_{\alpha_k}\}$ in $B_{\sigma}(x_1) \subset M$. Then, passing to a subsequence, there exist $u_0 : M \to N$ which is a smooth harmonic map and finitely many bubbles $v_j : S^2 \to N$ such that $u_{\alpha_k} \to u_0$ weakly in $W^{1,2}(M, N)$, and in $C^0_{\text{loc}}(B_{\sigma}(x_1) \setminus \{x_1\}, N)$; and the following identity holds
\[
\lim_{k \to +\infty} E_{\alpha_k}(u_{\alpha_k}, B_{\sigma}(x_1)) = E(u_0, B_{\sigma}(x_1)) + |B_{\sigma}(x_1)| + \sum_{j=1}^{n_0} \mu_j^2 E(v_j),
\]
(1.6) where $\mu_j$ is defined by (1.2) and $n_0$ is the number of bubbles at blow-up point $x_1$.

It is our another purpose to study the properties of the necks joining bubbles. Our main results are stated as follows:

**Theorem 1.3.** Let $B_{\sigma} = B_{\sigma}(0)$ be a ball in $\mathbb{R}^2$, $g_{\alpha} = e^{\psi_{\alpha}(x)}((dx^1)^2 + (dx^2)^2)$ and $g_0 = e^{\psi(x)}((dx^1)^2 + (dx^2)^2)$ be a family of metrics on $B_{\sigma}$, where $\psi_{\alpha} \in C^\infty(B_{\sigma})$, $\psi_{\alpha}(0) = 0$, and $\psi_{\alpha}$ converges smoothly to $\phi \in C^\infty(B_{\sigma})$. Assume $u_{\alpha} : (B_{\sigma}, g_{\alpha}) \to N$ is a sequence of $\alpha$-harmonic maps as stated in Theorem 1.1, and there is only one bubble $v^1 : S^2 \to N$ in $B_{\sigma}$ for such a sequence and the blow-up point is $\{0\}$. Then we have

1) when $v^1 = 1$, the set $u_0(B_{\sigma}) \cup v^1(S^2)$ is a connected subset of $N$;
2) when $v^1 \in (1, \infty)$, the set $u_0(B_{\sigma})$ and $v^1(S^2)$ are connected by a geodesic with length
\[
L = \sqrt{\frac{E(v^1)}{\pi}} \log v^1;
\]
3) when $v^1 = +\infty$, the neck contains at least an infinite length geodesic.

Here $v^1$ is defined by (1.3), i.e. $v^1 = \lim_{\alpha_k \to 1} (\lambda_{\alpha_k}^1)^{-\sqrt{\alpha_k - 1}}$.

As a direct corollary, we have

**Theorem 1.4.** Let $M$ be a smooth closed Riemann surface and $N$ be a smooth closed Riemannian manifold and $u_{\alpha_k} \in C^\infty(M, N)$ be a sequence of $\alpha_k$-harmonic maps with uniformly bounded energy, i.e. $E_{\alpha_k}(u_{\alpha_k}) \leq \Theta$, and $u_{\alpha_k}$ converges to a smooth harmonic map $u_0 : M \to N$ in $C^\infty_{\text{loc}}(B_{\sigma}(x_1) \setminus \{x_1\}, N)$ as $\alpha_k \to 1$. Assume there is only one bubble in $B_{\sigma}(x_1) \subset M$ for $\{u_{\alpha_k}\}$ and $v^1 : S^2 \to N$ is the bubbling map. Let $v^1 = \lim_{\alpha_k \to 1} (\lambda_{\alpha_k}^1)^{-\sqrt{\alpha_k - 1}}$. Then we have
1) when \( \nu^1 = 1 \), the set \( u_0(B_{\sigma}(x_1)) \cup v^1(S^2) \) is a connected subset of \( N \); 
2) when \( \nu^1 \in (1, \infty) \), the set \( u_0(B_{\sigma}(x_1)) \) and \( v^1(S^2) \) are connected by a geodesic with length

\[
L = \sqrt{\frac{E(v^1)}{\pi}} \log v^1;
\]

3) when \( \nu^1 = +\infty \), the neck contains at least an infinite length geodesic.

We should mention that after we completed the paper we found that Moore had proved that the same length formula holds true if a neck is of finite length \( L \) and \( \tilde{g} \geq 1 \) (the genus of \( M \)). Note that \( E(u) \) is defined by \( \frac{1}{2} \int_M |\nabla u|^2 \, dV_{\tilde{g}} \) in [14]. However, in this paper the arguments to prove Theorem 1.3 is completely different from Moore’s proof. The key estimations, which give the details of the necks, are established in Proposition 4.3 in Section 4.

We fail to find a sufficient condition such that \( \nu^i < +\infty \), but we will show that there are indeed many cases that the necks contain at least a geodesic of infinite length:

**Corollary 1.5.** Let \( \{u_{\alpha_k}\} (\alpha_k \to 1) \) be a sequence of maps from \( M \) into \( N \) each \( u_{\alpha_k} \) of which is a minimizer of \( E_{\alpha_k} \) in the homotopy class containing \( u_{\alpha_k} \). We assume that, for any \( \alpha_i \neq \alpha_j \), \( u_{\alpha_i} \) and \( u_{\alpha_j} \) do not belong to a homotopy class. If

\[
\sup_{\alpha_k} E_{\alpha_k}(u_{\alpha_k}) < +\infty,
\]

then \( \{u_{\alpha_k}\} \) blows up at some points, and the necks contain at least a geodesic of infinite length.

**Remark 1.3.** In Section 5, by constructing a compact manifold \( N \) we give an example of sequence of minimizing \( \alpha \)-harmonic maps, which satisfies the conditions in the above corollary. This indicates that there exists a neck joining bubbles which is a geodesic of infinite length.

As a consequence of Theorem 1.1, we have the following proposition which implies the result due to Chen–Tian that, if the necks consist of some geodesics of finite length, then the energy identity is true:

**Proposition 1.6.** The energy identity holds true for a subsequence of \( u_{\alpha} \) if and only if

\[
\lim_{\alpha \to 1} \liminf_{\alpha \to 1} \|\nabla u_{\alpha}\|_{C^0(M)}^{\alpha - 1} = 1. \tag{1.7}
\]

The limit set of such subsequence has no neck if and only if

\[
\lim_{\alpha \to 1} \liminf_{\alpha \to 1} \|\nabla u_{\alpha}\|_{C^0(M)}^{\alpha - 1} = 1.
\]

The bubbles in limit set of such subsequence are joined by some geodesics of finite length, if and only if

\[
\lim_{\alpha \to 1} \liminf_{\alpha \to 1} \|\nabla u_{\alpha}\|_{C^0(M)}^{\alpha - 1} < +\infty.
\]
Proof. We only prove the first claim. First, we prove that (1.7) implies $\mu_j = 1$. We assume $v_j^\alpha(x) = u_\alpha(x^j + \lambda_\alpha x)$ converges to $v_j$ in $C^1_{\text{loc}}(\mathbb{R}^n \setminus \{p_1, p_2, \ldots, p_s\})$. Then we have

$$
(\lambda_\alpha^j)^{1-\alpha} = \frac{|\nabla u_\alpha(x^j + \lambda_\alpha^j x)|^{\alpha-1}}{|\nabla v_j^\alpha(x)|^{\alpha-1}}
$$

for any $x$ with $|\nabla v_j^\alpha(x)| \neq 0$. Hence we get $\mu_j \leq 1$ and then $\mu_j = 1$.

Conversely, if $\mu_j = 1$ for all $j$, we need to verify (1.7). Let $x_\alpha \in M$ be the point such that $|\nabla u_\alpha(x_\alpha)| = \max_M |\nabla u_\alpha|$, and

$$
\lambda_\alpha = \frac{1}{|\nabla u_\alpha(x_\alpha)|}.
$$

We set $v_\alpha(x) = u_\alpha(x_\alpha + \lambda_\alpha x)$. One is easy to check that $v_\alpha$ will converge to a non-trivial harmonic map $v_0$ locally. By (H1) and (H2) we must have $j$, such that

$$
B_{R\lambda_\alpha^j}(x_\alpha^j) \cap B_{R\lambda_\alpha^j}(x_\alpha) \neq \emptyset, \quad \text{and} \quad \frac{1}{C} \lambda_\alpha^j < \lambda_\alpha < C \lambda_\alpha^j
$$

for some $C > 0$. Hence we get $|\lambda_\alpha|^{\alpha-1} \to 1$. \qed

The paper is organized as follows: In Section 2 we recall the well-known $\epsilon$-regularity theorem due to Sacks and Uhlenbeck, and establish some gradient estimates and integral estimates. We show Theorem 1.1 in Section 3. We analyze the asymptotic behaviors of the neck domains and give the length formula of the neck in Section 4 for the case only one bubble appears. In Section 5, an example of sequence of maps is given to show the necks contain at least a geodesic of infinite length. In Section 6, we indicate how to construct the bubble tree of a sequence of $\alpha$-harmonic maps.

2. Preliminary

In this section we intend to establish some integral formulas on $\alpha$-harmonic maps from a compact closed surfaces by the variations of domain. Of course, we need to choose some suitable variational vector fields on $M$ which generate the transformations of $M$. We will see that these integral relations will play an important role on the proofs of main theorems.

Now, we give the reasons and motivations to study the functionals $E_{\alpha, \epsilon_\alpha}$ stated in Section 1. Recall that the functional $E_{\alpha}$ is not of conformal invariance. For example, on an isothermal coordinate system around a point $p \in M$, if the metric is given by $g = e^{\phi}((dx^1)^2 + (dx^2)^2)$ with $\phi(p) = 0$ and $\tilde{u}_\alpha(x) = u_\alpha(\lambda x)$, then we have

$$
\int_{B_{\frac{\epsilon\lambda_\alpha}{\lambda}}(p)} (1 + |\nabla u_\alpha|^2)^\alpha dV_g = \int_{B_{\frac{\epsilon}{\lambda}}(p)} \lambda^{2-2\alpha}(\lambda^2 + |\nabla g' \tilde{u}_\alpha|^2)^\alpha dV_{g'},
$$

where $g' = e^{-\phi}((dx^1)^2 + (dx^2)^2)$. This shows that $E_{\alpha}$ is not invariant under conformal transformation.
where \( g' = e^{p+\lambda(x)}((dx_1)^2 + (dx_2)^2) \). On the other hand, we should also note that, for a sequence of \( \alpha \)-harmonic maps \( u_\alpha \), it is possible that there exist several bubbles near a blow-up point. For example, if there are sequences \( \{\lambda_1^{\alpha}\} \) and \( \{\lambda_2^{\alpha}\} \) such that
\[
\frac{\lambda_2^{\alpha}}{\lambda_1^{\alpha}} \to 0, \quad \lambda_1^{\alpha} \to 0,
\]
as \( \alpha \to 1 \), and
\[
v_1^{\alpha}(x) = u_\alpha(\lambda_1^{\alpha}x) \to v^1 \quad \text{in} \quad C^k_{\text{loc}}(\mathbb{R}^2), \quad v_2^{\alpha}(x) = u_\alpha(\lambda_2^{\alpha}x) \to v^2 \quad \text{in} \quad C^k_{\text{loc}}(\mathbb{R}^2 \setminus \{0\}),
\]
where \( v^1, v^2 \) are non-trivial harmonic maps from \( S^2 \) to \( N \). For this case, we have
\[
v_2^{\alpha}(x) = v_1^{\alpha}(\frac{\lambda_2^{\alpha}}{\lambda_1^{\alpha}}x),
\]
i.e. \( v^2(x) \) is in fact a bubble for the sequence \( u_\alpha \). Therefore, we have to consider the Euler–Lagrange equation satisfied by \( v_1^{\alpha} \). It is easy to check that \( v_1^{\alpha} \) is locally a critical point of the functional
\[
F(v) = \int_{B_\delta} \left((\lambda_1^{\alpha})^2 + |\nabla v|^2\right)^{\alpha} dV_{g_\alpha},
\]
where \( g_\alpha = e^{p+\lambda_1^{\alpha}x}((dx_1)^2 + (dx_2)^2) \). For the above reasons, we need to consider more general \( \alpha \)-energy functionals which are of the following forms:
\[
E_{\alpha,\epsilon_\alpha}(u) = \int_{B} \left(\epsilon_\alpha + |\nabla g u_\alpha|^2\right)^{\alpha} dV_{g_\alpha},
\]
where \( B = \{x: |x| < 1\} \) is a unit ball in \( \mathbb{R}^2 \) and \( \epsilon_\alpha \in (0, 1] \). If \( u_\alpha \) is a critical point of the above some functional \( E_{\alpha,\epsilon_\alpha}(u) \), then, \( u_\alpha \) satisfies the following elliptic system which is also called \( \alpha \)-harmonic map equation:
\[
\Delta_{g_\alpha} u_\alpha + (\alpha - 1) \frac{\nabla g_\alpha |\nabla g u_\alpha|^2}{\epsilon_\alpha + |\nabla g u_\alpha|^2} + A(u_\alpha)(du_\alpha, du_\alpha) = 0. \quad (2.1)
\]
Here we always assume that the sequence \( \epsilon_\alpha (\epsilon_\alpha \leq 1) \) satisfies
\[
\lim_{\alpha \to 1} \epsilon_\alpha^{\alpha-1} \beta_0 > 0. \quad (2.2)
\]
From (1.5) we can see that this assumption is reasonable.

Next, we recall the well-known \( \epsilon \)-regularity theorem due to Sacks and Uhlenbeck [17]:

**Theorem 2.1.** Let \( u_\alpha : (B, g_\alpha) \to N \) satisfies Eq. (2.1). Then, there exists \( \epsilon > 0 \) and \( \alpha_0 > 1 \) such that if \( E(u_\alpha, B) < \epsilon \) and \( 1 \leq \alpha \leq \alpha_0 \), then for all smaller \( r < 1 \), we have
\[
\|\nabla g_{\alpha} u_\alpha\|_{W^{1,p}(B_\delta)} \leq C(p, r)E(u_\alpha, B),
\]
here \( B_r \subset B \) is a ball with radius \( r \), \( 1 < p < \infty \).
Remark 2.1. It is worthy to point out that the constant \( C(p, r) \) in Theorem 2.1 does not depend on \( \alpha, \epsilon_\alpha \) and \( g_\alpha \) if \( g_\alpha = e^{\phi_\alpha}((dx^1)^2 + (dx^2)^2) \), where \( \phi_\alpha(0) = 0 \) and \( \phi_\alpha \to \varphi \in C^\infty(B) \) smoothly.

We also have the following a priori estimates:

Lemma 2.2. Let \((B, g_\alpha)\) be a unit disk in \( \mathbb{R}^2 \) with metric \( g_\alpha = e^{\phi_\alpha}((dx^1)^2 + (dx^2)^2) \), where \( \phi_\alpha(0) = 0 \) and \( \phi_\alpha \to \varphi \in C^\infty(B) \) smoothly. If \( u_\alpha: (B, g_\alpha) \to N \) is a critical point of \( E_{\alpha, \epsilon_\alpha}(u, B) \) given by

\[
E_{\alpha, \epsilon_\alpha}(u, B) = \int_B \left( \epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2 \right)^\alpha dV_{g_\alpha},
\]

with \( \lim_{\alpha \to 1} \epsilon_\alpha^{-1} > \beta_0 > 0 \) and \( E_{\alpha, \epsilon_\alpha}(u_\alpha) \leq \Theta \), then we have

\[
\beta_0 < \liminf_{\alpha \to 1} \| (\epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2)^{-1} \|_{C^0(B)} \leq \limsup_{\alpha \to 1} \| (\epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2)^{-1} \|_{C^0(B)} < \beta_1,
\]

where \( \beta_1 \) is independent of \( \alpha \).

Proof. Obviously, we only need to prove \( \| \nabla_{g_\alpha} u_\alpha \|_{C^0(B)} < C \). We assume that there is a sequence \( \{ u_\alpha_k \} \) such that \( \| \nabla_{g_\alpha_k} u_\alpha_k \|_{C^0(B)} \to + \infty \) as \( \alpha_k \to 1 \). Suppose that \( \max |\nabla_{g_\alpha_k} u_\alpha_k| \) is attained at \( x_{\alpha_k} \), i.e.

\[
|\nabla_{g_\alpha_k} u_{\alpha_k}(x_{\alpha_k})| = \max_{B} \{ |\nabla_{g_\alpha_k} u_{\alpha_k}| \}.
\]

Let \( \lambda_k = \frac{1}{|\nabla_{g_\alpha_k} u_{\alpha_k}(x_{\alpha_k})|} \) and \( v_k(x) = u_\alpha_k(x_{\alpha_k} + \lambda_k x) \). Obviously, we have \( E_{\alpha_k}(v_k) \leq C < \infty \). Then there exists a subsequence of \( \{ v_{k_j} \} \) which converges to a new non-trivial harmonic map from \( S^2 \) to \( N \). Hence, from (1.5) we can infer \( \lambda_{k_j}^{-1} < C \), which contradicts with the choice of \( \alpha_k \). \( \Box \)

Next we need to establish some general variational formulas for the functional \( E_{\alpha, \epsilon_\alpha}(u, B) \).

By virtue of these variational identities we can derive some important estimates on the energy on the necks of the critical points of \( E_{\alpha, \epsilon_\alpha} \). The integral identities in the following two lemmas, i.e. (2.6) and (2.8), will be used repeatedly in our arguments.

Lemma 2.3. Let \((B, g_\alpha)\) be a unit disk in \( \mathbb{R}^2 \) with metric \( g_\alpha = e^{\phi_\alpha}((dx^1)^2 + (dx^2)^2) \), where \( \phi_\alpha(0) = 0 \) and \( \phi_\alpha \to \varphi \in C^\infty(B) \) smoothly. If \( u_\alpha \) is a critical point of \( E_{\alpha, \epsilon_\alpha}(u, B) \) where \( 1 \geq \lim_{\alpha \to 1} \epsilon_\alpha^{-1} > \beta_0 > 0 \), then, there holds true for any \( t < 1 \)

\[
\left( 1 - \frac{1}{2\alpha} \right) \int_{\partial B_t} \left( \epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2 \right)^{\alpha-1} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 ds_0 - \frac{1}{2\alpha} \int_{\partial B_t} \left( \epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2 \right)^{\alpha-1} |u_\alpha|_\theta^2 ds_0
\]

\[
= \frac{(\alpha - 1)}{\alpha t} \int_{B_t} \left( \epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2 \right)^{\alpha-1} |\nabla_0 u_\alpha|^2 dx + O(t),
\]
where $ds_0$ is the volume element and $\nabla_0$ is the gradient operator with respect to Euclidean metric.

**Proof.** Take a 1-parameter family of transformations $\{\phi_s\}$ which is generated by the vector field $X$. If we assume $X$ is supported in $B$, then we have

\[
E_{\alpha, \epsilon_\alpha} (u \circ \phi_s) = \int_B \left( \epsilon_\alpha + |\nabla_g u \circ \phi_s|^2 \right)^\alpha dV_{g_\alpha} = \int_B \left( \epsilon_\alpha + \sum_\beta |d(u \circ \phi_s) (e_\beta(x))|^2 \right)^\alpha dV_{g_\alpha}
\]

\[
= \int_B \left( \epsilon_\alpha + \sum_\beta |d(u \circ \phi_s) (e_\beta(x))|^2 \right)^\alpha dV_{g_\alpha}
\]

\[
= \int_B \left( \epsilon_\alpha + \sum_\beta |d(u \circ \phi_s) (e_\beta(\phi_s^{-1}(x)))|^2 \right)^\alpha \text{Jac}(\phi_s^{-1}) dV_{g_\alpha},
\]

where $\{e_\alpha\}$ is a local orthonormal basis of $TB$. Noting

\[
\frac{d}{ds} \text{Jac}(\phi_s^{-1}) dV_{g_\alpha}|_{s=0} = -\text{div}(X) dV_{g_\alpha},
\]

by differentiating the above identity we obtain the formula

\[
dE_{\alpha, \epsilon_\alpha} (u)(u_s(X)) = -\int_B \left( \epsilon_\alpha + |\nabla_g u|^2 \right)^\alpha \text{div}(X) dV_{g_\alpha}
\]

\[
+ 2\alpha \sum_\beta \int_B \left( \epsilon_\alpha + |\nabla_g u|^2 \right)^{\alpha-1} \langle du(\nabla_{e_\beta} X), du(e_\beta) \rangle dV_{g_\alpha}.
\]

Now, we assume $u_{\alpha}$ to be the critical point of $E_{\alpha, \epsilon_\alpha}$. For any vector field $X$ on $B$, we have

\[
2\alpha \sum_\beta \int_B \left( \epsilon_\alpha + |\nabla_g u_{\alpha}|^2 \right)^{\alpha-1} \langle du_{\alpha}(\nabla_{e_\beta} X), du_{\alpha}(e_\beta) \rangle dV_{g_\alpha}
\]

\[
= \int_B \left( \epsilon_\alpha + |\nabla_g u_{\alpha}|^2 \right)^\alpha \text{div} X dV_{g_\alpha}. \tag{2.3}
\]

Next, for $0 < t' < t \leq \rho < 1$, we choose a vector field $X$ with compact support in $B_\rho$ by

\[
X = \eta(r) r \frac{\partial}{\partial r} = \eta(|x|) x^i \frac{\partial}{\partial x^i}, \text{ where } \eta \text{ is defined by}
\]

\[
\eta(r) = \begin{cases} 
1 & \text{if } r \leq t', \\
\frac{t-t'}{t-t} & \text{if } t' \leq r \leq t, \\
0 & \text{if } r \geq t,
\end{cases}
\]

where $r = \sqrt{(x^1)^2 + (x^2)^2}$. By a direct computation we obtain
\[ \nabla_{\vec{\partial}^T} X = \eta \frac{\partial}{\partial x^1} + \eta \left( x^1 \right)^2 \frac{\partial}{\partial x^1} + \eta x^1 x^2 \frac{\partial}{\partial x^2} + \eta x^1 \Gamma_{11} \frac{\partial}{\partial x^1} + \eta x^1 \Gamma_{12} \frac{\partial}{\partial x^2} + \eta x^2 \Gamma_{12} \frac{\partial}{\partial x^1} + \eta x^2 \Gamma_{12} \frac{\partial}{\partial x^2}. \]

Here \( \Gamma_{ij} \) are the coefficients of Levi-Civita connection with respect to \( g_\alpha \). Then,

\[ \sum_{\beta} \langle du_\alpha (\nabla_{e_\beta} X), du_\alpha (e_\beta) \rangle dV_{g_\alpha} = \langle du_\alpha \left| \nabla_0 u_\alpha \right|^2 + \eta \left| \frac{\partial u_\alpha}{\partial r} \right|^2 + O(\left| x \right|) \rangle dV_{g_\alpha}. \]  

(2.4)

where \( \nabla_0 \) is the gradient operator with respect to standard Euclidean metric.

It is also easy to check

\[ \text{div}(X) = 2\eta + r\eta' + r \frac{\partial \varphi}{\partial r}. \]

Hence, by substituting the above identity and (2.4) into (2.3) we derive

\[ 0 = (2\alpha - 2) \int_{B_t} \eta \left( \epsilon_\alpha + \left[ \nabla_{g_\alpha} u_\alpha \right]^2 \right)^{\alpha - 1} \left| \nabla_0 u_\alpha \right|^2 dx + \int_{B_r} O(\left| x \right|) \left( \epsilon_\alpha + \left[ \nabla_{g_\alpha} u_\alpha \right]^2 \right)^{\alpha - 1} \left| \nabla_0 u_\alpha \right|^2 dx \]

\[ \quad - \frac{2\epsilon_\alpha}{B_t} \int_{B_t} \eta \left( \epsilon_\alpha + \left[ \nabla_{g_\alpha} u_\alpha \right]^2 \right)^{\alpha - 1} dV_{g_\alpha} + \frac{\epsilon_\alpha}{t - t'} \int_{B_t \setminus B_{t'}} r \left( \epsilon_\alpha + \left[ \nabla_{g_\alpha} u_\alpha \right]^2 \right)^{\alpha - 1} dV_{g_\alpha} \]

\[ \quad + \frac{1}{t - t'} \int_{B_t \setminus B_{t'}} \left( \epsilon_\alpha + \left[ \nabla_{g_\alpha} u_\alpha \right]^2 \right)^{\alpha - 1} \left[ \nabla_0 u_\alpha \right]^2 r - 2\alpha r \left| \frac{\partial u_\alpha}{\partial r} \right|^2 \right] dx \]

\[ \quad - \int_{B_t} \epsilon_\alpha \left( \epsilon_\alpha + \left[ \nabla_{g_\alpha} u_\alpha \right]^2 \right)^{\alpha - 1} r \eta \left| \frac{\partial \varphi}{\partial r} \right| dV_{g_\alpha}. \]

Letting \( t' \to t \) in the above identity and using Lemma 2.2, we obtain the following

\[ \int_{\partial B_t} \left( \epsilon_\alpha + \left[ \nabla_{g_\alpha} u_\alpha \right]^2 \right)^{\alpha - 1} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 ds_0 - \frac{1}{2\alpha} \int_{\partial B_t} \left( \epsilon_\alpha + \left[ \nabla_{g_\alpha} u_\alpha \right]^2 \right)^{\alpha - 1} \left| \nabla_0 u_\alpha \right|^2 ds_0 \]

\[ = \frac{(\alpha - 1)}{\alpha t} \int_{B_t} \left( \epsilon_\alpha + \left[ \nabla_{g_\alpha} u_\alpha \right]^2 \right)^{\alpha - 1} \left| \nabla_0 u_\alpha \right|^2 dx + O(t), \]  

(2.5)

where \( ds_0 \) is the volume element of \( \partial B_t \) with respect to Euclidean metric.
Note the metric $g_\alpha$ can be written as $g_\alpha = e^{\varphi_\alpha}(dr^2 + r^2 d\theta^2)$ in the polar coordinate system. Set

$$u_{\alpha,\theta} = \frac{1}{r} \frac{\partial u_\alpha}{\partial \theta}.$$  

Obviously

$$|\nabla_0 u_\alpha|^2 = \left|\frac{\partial u_\alpha}{\partial r}\right|^2 + |u_{\alpha,\theta}|^2.$$  

Hence, we get from the above identity and (2.5)

$$\left(1 - \frac{1}{2\alpha}\right) \int_{\partial B_t} \left(\epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2\right)^{\alpha-1} \left|\frac{\partial u_\alpha}{\partial r}\right|^2 ds_0 - \frac{1}{2\alpha} \int_{\partial B_t} \left(\epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2\right)^{\alpha-1} |u_{\alpha,\theta}|^2 ds_0$$

$$= \frac{(\alpha - 1)}{\alpha t} \int_{B_t} \left(\epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2\right)^{\alpha-1} |\nabla_0 u_\alpha|^2 dx + O(t).$$  

(2.6)

Thus we finish the proof of the lemma. □

We also need to infer a special kind of variational formulas which is derived from the variations with respect to radial direction, i.e. the following Pohozaev identity.

**Lemma 2.4.** Let $(B, g_\alpha)$ be a unit disk in $\mathbb{R}^2$ with metric $g_\alpha = e^{\varphi_\alpha}((dx_1)^2 + (dx_2)^2)$, where $\varphi_\alpha(0) = 0$. If $u_\alpha$ is a critical point of $E_{\alpha,\epsilon_\alpha}(u, B)$, then there holds true for any $t < 1$

$$\int_{\partial B_t} \left(\left|\frac{\partial u_\alpha}{\partial r}\right|^2 - |u_{\alpha,\theta}|^2\right) ds_0 = -\frac{2(\alpha - 1)}{t} \int_{B_t} \frac{\nabla_0 |\nabla_{g_\alpha} u_\alpha|^2 \nabla_0 u_\alpha}{\epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2} r \frac{\partial u_\alpha}{\partial r} dx,$$

where $\frac{\partial}{\partial r}$ denotes the radial derivative.

**Proof.** Denote $\Delta_0 = \frac{\partial^2}{\partial (x_1)^2} + \frac{\partial^2}{\partial (x_2)^2}$. Since $u_\alpha$ is a critical point of $E_{\alpha,\epsilon_\alpha}(u, B)$, for each $\alpha \geq 1$, $u_\alpha$ satisfies the Euler–Lagrange equation (2.1), i.e.

$$\Delta_0 u_\alpha + (\alpha - 1) \frac{\nabla_0 |\nabla_{g_\alpha} u_\alpha|^2 \nabla_0 u_\alpha}{\epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2} + A(u_\alpha)(du_\alpha, du_\alpha) = 0.$$  

As in [13], we multiply the both sides of the above equation with $r \frac{\partial u_\alpha}{\partial r}$ to obtain

$$\int_{B_t} r \frac{\partial u_\alpha}{\partial r} \Delta_0 u_\alpha dx = -(\alpha - 1) \int_{B_t} \frac{\nabla_0 |\nabla_{g_\alpha} u_\alpha|^2 \nabla_0 u_\alpha}{\epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2} r \frac{\partial u_\alpha}{\partial r} dx.$$
It is easy to see

$$
\int_{B_t} r \frac{\partial u_\alpha}{\partial r} \Delta_0 u_\alpha \, dx = \int_{\partial B_t} r \left| \frac{\partial u_\alpha}{\partial r} \right|^2 \, ds_0 - \int_{B_t} \nabla_0 \left( r \frac{\partial u_\alpha}{\partial r} \right) \nabla_0 u_\alpha \, dx.
$$

Since

$$
\int_{B_t} \left( \frac{\partial u_\alpha}{\partial r} \right) \nabla_0 u_\alpha \, dx = \int_{B_t} \nabla_0 \left( x^k \frac{\partial u_\alpha}{\partial x^k} \right) \nabla_0 u_\alpha \, dx
$$

$$
= \int_{B_t} |\nabla_0 u_\alpha|^2 \, dx + \int_0^{2\pi} \int_0^t \frac{r}{2} \frac{\partial (|\nabla_0 u_\alpha|^2)}{\partial r} \, r \, d\theta \, dr
$$

$$
= \int_{B_t} |\nabla_0 u_\alpha|^2 \, dx + \int_{\partial B_t} |\nabla_0 u_\alpha|^2 \, ds_0 - \int_{B_t} |\nabla_0 u_\alpha|^2 \, dx
$$

$$
= \frac{1}{2} \int_{\partial B_t} |\nabla_0 u_\alpha|^2 \, ds_0.
$$

then, we have

$$
\int_{\partial B_t} \left( \left| \frac{\partial u_\alpha}{\partial r} \right|^2 - \frac{1}{2} |\nabla_0 u_\alpha|^2 \right) \, ds_0 = -\frac{\alpha - 1}{t} \int_{B_t} \frac{\nabla_0 |\nabla u_\alpha|^2 \nabla_0 u_\alpha}{\epsilon_\alpha + |\nabla u_\alpha|^2} r \frac{\partial u_\alpha}{\partial r} \, dx. \quad (2.7)
$$

Immediately, it follows

$$
\int_{\partial B_t} \left( \left| \frac{\partial u_\alpha}{\partial r} \right|^2 - |u_\alpha, \theta|^2 \right) \, ds_0 = -\frac{2(\alpha - 1)}{t} \int_{B_t} \frac{\nabla_0 |\nabla u_\alpha|^2 \nabla_0 u_\alpha}{\epsilon_\alpha + |\nabla u_\alpha|^2} r \frac{\partial u_\alpha}{\partial r} \, dx. \quad (2.8)
$$

Thus, we complete the proof. \(\square\)

3. The proof of Theorem 1.1

In this section, we make effort to establish a weak energy identity for a sequence of \(\alpha\)-harmonic maps. We will follow the idea of Ding and Tian in [7] and apply (2.6) and (2.8) to show Theorem 1.1. We lay emphasis on the arguments on the case only one bubble occurs for the sequence of \(\alpha\)-harmonic maps.

3.1. The weak energy identity for the case of only one bubble

In this subsection we prove Theorem 1.1 in the case of \(n_0 = 1\), where \(n_0\) is the number of the bubbles. The proof for the cases of several bubbles will be given in the next subsection.
By the assumptions in Theorem 1.1, we know that \( u_\alpha : (B_\sigma, g_\alpha) \to N \) is a sequence of \( \alpha \)-harmonic maps with

\[
E_{\alpha, \epsilon_\alpha}(u_\alpha, B_\sigma) < \Theta \quad \text{and} \quad 0 < \beta_0 < \lim_{\alpha \to 1} \epsilon_\alpha^{\alpha - 1} \leq 1.
\]

Each \( u_\alpha \) satisfies the corresponding equation (2.1).

Moreover, there exists a harmonic map \( u_0 \) from \( (B_\sigma, g_0) \) to \( N \) such that \( u_\alpha \to u_0 \) in \( C^k_{\text{loc}} (B_\sigma \setminus \{0\}) \) and \( 0 \) is the only blow-up point of \( \{u_\alpha\} \) in \( B_\sigma \). Therefore, by the discussions in Lemmas 2.3 and 2.4 we know that for any \( \alpha > 1 \), \( u_\alpha \) satisfy (2.6) and (2.8).

We need to define some important quantities related to the energy on necks. In the sequel, we always pick \( x_\alpha \) such that

\[
|\nabla g_\alpha u_\alpha(x_\alpha)| = \max_{B_\sigma} |\nabla g_\alpha u_\alpha|.
\]

Obviously, \( x_\alpha \to 0 \) for \( \{u_\alpha\} \) in Theorem 1.1.

Set \( \lambda_\alpha = \frac{1}{\max_{B_\sigma} |u_\alpha|} \) and \( v_\alpha = u_\alpha(x_\alpha + \lambda_\alpha x) \). Obviously, \( \|v_\alpha(0)\| = \|\nabla v_\alpha\|_{L^\infty} = 1 \). Hence, \( v_\alpha \) converges to a non-trivial harmonic map \( v \) from \( \mathbb{R}^2 \) to \( N \) locally and smoothly. \( v \) is the bubble.

Set

\[
\Lambda = \lim_{R \to +\infty} \lim_{\alpha \to 1} A_\alpha(R),
\]

where

\[
A_\alpha(R) = \int_{B_{R\lambda_\alpha}(x_\alpha)} |\nabla_{g_\alpha} u_\alpha|^{2\alpha} V_{g_\alpha}.
\]

In the present situation, \( \mu \) is written by \( \mu = \lim_{\alpha \to 1} \lambda_\alpha^{2-2\alpha} \). Then, from (1.5) we can see easily \( \Lambda \) and \( \mu \) satisfy the following relation

\[
\Lambda = \mu E(v).
\]

Moreover, we also have the following fact

\[
\Lambda = \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_{R\lambda_\alpha}} \left( \epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2 \right)^{\alpha - 1} |\nabla_{g_\alpha} u_\alpha|^2 \, dV_{g_\alpha}.
\]

Indeed,

\[
\lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_{R\lambda_\alpha}} \left( \epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2 \right)^{\alpha - 1} |\nabla_{g_\alpha} u_\alpha|^2 \, dV_{g}
\]

\[
= \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_R} \left( \epsilon_\alpha \lambda_\alpha^2 + |\nabla_{g_\alpha} v_\alpha|^2 \right)^{\alpha - 1} \lambda_\alpha^{2-2\alpha} |\nabla_0 v_\alpha|^2 \, dx = \mu \int_{\mathbb{R}^2} |\nabla_0 v|^2 \, dx = \Lambda.
\]
Furthermore, we claim that for any $\epsilon > 0$ there exist $\delta_1$ and $R$ such that, $\forall \lambda \in (\frac{R\lambda_2}{2}, \delta_1)$ where $4\delta_1 \leq \sigma$, as $\alpha - 1$ is small enough there holds

$$\int_{B_{2\lambda} \setminus B_{\lambda}(x_\alpha)} |\nabla_{g_\alpha} u_{\alpha}|^2 dV_{g_\alpha} \leq \epsilon. \quad (3.2)$$

Suppose that the claim is false, then there exist $\alpha_i \to 1$ and $\lambda_i \to 0$ satisfying $\lambda_i \alpha_i \to +\infty$ such that

$$\int_{B_{2\lambda_i} \setminus B_{\lambda_i}(x_{\alpha_i})} |\nabla_{g_{\alpha_i}} u_{\alpha_i}|^2 dV_{g_{\alpha_i}} \geq \epsilon. \quad (3.3)$$

Denote $v'_{\alpha_i}(x) = u_{\alpha_i}(\lambda_i x + x_{\alpha_i}).$ Then, there exists $v'$ such that $v'_{\alpha_i} \to v'$ in $C^k_{\text{loc}}(\mathbb{R}^2 \setminus \{0\} \cup \mathcal{A}), N)$, where $\mathcal{A}$ is a finite set which does not contain $0.$ If $\mathcal{A} = \emptyset$ then it follows from (3.3) that $v'$ is a non-constant harmonic sphere which is different from $v.$ This contradicts the assumption $n_0 = 1.$ Next, if there exists $x_1 \in \mathcal{A},$ then, by a similar argument with the previous to get $v_1 = v,$ we can still obtain sequences $x_i \to x_1$ and $\lambda_i \to 0$ such that $v'_i(x_i + \lambda_i x)$ converges to a harmonic map $v^*.$ Hence we get $u_{\alpha_i}(x_{\alpha_i} + \lambda_i (\lambda_i x + x))$ converges to $v^*$ strongly, and then $v^*$ is the second harmonic sphere. This shows that the claim (3.2) must be true.

Set

$$u^*_\alpha = \frac{1}{2\pi} \int_0^{2\pi} u_{\alpha}(x_{\alpha} + re^{i\theta}) d\theta.$$

One is easy to check that, for any $a < b$ and $B_b \setminus B_a(x_{\alpha}) \subset B_\sigma,$ the following inequality holds true

$$\int_{B_b \setminus B_a(x_{\alpha})} \left| \frac{\partial u^*_\alpha}{\partial r} \right|^2 \left| \int_a^{b} \frac{2\pi}{2\pi} \left| \frac{\partial u_{\alpha}}{\partial \theta} \right|^2 d\theta \right|^2 r dr d\theta \leq \int_a^{b} \left( \int_0^{2\pi} \left| \frac{\partial u_{\alpha}}{\partial \theta} \right|^2 d\theta \right) r dr \leq \int_a^{b} \left( \int_0^{2\pi} \left| \frac{\partial u_{\alpha}}{\partial r} \right|^2 d\theta \right) r dr.$$

$$= \int_a^{b} \left( \int_0^{2\pi} \left| \frac{\partial u_{\alpha}}{\partial r} \right|^2 d\theta \right) r dr = \int_{B_b \setminus B_a(x_{\alpha})} \left| \frac{\partial u_{\alpha}}{\partial r} \right|^2 dx. \quad (3.4)$$

By applying (3.2) and Sacks–Uhlenbeck $\epsilon$-regularity theorem (Theorem 2.1), we have the following

**Lemma 3.1.** Let $B_\sigma = B_\sigma (0)$ is a ball in $\mathbb{R}^2$ with metrics $g_\alpha = e^{\varphi_\alpha(x)}((dx_1)^2 + (dx_2)^2),$ where $\varphi_\alpha \in C^\infty(\overline{B_\sigma}),$ $\varphi_\alpha(0) = 0,$ and $\varphi_\alpha$ converges smoothly to a real function $\varphi \in C^\infty(\overline{B_\sigma}).$ Assume
\( u_\alpha : (B_\sigma , g_\alpha ) \rightarrow N \) be a sequence of \( \alpha \)-harmonic maps each of which satisfies the corresponding equation (2.1), and \( E_{\alpha, \epsilon_\alpha}(u_\alpha , B_\sigma ) < \Theta \). Then, as \( \alpha - 1 \) is small enough, for any \( a \) and \( b \) such that \( R\lambda_\alpha < a < b < \delta \leq \frac{\sigma}{4} \) we have

\[
\int_{B_b \setminus B_a(x_\alpha)} |\nabla^2 g_\alpha u_\alpha|^r |\nabla g_\alpha u_\alpha| \, dV_{g_\alpha} \leq C \int_{B_b \setminus B_\frac{a}{2}(x_\alpha)} |\nabla g_\alpha u_\alpha|^2 \, dV_{g_\alpha}
\]

and

\[
\int_{B_b \setminus B_a(x_\alpha)} |\nabla^2 g_\alpha u_\alpha| \cdot |u_\alpha - u_\alpha^*| \, dV_{g_\alpha} \leq C \int_{B_{4b} \setminus B_{\frac{a}{2}}(x_\alpha)} |\nabla g_\alpha u_\alpha|^2 \, dV_{g_\alpha},
\]

where \( C \) does not rely on \( \alpha \).

**Proof.** First, we prove the first inequality in the above lemma. Since \( E_{\alpha, \epsilon_\alpha}(u_\alpha , B_\sigma ) < \Theta \), it is easy to see that there exists a constant \( C_0 \) such that

\[
E(u_\alpha, B_\sigma ) < C_0.
\]

We assume that \( 2^K a \in (b, 2b) \) and set

\[
D_i = B_{2^i a} \setminus B_{2^{i-1} a}(x_\alpha).
\]

We scale \( D_i \) to \( B_2 \setminus B_1 \), and \( u_\alpha \) to \( \bar{u}_\alpha \). By Theorem 2.1 (the \( \epsilon \)-regularity theory), we have on \( D_i \)

\[
|\nabla g_\alpha u_\alpha| \leq \frac{1}{2^{i-1} a} |\nabla_0 \bar{u}_\alpha|_{C^0(B_2 \setminus B_1)} \leq \frac{C_1}{2^{i-1} a} \|
abla_0 \bar{u}_\alpha\|_{L^2(B_4 \setminus B_1/2)} = \frac{C_1}{2^{i-1} a} \|
abla g_\alpha u_\alpha\|_{L^2(D_{i+1} \cup D_i \cup D_{i-1})}.
\]

Hence, it follows

\[
\|r \nabla g_\alpha u_\alpha\|_{C^0(D_i)} \leq 2 \left(2^{i-1} a\right) \|
abla g_\alpha u_\alpha\|_{C^0(D_i)} \leq C_2 \|
abla g_\alpha u_\alpha\|_{L^2(D_{i+1} \cup D_i \cup D_{i-1})}.
\]

Similarly, we have

\[
\|r^2 \nabla^2 g_\alpha u_\alpha\|_{C^0(D_i)} \leq C'_2 \|
abla g_\alpha u_\alpha\|_{L^2(D_{i+1} \cup D_i \cup D_{i-1})}.
\]

Then, we have

\[
\int_{D_i} |\nabla^2 g_\alpha u_\alpha|^r |\nabla g_\alpha u_\alpha| \, dV_{g_\alpha} \leq C \int_{D_{i+1} \cup D_i \cup D_{i-1}} |\nabla g_\alpha u_\alpha|^2 \, dV_{g_\alpha} \int_{D_i} \frac{dV_{g_\alpha}}{r^2} \leq C' \int_{D_{i+1} \cup D_i \cup D_{i-1}} |\nabla g_\alpha u_\alpha|^2 \, dV_{g_\alpha}.
\]
Therefore, we get the first inequality in this lemma. The proof of the second inequality goes to almost the same.

To prove the energy identity, it is crucial to estimate the energy on necks. We first estimate the energy on neck domain $B_\delta \setminus B_{R_\lambda}(x_\alpha)$ with respect to the angle variable $\theta$, i.e. $\int_{B_\delta \setminus B_{R_\lambda}(x_\alpha)} |u_{\alpha, \theta}|^2 \, dx$.

**Lemma 3.2.** For $\alpha$-harmonic map sequence $\{u_\alpha\}$ ($\alpha \to 1$) stated in Theorem 1.1, there holds true for $4\delta \leq \sigma$

$$\lim_{\delta \to 0} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_\delta \setminus B_{R_\lambda}(x_\alpha)} |u_{\alpha, \theta}|^2 \, dx = 0.$$  

**Proof.** We adopt the techniques of Sacks–Uhlenbeck [17] and [12] to show the lemma. Using (3.2) we have

$$|u^*_\alpha(r) - u_\alpha(r, \theta)| \leq \epsilon_1. \quad (3.5)$$

We compute

$$I = \int_{B_\delta \setminus B_{R_\lambda}(x_\alpha)} |\nabla_g u_\alpha|^2 \, dV_g.$$ 

$$I = \int_{B_\delta \setminus B_{R_\lambda}(x_\alpha)} \nabla_0 u_\alpha \nabla_0 (u_\alpha - u^*_\alpha) \, dx + \int_{B_\delta \setminus B_{R_\lambda}(x_\alpha)} \nabla_g u_\alpha \nabla_g u^*_\alpha \, dV_g$$

$$= - \int_{B_\delta \setminus B_{R_\lambda}(x_\alpha)} \Delta_0 u_\alpha (u_\alpha - u^*_\alpha) \, dx + \int_{B_\delta \setminus B_{R_\lambda}(x_\alpha)} \nabla_0 u_\alpha \nabla_0 u^*_\alpha \, dx$$

$$+ \int_{\partial (B_\delta \setminus B_{R_\lambda}(x_\alpha))} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u^*_\alpha) \, ds_0$$

$$= \int_{B_\delta \setminus B_{R_\lambda}(x_\alpha)} A(u_\alpha)(\nabla_0 u_\alpha, \nabla_0 u_\alpha) (u_\alpha - u^*_\alpha) \, dx$$

$$+ (\alpha - 1) \int_{B_\delta \setminus B_{R_\lambda}(x_\alpha)} \frac{\nabla_{g_\alpha} |\nabla_{g_\alpha} u_\alpha|^2 \nabla_{g_\alpha} u_\alpha}{\epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2} (u_\alpha - u^*_\alpha) \, dV_{g_\alpha}$$

$$+ \int_{\partial (B_\delta \setminus B_{R_\lambda}(x_\alpha))} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u^*_\alpha) \, ds_0 + \int_{B_\delta \setminus B_{R_\lambda}(x_\alpha)} \frac{\partial u_\alpha}{\partial r} \frac{\partial u^*_\alpha}{\partial r} \, dx. \quad (3.6)$$
On the other hand, noting (3.4) we have
\[
\int_{B \setminus B_{R\lambda\alpha}(x_\alpha)} \frac{\partial u_\alpha}{\partial r} \frac{\partial u^*_\alpha}{\partial r} \, dx \leq \left( \int_{B \setminus B_{R\lambda\alpha}(x_\alpha)} \frac{\partial u_\alpha}{\partial r} \, dx \right)^2 \int_{B \setminus B_{R\lambda\alpha}(x_\alpha)} \frac{\partial u^*_\alpha}{\partial r} \, dx
\]
\[
\leq \int_{B \setminus B_{R\lambda\alpha}(x_\alpha)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 \, dx.
\]
(3.7)
Hence, by using Lemma 3.1, (3.5), (3.7) and noting the following fact
\[
\left| \nabla g_\alpha \nabla g_\alpha u_\alpha \right| \left| \frac{\partial u_\alpha}{\partial r} \right|^2 \left| \frac{\partial u^*_\alpha}{\partial r} \right| \leq 2 \left| \nabla^2 g_\alpha u_\alpha \right|
\]
we can infer from (3.6)
\[
\int_{B \setminus B_{R\lambda\alpha}(x_\alpha)} |\nabla u_\alpha|^2 \, dx \leq \int_{B \setminus B_{R\lambda\alpha}(x_\alpha)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 \, dx + 3C(\alpha - 1) \int_{B_\delta} |\nabla g_\alpha u_\alpha|^2 \, dV_g
\]
\[
+ \int_{\partial B_\delta(x_\alpha)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u^*_\alpha) \, ds_0 - \int_{\partial B_{R\lambda\alpha}(x_\alpha)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u^*_\alpha) \, ds_0
\]
\[
+ \epsilon'_1 \int_{B \setminus B_{R\lambda\alpha}(x_\alpha)} |\nabla u_\alpha|^2 \, dx,
\]
where \( \epsilon'_1 = \epsilon_1 \| A \|_{L^\infty(M)} \).
Since
\[
|\nabla u_\alpha|^2 = \left| \frac{\partial u_\alpha}{\partial r} \right|^2 + |u_{\alpha,\theta}|^2,
\]
we have
\[
\int_{B_\delta \setminus B_{R\lambda\alpha}(x_\alpha)} |u_{\alpha,\theta}|^2 \, dx \leq \int_{\partial B_\delta(x_\alpha)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u^*_\alpha) \, ds_0
\]
\[
- \int_{\partial B_{R\lambda\alpha}(x_\alpha)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u^*_\alpha) \, ds_0 + C'((\alpha - 1) + \epsilon).
\]
Keeping (3.2) in mind, we have
\[
\lim_{\delta \to 0} \lim_{\alpha \to 1} \int_{\partial B_\delta(x_\alpha)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u^*_\alpha) \, ds_0 = 0
\]
and

\[
\lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{\partial B_R(x_0)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u^{*}_\alpha) \, ds_0 = 0.
\]

Hence, we can see the above inequality implies the conclusion of Lemma 3.2. \qed

Immediately we infer from Lemmas 2.2 and 3.2.

**Corollary 3.3.** For \(\alpha\)-harmonic map sequence \(\{u_\alpha\}\) (\(\alpha \to 1\)) stated in Theorem 1.1, there holds true

\[
\lim_{\delta \to 0} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_\delta \setminus B_R(x_0)} \left( \epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2 \right)^{\alpha - 1} \frac{1}{r^2} \, dx = 0.
\]

Next, we will be concerned with the total energy on the neck domain of \(\{u_\alpha\}\). We need to study some differential relations of some functionals which are related closely to \(\alpha\)-energy \(E_{\alpha, \epsilon_\alpha}\). Now we define these functionals as follows

\[
F_{\alpha}(t) = \int_{B_{\rho t}(x_0)} \left( \epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2 \right)^{\alpha - 1} |\nabla_0 u_\alpha|^2 \, dx,
\]

\[
E_{r,\alpha}(t) = \int_{B_{\rho t} \setminus B_{\rho 0}(x_0)} \left( \epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2 \right)^{\alpha - 1} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 \, dx,
\]

and

\[
E_{\theta,\alpha}(t) = \int_{B_{\rho t} \setminus B_{\rho 0}(x_0)} \left( \epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2 \right)^{\alpha - 1} |u_\alpha,\theta|^2 \, dx.
\]

By (2.6), for \(t \in [\epsilon, t_0]\), we have

\[
\left(1 - \frac{1}{2\alpha}\right) E'_{r,\alpha} - \frac{1}{2\alpha} E'_{\theta,\alpha} = \frac{\alpha - 1}{\alpha} \log \lambda_\alpha F_{\alpha}(t) + O(\lambda_\alpha \log \lambda_\alpha).
\]

Then

\[
\left(1 - \frac{1}{2\alpha}\right) E_{r,\alpha}(t) - \frac{1}{2\alpha} E_{\theta,\alpha}(t) = \frac{1}{2} \int_{\epsilon}^{t} \left[ \frac{1}{\alpha} \log \lambda_\alpha^{2(\alpha - 1)} F_{\alpha}(t) + O(\lambda_\alpha \log \lambda_\alpha) \right] \, dt. \quad (3.8)
\]
It is easy to check that the sequences \( \{(1 - \frac{1}{2\alpha})E_{r,\alpha}(t) - \frac{1}{2\alpha}E_{\theta,\alpha}(t)\} \) and \( \{F_{\alpha}(t)\} \) are compact in \( C^0([\epsilon, t_0]) \) topology for any \( \epsilon > 0 \). Therefore, there exist two functions \( F \) and \( E_r \) which belong to \( C^0([\epsilon, t_0]) \) such that, as \( \alpha \to 1 \),

\[
F_{\alpha} \to F, \quad E_{r,\alpha} \to E_r \quad \text{in} \quad C^0([\epsilon, t_0]),
\]

since we have from Corollary 3.3 that \( E_{\theta,\alpha}(t) \to 0 \) in \( C^0([\epsilon, t_0]) \).

**Lemma 3.4.** The functionals \( E_r(t) \) and \( F(t) \), associated with sequence \( \{u_{\alpha}\} \) \( (\alpha \to 1) \) stated in Theorem 1.1, satisfy the following identities

\[
E_r(t) = \mu t_0 - tF(t_0) - F(t_0) \quad (3.9)
\]

and

\[
\lim_{t_0 \to 1} F(t_0) = \Lambda. \quad (3.10)
\]

**Proof.** By letting \( \alpha \to 1 \) in (3.8) we infer

\[
E_r(t) = -\log \mu \int_{t_0}^{t} F \, dt. \quad (3.11)
\]

By the definitions of \( F_{\alpha} \), \( E_{r,\alpha} \) and \( E_{\theta,\alpha} \) we have

\[
F_{\alpha}(t) = F_{\alpha}(t_0) + E_{r,\alpha}(t) + E_{\theta,\alpha}(t).
\]

Since Corollary 3.3 claims \( \lim_{\alpha \to 1} E_{\theta,\alpha}(t) = 0 \), as \( \alpha \to 1 \) the above identity leads to

\[
F(t) = E_r(t) + F(t_0).
\]

Hence, combining the above identity and (3.11) yields

\[
E_r(t) = -\log \mu \int_{t_0}^{t} (E_r(t) + F(t_0)) \, dt.
\]

This implies that \( E_r(t) \in C^1 \) and

\[
E'_r = -(\log \mu)(E_r + F(t_0)).
\]

It follows

\[
E_r(t) = \mu t_0 - tF(t_0) - F(t_0).
\]

Now we turn to prove (3.10). Integrating (2.5) with respect to \( t \) on the interval \([R\lambda_{\alpha}, \lambda_{\alpha} t_0]\) we obtain
\[ F_\alpha(t_0) - \int_{B_{R\lambda_\alpha}(x_\alpha)} \left( \epsilon_\alpha + |\nabla g_\alpha u_\alpha|^2 \right)^{\alpha-1} |\nabla g_\alpha u_\alpha|^2 \, dx \]

\[ \leq C \int_{B_{t_0}(x_\alpha) \setminus B_{R\lambda_\alpha}(x_\alpha)} \left( \epsilon_\alpha + |\nabla g_\alpha u_\alpha|^2 \right)^{\alpha-1} |u_{\alpha,0}|^2 \, dx \]

\[ + C \int_{R_{t_0}^\alpha} \frac{\alpha - 1}{r} \, dr + C \left( \lambda_\alpha^{t_0} - R\lambda_\alpha \right). \]  

(3.12)

From Corollary 3.3 we have

\[ \lim_{t_0 \to 1} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_{t_0}(x_\alpha) \setminus B_{R\lambda_\alpha}(x_\alpha)} \left( \epsilon_\alpha + |\nabla g_\alpha u_\alpha|^2 \right)^{\alpha-1} |u_{\alpha,0}|^2 \, dx = 0. \]

Also note the following holds true

\[ \lim_{t_0 \to 1} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{R_{t_0}^\alpha} \frac{\alpha - 1}{r} \, dr = \lim_{t_0 \to 1} \frac{(1 - t_0)}{2} \log \mu = 0. \]

Thus, taking limit with respect to \( \alpha, R \) and \( t_0 \) in (3.12) leads to

\[ \lim_{t_0 \to 1} F(t_0) = \lim_{t_0 \to 1} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_{R\lambda_\alpha}(x_\alpha)} \left( \epsilon_\alpha + |\nabla g_\alpha u_\alpha|^2 \right)^{\alpha-1} |\nabla g_\alpha u_\alpha|^2 \, dx. \]

In view of (3.1), (3.10) follows from the above identity. Thus we complete the proof of the lemma.

On the other hand, we also have

**Lemma 3.5.** For any \( t \in (0, 1) \), the sequence \( \{u_\alpha\} (\alpha \to 1) \) stated in Theorem 1.1 satisfies

\[ \lim_{\alpha \to 1} \int_{B_{R_t}(x_\alpha)} \left( \epsilon_\alpha + |\nabla g_\alpha u_\alpha|^2 \right)^{\alpha-1} |\nabla g_\alpha u_\alpha|^2 \, dV_{g_\alpha} = \mu^{1-t} \Lambda. \]

**Proof.** It is easy to verify

\[ \int_{B_{R_t}(x_\alpha)} \left( \epsilon_\alpha + |\nabla g_\alpha u_\alpha|^2 \right)^{\alpha-1} |\nabla g_\alpha u_\alpha|^2 \, dV_{g_\alpha} = E_{r,\alpha}(t) + E_{\delta,\alpha}(t) + F_\alpha(t_0). \]
By Corollary 3.3 we have \( \lim_{\alpha \to 1} E_{\theta, \alpha}(t) = 0 \). Hence, in view of (3.9) we can deduce the following

\[
\lim_{\alpha \to 1} \int_{B_{\lambda t}^{\alpha}(x_{\alpha})} \left( \epsilon_{\alpha} + |\nabla_{g_{\alpha}} u_{\alpha}|^2 \right)^{\alpha - 1} |\nabla_{g_{\alpha}} u_{\alpha}|^2 \, dV_{g_{\alpha}} = E_{r}(t) + F(t_0) = \mu t_0^{1-t} F(t_0). \quad \square
\]

**Proof of Theorem 1.1 for the case only one bubble.** In the present situation (2.5) is written by

\[
\left(1 - \frac{1}{2\alpha}\right) \int_{\partial B_{\lambda t}^{\alpha}} \left( \epsilon_{\alpha} + |\nabla_{g_{\alpha}} u_{\alpha}|^2 \right)^{\alpha - 1} |\nabla_{g_{\alpha}} u_{\alpha}|^2 \, ds_0 - \int_{\partial B_{\lambda t}^{\alpha}} \left( \epsilon_{\alpha} + |\nabla_{g_{\alpha}} u_{\alpha}|^2 \right)^{\alpha - 1} |u_{\alpha, \theta}|^2 \, dx
\]

\[
= \frac{(\alpha - 1)}{\alpha t} \int_{B_{\lambda t}^{\alpha}} \left( \epsilon_{\alpha} + |\nabla_{g_{\alpha}} u_{\alpha}|^2 \right)^{\alpha - 1} |\nabla_{g_{\alpha}} u_{\alpha}|^2 \, dx + O(t).
\]

Integrating the above equality with respect to \( t \) on the interval \([\lambda_{\alpha}^t, \delta]\) gives

\[
\int_{B_{\delta} \setminus B_{\lambda t}^{\alpha}(x_{\alpha})} \left( \epsilon_{\alpha} + |\nabla_{g_{\alpha}} u_{\alpha}|^2 \right)^{\alpha - 1} |\nabla_{g_{\alpha}} u_{\alpha}|^2 \, dx \leq C \int_{B_{\delta} \setminus B_{\lambda t}^{\alpha}(x_{\alpha})} \left( \epsilon_{\alpha} + |\nabla_{g_{\alpha}} u_{\alpha}|^2 \right)^{\alpha - 1} |u_{\alpha, \theta}|^2 \, dx + C \int_{\lambda_{\alpha}^t}^{\delta} \frac{\alpha - 1}{r} \, dr + C(\delta - \lambda_{\alpha}^t).
\]

Taking the same argument as we proved (3.10), we find that

\[
\lim_{\delta \to 0} \lim_{\alpha \to 1} \lim_{t \to 0} \int_{B_{\delta} \setminus B_{\lambda t}^{\alpha}(x_{\alpha})} \left( \epsilon_{\alpha} + |\nabla_{g_{\alpha}} u_{\alpha}|^2 \right)^{\alpha - 1} |\nabla_{g_{\alpha}} u_{\alpha}|^2 \, dV_{g_{\alpha}} = 0. \tag{3.13}
\]

By Lemma 3.5 we have

\[
\lim_{t \to 0} \lim_{\alpha \to 1} \int_{B_{\lambda t}^{\alpha}(x_{\alpha})} \left( \epsilon_{\alpha} + |\nabla_{g_{\alpha}} u_{\alpha}|^2 \right)^{\alpha - 1} |\nabla_{g_{\alpha}} u_{\alpha}|^2 \, dV_{g_{\alpha}} = \mu \Lambda.
\]

The above equality and (3.13) lead to

\[
\lim_{\delta \to 0} \lim_{\alpha \to 1} \int_{B_{\delta}(x_{\alpha})} \left( \epsilon_{\alpha} + |\nabla_{g_{\alpha}} u_{\alpha}|^2 \right)^{\alpha - 1} \left|\nabla_{g_{\alpha}} u_{\alpha}\right|^2 \, dV_{g_{\alpha}} = \mu \Lambda = \mu^2 E(v).
\]

Here we have used the fact \( \Lambda = \mu E(v) \). Noting Lemma 2.2, we conclude

\[
\lim_{\delta \to 0} \lim_{\alpha \to 1} \int_{B_{\delta}(x_{\alpha})} \left( \epsilon_{\alpha} + |\nabla_{g_{\alpha}} u_{\alpha}|^2 \right)^{\alpha} \, dV_{g_{\alpha}} = \mu^2 E(v).
\]

So we have completed the proof of Theorem 1.1 in the case that \( n_0 = 1 \). \quad \square
3.2. The weak energy identity for the cases of several bubbles

For the case $n_0 > 1$, the proof of Theorem 1.1 can be completed by induction in $n_0$, the number of bubbles. One just has to distinguish more bubbles domains and neck domains.

**Proof of Theorem 1.1 in the case $n_0 > 1$.** Assume that we have shown the weak energy identity for the case of $n_0 - 1$ bubbles. We need to reduce the case of $n_0$ bubbles to the cases of bubbles of $m$ bubbles where $m < n_0$. By the assumptions of Theorem 1.1, \( \{u_\alpha\} \subset C^\infty(B_\sigma, N) \ (\alpha \to 1) \) is a sequence of $\alpha$-harmonic maps, each $u_\alpha$ of which satisfies

\[
\Delta_0 u_\alpha + (\alpha - 1) \frac{\nabla g_\alpha u_\alpha^2 \nabla_0 u_\alpha}{\epsilon_\alpha + |\nabla g_\alpha u_\alpha|^2} + A(u_\alpha)(\nabla_0 u_\alpha, \nabla_0 u_\alpha) = 0,
\]

where $g_\alpha = e^{\varphi_\alpha}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$ with $\varphi_\alpha(0) = 0$ and $\varphi_\alpha$ converges smoothly to $\varphi$.

First we show how to pick the first bubble of $\{u_\alpha\}$. Let $x^{1}_\alpha \in B_\delta$ such that

\[
|\nabla g_\alpha u_\alpha(x^{1}_\alpha)| = \max_{B_\delta} |\nabla g_\alpha u_\alpha|,
\]

and $\lambda^{1}_\alpha = \frac{1}{\max_{B_\delta} |\nabla g_\alpha u_\alpha|}$. Then, there exists a harmonic map $v^1$ such that in $C^k_{\text{loc}}(\mathbb{R}^2)$

\[
u_\alpha(x^{1}_\alpha + \lambda^{1}_\alpha x) \to v^1.
\]

Let $v^{1}_\alpha = u_\alpha(x^{1}_\alpha + \lambda^{1}_\alpha x)$. Then, $v^{1}_\alpha$ satisfies

\[
\Delta_0 v^{1}_\alpha + (\alpha - 1) \frac{\nabla g_\alpha v^{1}_\alpha^2 \nabla_0 v^{1}_\alpha}{\epsilon_\alpha (\lambda^{1}_\alpha)^2 + |\nabla g_\alpha v^{1}_\alpha|^2} + A(v^{1}_\alpha)(\nabla_0 v^{1}_\alpha, \nabla_0 v^{1}_\alpha) = 0,
\]

where $\Delta_0$ and $\nabla_0$ are the Laplace operator and gradient operator on Euclidean plane $\mathbb{R}^2$, and

\[
g^{1}_\alpha = e^{\varphi_\alpha(x^{1}_\alpha + \lambda^{1}_\alpha x)}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2).
\]

Since $\|\nabla g_\alpha v^{1}_\alpha\|_{L^\infty(B_\delta)} = 1$, we have that $v^{1}_\alpha$ converges to a harmonic map $v^1$ from $\mathbb{R}^2$ to $N$ locally and smoothly. As $|\nabla g_\alpha v^1(0)| = 1$, we know that $v^1$ is not trivial. $v^1$ is the first bubble.

Since at point $\{0\}$ there exist another $n_0 - 1$ bubbles $v^2, \ldots, v^{n_0}$, and sequences $x^i_\alpha, \lambda^i_\alpha$ such that for each $i$

\[
u_\alpha(x^i_\alpha + \lambda^i_\alpha x) \to v^i
\]

in $C^k_{\text{loc}}(\mathbb{R}^2 \setminus A^i)$, where $A^i$ are some finite sets. Moreover, for any $i \neq j$, one of (H1) and (H2) holds. By the definition of $\lambda^1_\alpha$ we have (also see Appendix A of this paper)

\[
\lambda^1_\alpha = \min_{i \in \{1, \ldots, n_0\}} \{\lambda^i_\alpha\}.
\]
Without loss of generality we may assume
\[ \lambda_{\alpha}^{n_0} = \max_{i \in \{1, \ldots, n_0\}} \{ \lambda_{\alpha}^i \}. \]

Let
\[ r_{\alpha}^{n_0} = \lambda_{\alpha}^{n_0} + \sum_{j=1}^{n_0} |x_{\alpha}^{n_0} - x_{\alpha}^j| \frac{1}{(n_0 - 1)}. \]

Similar to the proof of (3.2), we have that, for any \( \epsilon > 0 \), there are \( \delta_0 \) and \( R \) such that
\[ \int_{B_{2R} \setminus B_\delta(x_\alpha)} |\nabla_{g_\alpha} u_\alpha|^2 dV_{g_\alpha} \leq \epsilon, \quad \forall \lambda \in (R r_{\alpha}^{n_0}, \delta_0). \] (3.14)

Then, using the same arguments as in the above subsection, we infer
\[ \lim_{\alpha \to 1} \int_{B_\delta} (\epsilon_\alpha + |\nabla_{g_\alpha} u_\alpha|^2)^{\alpha} dV_{g_\alpha} = \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_R} (r_{\alpha}^{n_0})^{2-2\alpha} (\epsilon_\alpha (r_{\alpha}^{n_0})^2 + |\nabla_{g_\alpha} u_\alpha^{n_0}|^2)^{\alpha} dV_{g_\alpha}^{n_0} \]
\[ + |B_\delta| \lim_{\alpha \to 1} \epsilon_\alpha + \int_{B_\delta} |\nabla_{g_\alpha} u_0|^2 dV_{g_\alpha}, \] (3.15)

where \( u_\alpha^{n_0}(x) = u_\alpha(r_{\alpha}^{n_0} x + x_\alpha^1) \) and \( g_\alpha^{n_0} = e^{\phi_{\alpha}(r_{\alpha}^{n_0} x)}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2). \) Moreover, (3.14) implies that all the blow-up points of \( \{u_\alpha^{n_0}\} \) lie on \( B_R \) for some \( R > 0. \)

Denote the limiting function of \( u_\alpha^{n_0} \) by \( v_0. \) We have the following cases: (i) \( v_0 \) is a non-trivial harmonic map, (ii) \( v_0 \) is trivial.

For case (i), \( v_0 \) is a bubble, we denote \( v_0 = v_0^{n_0} \) and \( E(v_0) = E(v_0^{n_0}). \) Then, we always have
\[ \limsup_{\alpha \to 1} \sum_{j=1}^{n_0} |x_{\alpha}^{n_0} - x_{\alpha}^j| \frac{1}{(n_0 - 1)} < \infty. \]

Otherwise, there are \( n_0 + 1 \) bubbles. This implies \( \lim_{\alpha \to 1} (r_{\alpha}^{n_0})^{2-2\alpha} = \mu_{n_0}. \) On the other hand, since one of (H1) and (H2) always holds, we have \( \lambda_{\alpha}^{i}/r_{\alpha}^{n_0} \to 0 \) for any \( i \neq n_0. \) Obviously, for
\[ \lambda_{\alpha}^{n_0} \frac{x_{\alpha}^{n_0} - x_{\alpha}^i}{r_{\alpha}^{n_0}} = \frac{1}{r_{\alpha}^{n_0}} \left( \lambda_{\alpha}^{n_0} \frac{x_{\alpha}^{n_0} - x_{\alpha}^i}{r_{\alpha}^{n_0}} \right), \]
we have
\[ u_\alpha^{n_0} \left( \frac{x_{\alpha}^{n_0} + \frac{\lambda_{\alpha}^{n_0} x_{\alpha}^i}{r_{\alpha}^{n_0}}}{x_{\alpha} - x_{\alpha}^i} \right) = u_\alpha \left( \frac{x_{\alpha}^{n_0} + \lambda_{\alpha}^{n_0} x_{\alpha}^i}{x_{\alpha}^i} \right) \to v^i(x). \]

Therefore, \( v^1, \ldots, v^{n_0-1} \) are all bubbles for \( u_\alpha^{n_0}. \) Now we consider the following functionals
\[ E_{\alpha, \epsilon_\alpha(r_{\alpha}^{n_0})^2}(v) \equiv \int_{B_R} ((r_{\alpha}^{n_0})^{2-2\alpha} + |\nabla_{g_\alpha} v|^2)^{\alpha} dV_{g_\alpha}. \]
Since there is only \( n_0 - 1 \) bubbles for the sequence \( \{u_n^{\alpha}\} \), by induction we have

\[
\lim_{\alpha \to 1} \int_{B_R} \left( \left( r_0^{\alpha}\right)^2 \epsilon + \left| \nabla g_{\alpha} u_0^{\alpha}\right|^2 \right)^{\alpha - 1} \left| \nabla g_{\alpha} n_0^{\alpha} u_n^{\alpha}\right|^2 dV_{g_{\alpha}} = E(v_0, B_R) + \sum_{i=1}^{n_0-1} \lim_{\alpha \to 1} \left( \frac{l_0^{\alpha}}{r_0^{\alpha}}\right)^{2-2\alpha} E(v_i).
\]

Noting

\[
\int_{B_R} \left( \left( r_0^{\alpha}\right)^2 - 2\alpha \right) \left( \left( r_0^{\alpha}\right)^2 \epsilon + \left| \nabla g_{\alpha} u_0^{\alpha}\right|^2 \right)^{\alpha - 1} \left| \nabla g_{\alpha} n_0^{\alpha} u_n^{\alpha}\right|^2 dV_{g_{\alpha}} = \int_{B_{R_0^{\alpha}}} \left( \epsilon_{\alpha} + \left| \nabla g_{\alpha} u_0^{\alpha}\right|^2 \right)^{\alpha - 1} \left| \nabla g_{\alpha} u_0^{\alpha}\right|^2 dV_{g_{\alpha}},
\]

it follows that

\[
\lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_{R_0^{\alpha}}} \left( \epsilon_{\alpha} + \left| \nabla g_{\alpha} u_0^{\alpha}\right|^2 \right)^{\alpha - 1} \left| \nabla g_{\alpha} u_0^{\alpha}\right|^2 dV_{g_{\alpha}} = \mu_2^2 E(v^m) + \sum_{i=1}^{m-1} \mu_2^i E(v^i).
\]

Combining (3.15) and (3.16) leads to

\[
\lim_{\alpha \to 1} \int_{B_\delta} \left( \epsilon_{\alpha} + \left| \nabla g_{\alpha} u_0^{\alpha}\right|^2 \right)^{\alpha} dV_{g_{\alpha}} = \sum_{i=1}^{n_0} \mu_2^i E(v^i) + |B_\delta| \lim_{\alpha \to 1} \epsilon_{\alpha} + \int_{B_\delta} \left| \nabla g_{\alpha} u_0^{\alpha}\right|^2 dV_{g_{\alpha}}.
\]

For case (ii), there are at least two concentration points of \( \{u_n^{\alpha}\} \). So, at any concentration point for \( u_n^{\alpha} \), there are at most \( n_0 - 1 \) bubbles, and then we can make induction to obtain the required identity. Thus, we complete the proof of Theorem 1.1. □

4. Description and further analysis of the necks

In this section, we always assume there is only one bubble on some small ball \( B_\delta \).

4.1. The description of no neck property

In this subsection, we consider the case \( \nu = 1 \). We will show that there is no neck for this case, i.e. the length of neck is zero. Following the arguments of Ding [5], we need to establish some asymptotic estimations on energy on neck domains \( B_\delta \setminus B_{R_{\lambda_{\alpha}}} \). We must make a detailed analysis on an integral

\[
f_{\alpha,k}(t) \equiv \int_{B_{2^{-k+R_{\lambda_{\alpha}}} \setminus B_{2^{-k+R_{\lambda_{\alpha}}} \setminus R_{\lambda_{\alpha}}}(x_{\alpha})}} \left| \nabla g_{\alpha} u_0^{\alpha}\right|^2 dV_{g_{\alpha}},
\]
and then derive an ordinary differential inequality on the integral by which the no neck property and the asymptotic behaviors of neck domains can be shown.

**Proof of Theorem 1.3 for the case \( \nu = 1 \).** Let \( x_\alpha \in B_\sigma \) such that

\[
|\nabla_{g_\sigma} u_\alpha(x_\alpha)| = \max_{B_\sigma} |\nabla_{g_\sigma} u_\alpha|.
\]

Obviously, by assumptions in Theorem 1.3 we have \( \lim_{\alpha \to 1} x_\alpha = 0 \). Since \( \nu = 1 \), then we also have \( \mu = 1 \) and

\[
\lim_{\delta \to 0} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} |\nabla_{g_\sigma} u_\alpha|^2 dV_{g_\sigma} = 0. \tag{4.1}
\]

For simplicity, we assume

\[
P = \frac{\log \delta - \log R\lambda_\alpha}{\log 2}
\]

is an integer. For any integer \( k \in [1, P - 1] \), we set

\[
Q_k(t) = B_{2^{k+t}R\lambda_\alpha} \setminus B_{2^{k-t}R\lambda_\alpha}(x_\alpha),
\]

where \( t + k \leq P \) and \( k - t \geq 0 \).

Using the same approximate method as in the arguments on Lemma 3.2, from (2.1) we can infer that on \( Q_k(t) \) the following inequality holds

\[
\int_{Q_k(t)} |\nabla_0 u_\alpha|^2 \, dx \leq \int_{Q_k(t)} A(u_\alpha)(\nabla_0 u_\alpha, \nabla_0 u_\alpha)(u_\alpha - u_\alpha^*) \, dx + \int_{\partial Q_k(t)} \frac{\partial u_\alpha}{\partial r}(u_\alpha - u_\alpha^*) \, ds_0 + \int_{Q_k(t)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 \, dx + C(\alpha - 1) \int_{Q_k(t+2)} |\nabla_0 u_\alpha|^2 \, dx. \tag{4.2}
\]

Next, we will apply Pohozaev identity (2.7) to control the term \( \int_{Q_k(t)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 \, dx \) in the above inequality. For the sake of convenience, we set

\[
H(r) = - \int_{B_r(x_\alpha)} \frac{\nabla_{g_\sigma} |\nabla_{g_\sigma} u_\alpha|^2 |\nabla_{g_\sigma} u_\alpha|}{\epsilon_\alpha + |\nabla_{g_\sigma} u_\alpha|^2} r \frac{\partial u_\alpha}{\partial r} \, dV_{g_\sigma} = - \int_{B_r(x_\alpha)} \frac{\nabla_0 |\nabla_{g_\sigma} u_\alpha|^2 \nabla_0 u_\alpha}{\epsilon_\alpha + |\nabla_{g_\sigma} u_\alpha|^2} \frac{\partial u_\alpha}{\partial r} \, dx.
\]

Using Lemma 3.1, we have
\[
|H(r)| \leq C \int \left| \nabla ^2 u_\alpha \right| r \left| \frac{\partial u_\alpha}{\partial r} \right| \, dx + |H(R\lambda_\alpha)| \\
\leq C \int \left| \nabla _0 u_\alpha \right|^2 \, dx + |H(R\lambda_\alpha)| < C',
\]

where we use the fact
\[
\lim_{\alpha \to 1} |H(R\lambda_\alpha)| \leq \lim_{\alpha \to 1} \int_{B_{R\lambda_\alpha}(x_\alpha)} \left| \nabla ^2 u_\alpha \right| r \left| \nabla _0 u_\alpha \right| \, dV_g = \int \left| \nabla ^2 v_\alpha \right| r \left| \nabla _0 v_\alpha \right| \, dx < C(R),
\]
where \( v_\alpha \) is defined as before. Therefore, integrating (2.7) and noting (4.3) we obtain
\[
\int_{Q_k(t)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 \, dx - \frac{1}{2} \int_{Q_k(t)} \left| \nabla _0 u_\alpha \right|^2 \, dx \leq C \int \frac{2^{k+1} R\lambda_\alpha}{r} \, dr \leq C(\alpha - 1)t.
\]
It follows (4.2) and the above inequality
\[
\left( \frac{1}{2} - \epsilon_1 \right) \int_{Q_k(t)} \left| \nabla _0 u_\alpha \right|^2 \, dx \leq C(\alpha - 1)(t + 1) + \int_{\partial Q_k(t)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u^*_\alpha) \, ds_0.
\]

On the other hand, we have
\[
\left| \int_{\partial Q_k(t)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u^*_\alpha) \, ds_0 \right| \leq \int_{\partial Q_k(t)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 r \, ds_0 \int_{\partial Q_k(t)} \left| u_\alpha - u^*_\alpha \right|^2 \, ds_0
\]
\[
\leq \int_{\partial Q_k(t)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 r \, ds_0 \int_{\partial Q_k(t)} \left| u_{\alpha,\theta} \right|^2 r^2 \, ds_0
\]
\[
\leq \frac{1}{2} \left[ \int_{\partial Q_k(t)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 r \, ds_0 + \int_{\partial Q_k(t)} \left| u_{\alpha,\theta} \right|^2 r \, ds_0 \right]
\]
\[
= \frac{1}{2} \int_{\partial Q_k(t)} r \left| \nabla _0 u_\alpha \right|^2 \, ds_0
\]
\[
= 2^{t+k-1} R\lambda_\alpha \int_{\partial B_{2^{t+k} R\lambda_\alpha}(x_\alpha)} \left| \nabla _0 u_\alpha \right|^2 \, ds_0
\]
\[
- 2^{k-t-1} R\lambda_\alpha \int_{\partial B_{2^{k-t} R\lambda_\alpha}(x_\alpha)} \left| \nabla _0 u_\alpha \right|^2 \, ds_0.
\]
Let
\[
f_{\alpha,k}(t) = \int_{Q_k(t)} |\nabla g_{\alpha} u_{\alpha}|^2 dV_{g_{\alpha}} = \int_{Q_k(t)} |\nabla_0 u_{\alpha}|^2 dx.
\]

From (4.5) we know
\[
\int_{\partial Q_k(t)} \frac{\partial u_{\alpha}}{\partial r} (u_{\alpha} - u^*_\alpha) ds_0 \leq \frac{1}{2 \log 2} f'_k(t).
\]
Hence, by combining (4.4) and the above inequality we have
\[
(1 - 2\epsilon_1) f_{\alpha,k}(t) \leq \frac{1}{\log 2} f'_k(t) + C(\alpha - 1)(t + 1).
\]

Multiplying the two sides of the above inequality by \(2^{-(1-2\epsilon_1)t}\) and integrating we obtain
\[
f_{\alpha,k}(1) \leq C2^{-(1-2\epsilon_1)t_1} f_{\alpha,k}(t_1) + C(\alpha - 1).
\]

It is easy to check that, if we set
\[
t_1 = L_k = \begin{cases} k & \text{if } 2k - 1 \leq P, \\ P - k & \text{if } 2k - 1 > P \end{cases}
\]
then, we get
\[
\sqrt{E(u_{\alpha}, Q_k(1))} \leq C2^{-aL_k} \sqrt{E(u_{\alpha}, B_\delta \setminus B_{R\lambda_{\alpha}}(x_{\alpha}))} + C\sqrt{\alpha - 1}
\]
for some positive \(a\) and \(C\).

Since \(u_{\alpha}\) satisfies (2.1), by the standard \(L^p\) estimate, we have
\[
\text{osc}_{B_{2k+1}R_{\alpha} \setminus B_{2k-1}R_{\alpha}} u_{\alpha} \leq C2^{-aL_k} \sqrt{E(u_{\alpha}, B_\delta \setminus B_{R\lambda_{\alpha}}(x_{\alpha}))} + C\sqrt{\alpha - 1}.
\]

These inequalities imply
\[
\text{osc}_{B_\delta \setminus B_{R_{\lambda_{\alpha}}}} u_{\alpha} \leq C \sqrt{E(u_{\alpha}, B_\delta \setminus B_{R_{\lambda_{\alpha}}}(x_{\alpha}))} \sum 2^{-aL_k} + C\sqrt{\alpha - 1}P
\]
\[
\leq C \sqrt{E(u_{\alpha}, B_\delta \setminus B_{R_{\lambda_{\alpha}}}(x_{\alpha}))} + C(R, \delta)\sqrt{\alpha - 1} + C \log \lambda_{\alpha}^{-\sqrt{\alpha - 1}}.
\]

Letting \(\alpha \to 1\), and then \(R \to +\infty, \delta \to 0\), we get
\[
\text{osc}_{B_\delta \setminus B_{R_{\lambda_{\alpha}}}} u_{\alpha} \to 0.
\]

This shows that the set \(u_0(B_\sigma) \cup v^1(S^2)\) is a connected subset of \(N\). Thus we complete the proof of Theorem 1.3 in the case \(v = 1\). \(\Box\)
4.2. Length formula of the neck when $\nu > 1$

The goal of this section is to show the neck domain converges to a geodesic in $N$ and furthermore calculate the length of the geodesic. As before, let $x_\alpha \in B_\sigma$ such that $|\nabla g_\alpha u_\alpha(x_\alpha)| = \max_{B_\sigma} |\nabla g_\alpha u_\alpha|$. By assumptions in Theorem 1.3 we have $\lim_{\alpha \to 1} x_\alpha = 0$.

4.2.1. Asymptotic behaviors of the sequence on the neck domain

In order to analyze the convergence behavior of $\{u_\alpha\}$ stated in Theorem 1.3 around a blow-up point, we need to consider the asymptotic behaviors of $\{u_\alpha\}$ on $\partial B_{\lambda t}(x_\alpha)$ with $t \in [t_2, t_1]$, where $0 < t_2 < t_1 < 1$. By the arguments in Lemma 3.5, we have

$$\lim_{\alpha \to 1} \int_{B_{\lambda t}(x_\alpha)} \left( \epsilon_\alpha + |\nabla g_\alpha u_\alpha|^2 \right)^{\alpha-1} |\nabla g_\alpha u_\alpha|^2 dV_{g_\alpha} \to \frac{\mu}{2} t E(v^1)$$

in $C^0([t_2, t_1])$. This implies

$$\lim_{\alpha \to 1} \int_{B_{\lambda t}(x_\alpha)} \left( \epsilon_\alpha + |\nabla g_\alpha u_\alpha| \right)^{\alpha-1} |\nabla g_\alpha u_\alpha|^2 dV_{g_\alpha} \to 0.$$

Then, we apply Lemma 2.2 to the above equality to derive

$$\lim_{\alpha \to 1} \int_{B_{\lambda t}(x_\alpha)} |\nabla g_\alpha u_\alpha|^2 dV_{g_\alpha} \to 0$$

in $C^0([t_2, t_1])$. Therefore, for any $t \in [t_2, t_1]$, we have

$$\text{osc}_{\partial B_{\lambda t}(x_\alpha)} u_\alpha \leq C \int_{B_{\lambda t}(x_\alpha)} |\nabla g_\alpha u_\alpha|^2 dV_{g_\alpha} \to 0,$$

i.e. $u_\alpha|_{\partial B_{\lambda t}(x_\alpha)}$ will converge to a point belonging to $N$ after passing to a subsequence. Especially, we have that, as $\alpha \to 1$,

$$u_\alpha(\partial B_{\lambda t}(x_\alpha)) \to y_1 \in N \quad \text{and} \quad u_\alpha(\partial B_{\lambda t}(x_\alpha)) \to y_2 \in N.$$

For simplicity, we will use “$(r, \theta)$” to denote “$x_\alpha + r(\cos \theta, \sin \theta)$.” First, we need to establish some estimates on the derivatives of $u_\alpha$ with respect to angle variable $\theta$.

Lemma 4.1. Let $\{u_\alpha\}$ be the sequence stated in Theorem 1.3. Assume that $\nu > 1$. Then, for $t_\alpha \in [t_2, t_1]$ where $0 < t_2 < t_1 < 1$, we have

$$\lim_{\alpha \to 1} \frac{1}{\alpha - 1} \int_{B_{\lambda t}(x_\alpha)} |u_{\alpha, \theta}|^2 dx < C,$$

where $C$ does not depend on $R$. 
Proof. We set
\[ Q(t) = B_{2t^{\frac{t_0}{\lambda_0^\alpha}}}(x_\alpha) \setminus B_{2^{-t_1}t^{\frac{t_0}{\lambda_0^\alpha}}}(x_\alpha). \]

Here we assume \( 2t \leq \lambda - \epsilon \), where
\[ \epsilon < \min\{t_2, 1 - t_1\}. \]

Applying (2.7), we get from (4.2) the following
\[
\left( 1 - \frac{1}{2\alpha} \right) \int_{Q(t)} |\nabla_0 u_\alpha|^2 \, dx \leq (\alpha - 1) \left( \int_{B_{2t^{\frac{t_0}{\lambda_0^\alpha}}}(x_\alpha)} \frac{1}{r} H(r) \, dr + C \int_{B_{2^{-t_1}t^{\frac{t_0}{\lambda_0^\alpha}}}(x_\alpha)} |\nabla_0 u_\alpha|^2 \, dx \right)
- \int_{\partial Q(t)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u_\alpha^*) \, ds. \tag{4.8}
\]

For any \( r \in [\lambda^{\frac{t_0}{\lambda_0^\alpha}} - \epsilon, \lambda^{\frac{t_0}{\lambda_0^\alpha}}] \), it is easy to check that
\[
H(r) - H(\lambda^{\frac{t_0}{\lambda_0^\alpha}}) \leq \int_{B_{\lambda^{\frac{t_0}{\lambda_0^\alpha}}}(x_\alpha)} |\nabla_\alpha^2 u_\alpha| \left| r \frac{\partial u_\alpha}{\partial r} \right| \, dx.
\]

Using Lemma 3.1, we can get
\[
\int_{B_{\lambda^{\frac{t_0}{\lambda_0^\alpha}}}(x_\alpha)} |\nabla_\alpha^2 u_\alpha| \left| r \frac{\partial u_\alpha}{\partial r} \right| \, dx \leq C \int_{B_{2\lambda^{\frac{t_0}{\lambda_0^\alpha}}}(x_\alpha)} |\nabla_0 u_\alpha|^2 \, dx.
\]

By integrating (2.5) we obtain
\[
\left( 1 - \frac{1}{2\alpha} \right) \int_{\frac{1}{2} \lambda^{\frac{t_0}{\lambda_0^\alpha}} + \epsilon}^{2 \lambda^{\frac{t_0}{\lambda_0^\alpha}} - \epsilon} ds \int_{\partial B_{s}(x_\alpha)} (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha - 1} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 \, ds_0
- \frac{1}{2\alpha} \int_{\frac{1}{2} \lambda^{\frac{t_0}{\lambda_0^\alpha}} + \epsilon}^{2 \lambda^{\frac{t_0}{\lambda_0^\alpha}} - \epsilon} ds \int_{\partial B_{s}(x_\alpha)} (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha - 1} \frac{1}{s^2} \left| \frac{\partial u_\alpha}{\partial \theta} \right|^2 \, ds_0
= \int_{\frac{1}{2} \lambda^{\frac{t_0}{\lambda_0^\alpha}} + \epsilon}^{2 \lambda^{\frac{t_0}{\lambda_0^\alpha}} - \epsilon} \left( (\alpha - 1) \int_{B_{s}(x_\alpha)} (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha - 1} |\nabla_0 u_\alpha|^2 \, dx \right) ds + O(\lambda^{2(t_0 - \epsilon)}). \tag{4.9}
\]

By Corollary 3.3 we know that the second term on the left-hand side of the above inequality vanishes as \( \alpha \to 0 \). On the other hand, noting the fact \( (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha - 1} \) is bounded, we have
\[
\int_{\frac{1}{2} \lambda t^\alpha + \epsilon}^{2 \lambda t^\alpha - \epsilon} \left( \frac{\alpha - 1}{\alpha s} \right) \int_{B_s(x_{\alpha})} \left( \epsilon_{\alpha} + |\nabla_{g_\alpha} u_{\alpha}|^2 \right)^{\alpha - 1} |\nabla_0 u_{\alpha}|^2 \, dx \, ds \\
\leq C \int_{\frac{1}{2} \lambda t^\alpha + \epsilon}^{2 \lambda t^\alpha - \epsilon} \frac{\alpha - 1}{\alpha s} \, ds = \frac{Ce}{\alpha} \log \frac{2}{\lambda^{2 - 2\alpha}}.
\]

Noting the assumption \(\lim_{\alpha \to 1} \epsilon_{\alpha}^{\alpha - 1} = \beta_0 > 0\), we can infer from (4.9) that, as \(\alpha\) is close to 1 enough,

\[
\int_{B_2(\lambda t^\alpha)^- \setminus B_1(\lambda t^\alpha + \epsilon)(x_{\alpha})} |\nabla_0 u_{\alpha}|^2 \, dx \leq C \epsilon \left( \log \frac{2}{\lambda^{2 - 2\alpha}} + 1 \right) + O \left( \lambda^{2(t - \epsilon)} \right).
\]

So, when \(\alpha\) is close to 1 enough we can always choose \(\epsilon\) such that

\[
C \int_{B_2(\lambda t^\alpha)^- \setminus B_1(\lambda t^\alpha + \epsilon)(x_{\alpha})} |\nabla_0 u_{\alpha}|^2 \, dx \leq C \epsilon (\log \mu + 1) < \epsilon_1.
\]

Hence, we have

\[
H(\lambda t_{\alpha}) - \epsilon_1 \leq H(r) \leq H(\lambda t_{\alpha}) + \epsilon_1. \tag{4.10}
\]

Let

\[
f_{\alpha}(t) = \int_{Q(t)} |\nabla_{g_\alpha} u_{\alpha}|^2 \, dV_{g_\alpha} = \int_{\overline{Q(t)}} |\nabla_0 u_{\alpha}|^2 \, dx.
\]

By using a similar estimate with (4.5) and (4.10), we infer that as \(\alpha\) is close to 1 enough there holds

\[
(1 - 2\epsilon_1) f_{\alpha}(t) \leq \frac{1}{\log 2} f''_{\alpha}(t) + (\alpha - 1)(at + \epsilon_1),
\]

where

\[
a = 4 \log 2 H(\lambda t_{\alpha}) + \epsilon_1.
\]

Then, it is easy to see

\[
(2^{-(1-2\epsilon_1)} f_{\alpha})' \geq -(\alpha - 1)(at + \epsilon_1)2^{-(1-2\epsilon_1)} t \log 2.
\]

Hence, we get
\[ f_\alpha(t) \leq 2^{-(1-2\epsilon_1)(\tau-t)} f_\alpha(\tau) \]
\[ + \frac{\alpha - 1}{1 - 2\epsilon_1} \left( \epsilon_1 + at + \frac{a}{\log 2} - a\tau 2^{-(1-2\epsilon_1)(\tau-t)} - \frac{2^{-(1-2\epsilon_1)(\tau-t)}}{(1 - 2\epsilon_1) \log 2} \right). \]

Then, it follows
\[ f_\alpha(k) \leq C_1(k) 2^{-(1-2\epsilon_1)\tau} f_\alpha(\tau) + \frac{\alpha - 1}{1 - 2\epsilon_1} \left( \epsilon_1 + ak + \frac{a}{\log 2} + aC_2(k)a\tau 2^{-(1-2\epsilon_1)\tau} \right). \]

Let \( 2^\tau = \lambda_\alpha^{-\epsilon} \). Then
\[
\int_{B_{2k \lambda_\alpha} \setminus B_{\frac{1}{2} \lambda_\alpha} (x_\alpha)} |\nabla_0 u_\alpha|^2 \, dx \leq C(k)\lambda_\alpha^{\epsilon(1-2\epsilon_1)} + \frac{\alpha - 1}{1 - 2\epsilon_1} \left( H(\lambda_\alpha^{\epsilon})4k \log 2 + \frac{a}{\log 2} \right)

+ C(k)\lambda_\alpha^{\epsilon(1-2\epsilon_1) \log \lambda_\alpha}. \tag{4.11}
\]

On the other hand, by (2.8) and (4.10), we get
\[
\int_{B_{2k \lambda_\alpha} \setminus B_{\frac{1}{2} \lambda_\alpha} (x_\alpha)} \left( |\frac{\partial u_\alpha}{\partial r}|^2 - |u_\alpha,\theta|^2 \right) \, dx \geq (\alpha - 1)4k \log 2 \left( H(\lambda_\alpha^{\epsilon}) - \epsilon_1 \right). \tag{4.12}
\]

Therefore, subtracting (4.11) by (4.12) we obtain
\[
2 \int_{B_{2k \lambda_\alpha} \setminus B_{\frac{1}{2} \lambda_\alpha} (x_\alpha)} |u_\alpha,\theta|^2 \, dx \leq C(k)\lambda_\alpha^{\epsilon(1-2\epsilon_1)} + \frac{\alpha - 1}{1 - 2\epsilon_1} \left( 2\epsilon_1 H(\lambda_\alpha^{\epsilon})4k \log 2 + \frac{a}{\log 2} \right)

+ C(k)\lambda_\alpha^{\epsilon(1-2\epsilon_1) \log \lambda_\alpha} + \epsilon_1 (\alpha - 1)4k \log 2. \tag{4.13}
\]

Since
\[
\nu = \lim_{\alpha \to 1} \lambda_\alpha^{-\sqrt{\alpha-1}} > 1,
\]
it is easy to see that, for any \( m \geq 0 \),
\[
\lambda_\alpha^{\epsilon(1-2\epsilon_1)} = o((\alpha - 1)^m). \tag{4.14}
\]

Then, noting (4.14) and letting \( \epsilon_1 \to 0 \) in the above inequality (4.13), we get
\[
\lim_{\alpha \to 1} \frac{1}{\alpha - 1} \int_{B_{2k \lambda_\alpha} \setminus B_{\frac{1}{2} \lambda_\alpha} (x_\alpha)} |u_\alpha,\theta|^2 \, dx \leq \frac{a'}{2 \log 2},
\]

where \( a' \) is a constant which does not depend on \( R \). Thus we complete the proof. \( \square \)
By virtue of the above lemma, furthermore we can establish the following

**Proposition 4.2.** Let \( \{u_\alpha\} \) be the sequence stated in Theorem 1.3. When \( \nu > 1 \) and \( 0 < t_2 \leq t_\alpha \leq t_1 < 1 \), we have that for any \( R > 0 \)

\[
\lim_{\alpha \to 1} \frac{1}{\alpha - 1} \int_{B_{r_\alpha} \setminus \pi_{r_\alpha} (x_\alpha)} |u_{\alpha, \theta}|^2 \, dx = 0. \tag{4.15}
\]

**Proof.** Since Lemma 4.1 says

\[
\int_{B_{2k_\alpha} \setminus B_{1/2k_\alpha} (x_\alpha)} \frac{|u_{\alpha, \theta}|^2}{\alpha - 1} \, dx = \int \frac{1}{r} \left( \int_0^{2\pi} \frac{\partial u_{\alpha}}{\partial \theta} (L_{\alpha, \lambda_{\alpha}}, \theta) \right)^2 \, d\theta \leq \frac{a'}{2 \log 2},
\]

for any \( \epsilon > 0 \), we can always find \( k_0 \), which is independent of \( \alpha \), and \( L_\alpha \in [2^{k_0}, 2^{k_0}] \) such that

\[
\frac{1}{\alpha - 1} \int_{\partial B_{r_\alpha} \setminus \pi_{r_\alpha} (x_\alpha)} |u_{\alpha, \theta}|^2 r \, ds = \frac{1}{\alpha - 1} \int_0^{2\pi} \left( \frac{\partial u_{\alpha}}{\partial \theta} (L_{\alpha, \lambda_{\alpha}}, \theta) \right)^2 \, d\theta < \epsilon
\]

and

\[
\frac{1}{\alpha - 1} \int_{\partial B_{1/2k_\alpha} \setminus \pi_{1/2k_\alpha} (x_\alpha)} |u_{\alpha, \theta}|^2 r \, ds = \frac{1}{\alpha - 1} \int_0^{2\pi} \left( \frac{\partial u_{\alpha}}{\partial \theta} \right)^2 \, d\theta < \epsilon.
\]

Then

\[
\left| \int_{\partial Q(\log L_\alpha / \log 2)} \frac{\partial u_{\alpha}}{\partial r} (u_{\alpha} - u^*_\alpha) \, ds \right| \leq \left( \int_{\partial Q(\log L_\alpha / \log 2)} r \left| \frac{\partial u_{\alpha}}{\partial r} \right|^2 \, ds \right)^{1/2} \left( \int_{\partial Q(\log L_\alpha / \log 2)} \left( \frac{\partial u_{\alpha}}{\partial \theta} \right)^2 \, d\theta \right)^{1/2}
\]

\[
\leq \epsilon (\alpha - 1) \int_{\partial Q(\log L_\alpha / \log 2)} r \left| \frac{\partial u_{\alpha}}{\partial r} \right|^2 \, ds.
\]

From Lemma 2.2 and (2.5), we get

\[
\int_{\partial Q(\log L_\alpha / \log 2)} r \left| \frac{\partial u_{\alpha}}{\partial r} \right|^2 \, ds \leq C \int_{\partial Q(\log L_\alpha / \log 2)} r |u_{\alpha, \theta}|^2 \, ds + C (\alpha - 1) + C \lambda_{\alpha}^{t_\alpha}
\]

\[
\leq (C + \epsilon) (\alpha - 1).
\]
By (4.8) and (4.10), we get

\[
(1 - 2\epsilon_1) \int_{B_{L_0^{\lambda_\alpha}} \setminus B_{1/L_0^{\lambda_\alpha}}(x_\alpha)} |\nabla u_\alpha|^2 \, dx \\
\leq \epsilon_1 (\alpha - 1) + 2(\alpha - 1)(2H(\lambda_\alpha^{E_\alpha}) \log L_\alpha + \epsilon).
\]

(4.16)

Noting (4.10) we can infer from (2.8)

\[
\frac{1}{\alpha - 1} \int_{B_{L_0^{\lambda_\alpha}} \setminus B_{1/L_0^{\lambda_\alpha}}(x_\alpha)} \left( \left| \frac{\partial u_\alpha}{\partial r} \right|^2 - |u_\alpha,\theta|^2 \right) \, dx = \int_{1/L_0^{\lambda_\alpha}}^{L_0^{\lambda_\alpha}} \frac{2}{r} H(r) \, dr,
\]

which implies that

\[
\frac{1}{\alpha - 1} \int_{B_{L_0^{\lambda_\alpha}} \setminus B_{1/L_0^{\lambda_\alpha}}(x_\alpha)} \left( \left| \frac{\partial u_\alpha}{\partial r} \right|^2 - |u_\alpha,\theta|^2 \right) \, dx \geq 4 \log L_0 (H(\lambda_\alpha^{E_\alpha}) - \epsilon).
\]

Combining (4.16) with the above inequality we conclude the following inequality holds true as \( \alpha \) is close to 1 sufficiently

\[
\frac{1}{\alpha - 1} \int_{B_{L_0^{\lambda_\alpha}} \setminus B_{1/L_0^{\lambda_\alpha}}(x_\alpha)} |u_\alpha,\theta|^2 \, dx \leq \epsilon_1 + C \epsilon.
\]

Thus, we have shown (4.15). \( \square \)

**Proposition 4.3.** Let \( \{u_\alpha\} \) be the sequence stated in Theorem 1.3. Assume that \( \nu > 1 \) and \( t_\alpha \) is a positive number such that \( 0 < t_2 \leq t_\alpha \leq t_1 < 1 \). Then, we have

\[
\frac{1}{\sqrt{\alpha - 1}} (u_\alpha(\lambda_\alpha^{E_\alpha} r, \theta) - u_\alpha(\lambda_\alpha^{E_\alpha}, 0)) \to \bar{a} \log r
\]

strongly in \( C^k(S^1 \times [1/\epsilon, \epsilon], \mathbb{R}^K) \), where \( \theta \) is the angle parameter of the ball centered at \( x_\alpha \), \( y = \lim_{\alpha \to 1} u_\alpha(\lambda_\alpha^{E_\alpha}, \theta) \) and \( \bar{a} \in T_y N \) is a vector in \( \mathbb{R}^K \) with

\[
|\bar{a}| = \mu^{1-\lim_{\alpha \to 1} t_\alpha} \frac{E(\nu)}{\pi}.
\]

**Proof.** For any \( t_\alpha \in [t_2, t_1] \), we have known that \( \lim_{\alpha \to 1} u_\alpha(\partial B_{\lambda_\alpha^{E_\alpha}}) \in N \) and denote

\[
y = u_\alpha(\partial B_{\lambda_\alpha^{E_\alpha}}).
\]
As $N$ is regarded as an embedded submanifold in $\mathbb{R}^K$, for simplicity, we may assume $y = 0 \in N$ and $T_yN = \mathbb{R}^n$, where $\mathbb{R}^K = \mathbb{R}^n \times \mathbb{R}^{K-n}$. We also let $\lambda'_\alpha = \lambda'_\alpha^0$, $x'_\alpha = (\lambda'_\alpha^0, 0) + x_\alpha$ and

$$u'_\alpha(x) = u_\alpha(\lambda'_\alpha x + x_\alpha), \quad v_\alpha(x) = \frac{1}{\sqrt{\alpha - 1}}[u_\alpha(\lambda'_\alpha x + x_\alpha) - u_\alpha(x'_\alpha)].$$

By (4.11) and Theorem 2.1, we get

$$\|\nabla u'_\alpha\|_{C^0(B_{2^k} \setminus B_{2^{k-1}})} + \|\nabla^2 u'_\alpha\|_{C^0(B_{2^k} \setminus B_{2^{k-1}})} < C(k) \sqrt{\alpha - 1},$$

and then

$$\|\nabla v_\alpha\|_{C^0(B_{2^k} \setminus B_{2^{k-1}})} + \|\nabla^2 v_\alpha\|_{C^0(B_{2^k} \setminus B_{2^{k-1}})} < C(k). \quad (4.17)$$

Noting that $v_\alpha(1, 0) = 0$, we get

$$\|v_\alpha\|_{C^0(B_{2^k} \setminus B_{2^{k-1}})} < C'(k).$$

Obviously, we have the equation:

$$\Delta_0 v_\alpha + \sqrt{\alpha - 1}(A(y) + o(1))(dv_\alpha, dv_\alpha) + (\alpha - 1)O(\|\nabla^2 v_\alpha\|) = 0,$$

hence, the sequence

$$v_\alpha \to v_0 \quad \text{in} \quad C^k_{\text{loc}}(\mathbb{R}^2 \setminus \{0\})$$

where $v_0$ satisfies

$$\Delta_0 v_0 = 0 \quad \text{with} \quad v_0 = v_0(|x|).$$

Set

$$v = (a_1, a_2, \ldots, a_n, 0, \ldots, 0) \log r.$$

We deduce from (2.6) that

$$\int_{\partial B_t} (\epsilon_\alpha + |\nabla u_\alpha|^2)^{\alpha - 1} |\nabla_0 v_\alpha|^2 \, ds_0 = \frac{2\alpha}{2\alpha - 1} \int_{\partial B_t} (\epsilon_\alpha + |\nabla u_\alpha|^2)^{\alpha - 1} |v_\alpha,\theta|^2 \, ds_0$$

$$+ \frac{2}{(2\alpha - 1)t} \int_{B_t} (\epsilon_\alpha + |\nabla u_\alpha|^2)^{\alpha - 1} |\nabla_0 u_\alpha|^2 \, dx + O(t).$$

Recalling that

$$F_\alpha(t) = \int_{B_t} (\epsilon_\alpha + |\nabla u_\alpha|^2)^{\alpha - 1} |\nabla_0 u_\alpha|^2 \, dx$$
and keeping (4.15) in our minds, we infer from the above identity and Lemma 2.2 that, as $\alpha \to 1$,
\[
\int_{B_{2\lambda_\alpha}' \setminus B_{\lambda_\alpha}'} \left( \epsilon_\alpha + |\nabla g_\alpha u_\alpha|^2 \right)^{\alpha-1} |\nabla_0 v_\alpha|^2 \, dx = \frac{2\alpha}{2\alpha - 1} \int_{\lambda_\alpha}^{2\lambda_\alpha} \frac{1}{t} F_\alpha(\log_2 t) \, dt + o(1)
\]
\[
= \frac{2\alpha}{2\alpha - 1} \log 2 F(\alpha) + o(1)
\]
\[
\to 2 \log 2 \left( \lim_{\alpha \to 1} t_\alpha \right).
\]

On the other hand, we have that, as $\alpha \to 1$, there holds
\[
\int_{B_{2\lambda_\alpha}' \setminus B_{\lambda_\alpha}'} \left( \epsilon_\alpha + |\nabla g_\alpha u_\alpha|^2 \right)^{\alpha-1} |\nabla_0 v_\alpha|^2 \, dx = \int_{B_1} \left( \epsilon_\alpha + |\nabla g_\alpha v_\alpha|^2 \right)^{\alpha-1} |\nabla_0 v_\alpha|^2 \, dx
\]
\[
\to 2\pi \mu \lim_{\alpha \to 1} t_\alpha |a|^2 \log 2.
\]

Hence, we get
\[
\lim_{\alpha \to 1} v_\alpha = (a_1, \ldots, a_n, 0, \ldots, 0) \log r
\]
with
\[
\sum_{i=1}^m a_i^2 = \frac{\Lambda}{\pi} \mu^{1-2\lim_{\alpha \to 1} t_\alpha}.
\]

As $v : S^2 \to N$ is the corresponding only bubble, then the above identity can be written as
\[
|\tilde{a}|^2 = \frac{E(v, S^2)}{\pi} \mu^{2-2\lim_{\alpha \to 1} t_\alpha}.
\]

Thus, we complete the proof of Proposition 4.3. $\square$

**Corollary 4.4.** Let $\{u_\alpha\}$ be the sequence stated in Theorem 1.3. Assume that $u_{\alpha_k}$ is a subsequence of $\{u_\alpha\}$ satisfying
\[
E_{\alpha_k}(u_{\alpha_k}, B_{\lambda_\alpha_k}'(x_\alpha)) \to \mu^{2-\nu} E(v)
\]
in $C^0([t_2, t_1])$ with respect to $C^0$-norm. If $\nu > 1$, then
\[
\int_{\lambda_\alpha_k}^{2\lambda_\alpha_k} \frac{1}{\sqrt{\alpha_k - 1}} \left| \frac{\partial u_{\alpha_k}}{\partial r} \right| \, dr \to 2\mu^{1-\nu} \sqrt{\frac{E(v)}{\pi}} \text{ in } C^0([t_2, t_1]);
\]
and
\[
\frac{1}{\sqrt{\alpha_k} - 1} \left( r \left| \frac{\partial u_{\alpha_k}}{\partial r} \right| \right) (\lambda_{\alpha_k}^t, \theta) \to \mu^{1-t} \sqrt{\frac{E(v)}{\pi}} \quad \text{in} \ C^0([t_2, t_1]).
\]

**Proof.** We need only to prove the first claim, since the proof of the second claim is similar. If the first claim was not true, then we assumed that there was a subsequence \( \alpha_{k_i}, t_i \to t_0 \) such that
\[
\left| \int_{\lambda_{\alpha_{k_i}}^{t_i}} \frac{1}{\sqrt{\alpha_{k_i} - 1}} \left| \frac{\partial u_{\alpha_{k_i}}}{\partial r} \right| \, dr - \log 2 \mu^{1-t_i} \sqrt{\frac{E(v)}{\pi}} \right| \geq \epsilon > 0.
\]
On the other hand, from the above arguments on Proposition 4.3 we know that, after passing to a subsequence, there holds
\[
\frac{u_{\alpha_{k_i}}(\lambda_{\alpha_{k_i}} x) - u_{\alpha_{k_i}}(\lambda_{\alpha_{k_i}}, 0)}{\sqrt{\alpha_{k_i} - 1}} \to \tilde{a} \log r,
\]
with \( |\tilde{a}| = |\mu^{1-t_0} \sqrt{\frac{E(v)}{\pi}}| \). Hence we derive the following
\[
\lim_{i \to +\infty} \int_{\lambda_{\alpha_{k_i}}^{t_i}} \frac{1}{\sqrt{\alpha_{k_i} - 1}} \left| \frac{\partial u_{\alpha_{k_i}}}{\partial r} \right| \, dr = |\tilde{a}| \int_{1}^{2} \frac{1}{r} \, dr = \log 2 \mu^{1-t_0} \sqrt{\frac{E(v)}{\pi}}.
\]
This is a contradiction. \( \square \)

4.2.2. The neck is a geodesic

To show the neck for the \( \alpha \)-harmonic map sequence is just a geodesic in \( N \) which joins \( N \) and the bubble, we define the following curve
\[
\omega_{\alpha}(r) \equiv \frac{1}{2\pi} \int_{0}^{2\pi} u_{\alpha}(r, \theta) \, d\theta : [\lambda_{\alpha}^{t_1}, \lambda_{\alpha}^{t_2}] \to \mathbb{R}^K,
\]
which is denoted by \( \Gamma_{\alpha} \). From (4.6), it is easy to see that for any \( r_k \to 0 \), \( \omega_{\alpha}(r_k) \) converges to a point \( p \in N \) after passing to a subsequence. Moreover, for any fixed \( R \), it is easy to see \( \omega_{\alpha}(r) \) converges to the point \( p \) in \([ \frac{1}{R} r_k, R r_k] \) as \( r_k \to 0 \). However, if we use the arc length of each curve \( \Gamma_{\alpha} \) to parameterize the curve \( \omega_{\alpha} \), then we will see that \( \omega_{\alpha}(\cdot) \) converges locally. For the sake of convenience, sometimes we denote the derivatives \( \frac{\partial f(r, \theta)}{\partial r} \) and \( \frac{\partial f(r, \theta)}{\partial \theta} \) of \( f(r, \theta) \) by \( \dot{f} \) and \( f, \theta \) respectively.
By computation we have

\[ \ddot{\omega}_\alpha = \frac{1}{2\pi} \int_0^{2\pi} \ddot{u}_\alpha(r, \theta) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left( \ddot{u}_\alpha(r, \theta) + \frac{u_{\alpha,\theta\theta}}{r^2} \right) \, d\theta \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \Delta_0 u_\alpha \, d\theta - \frac{1}{2\pi} \int_0^{2\pi} \frac{\ddot{u}_\alpha}{r} \, d\theta \]

\[ = -\frac{1}{2\pi} \int_0^{2\pi} A(u_\alpha)(du_\alpha, du_\alpha) - \alpha - 1 \frac{1}{2\pi} \int_0^{2\pi} \frac{\nabla_0|\nabla_g u_\alpha|^2 - \nabla_0 u_\alpha \cdot \dot{\omega}_\alpha}{\epsilon_\alpha^2 + |\nabla_g u_\alpha|^2} \, d\theta - \frac{\dot{\omega}_\alpha}{r}, \]

where we have used the fact \( \int_0^{2\pi} u_{\alpha,\theta\theta}(r, \theta) \, d\theta = 0. \)

In order to prove \( \Gamma_\alpha \) converges to a geodesic we need to analyze the asymptotic behavior of \( \Gamma_\alpha \) as \( \alpha \to 1. \) By virtue of Proposition 4.3, we can derive the following asymptotic estimates on the metric and the second fundamental form of \( \Gamma_\alpha. \)

**Lemma 4.5.** Denote the induced metric of \( \Gamma_\alpha \) in \( \mathbb{R}^K \) by \( h_\alpha, \) and the second fundamental form by \( A_{\Gamma_\alpha}. \) Then, after passing to a subsequence, for \( \lambda^{t_\alpha}_\alpha \in [\lambda_{t_\alpha}^{l_\alpha}, \lambda_{t_\alpha}^{l_\alpha}] \) there holds

\[ \dot{\omega}_\alpha(\lambda^{t_\alpha}_\alpha) = \frac{\sqrt{\alpha - 1}}{\lambda^{t_\alpha}_\alpha}(\tilde{a} + o(1)), \]

\[ h_\alpha \left( \frac{d}{dr}, \frac{d}{dr} \right) = |\dot{\omega}_\alpha|^2 = \frac{\alpha - 1}{\lambda^{2t_\alpha}_\alpha} (|\tilde{a}|^2 + o(1)); \]

(4.18)

and

\[ A_{\Gamma_\alpha}(d\omega_\alpha, d\omega_\alpha) = \frac{\alpha - 1}{\lambda^{2t_\alpha}_\alpha} (A(y)(\tilde{a}, \tilde{a}) + o(1)). \]

(4.19)

Moreover, for any \( t \in [t_2, t_1] \) there exists a constant such that

\[ \|A_{\Gamma_\alpha}\|^2_{h_\alpha(\lambda^{t_\alpha}_\alpha)} < C. \]

**Proof.** Let

\[ G_{\alpha} = -\ddot{\omega}_\alpha - \frac{\dot{\omega}_\alpha}{r}. \]

Given \( \lambda^{t_\alpha}_\alpha \in [\lambda_{t_\alpha}^{l_\alpha}, \lambda_{t_\alpha}^{l_\alpha}] \). As before, we always have

\[ \frac{u_\alpha(\lambda^{t_\alpha}_\alpha r, \theta) - u_\alpha(\lambda^{t_\alpha}_\alpha, 0)}{\sqrt{\alpha - 1}} \to \tilde{a} \log r, \]
where \( \tilde{a} \in T_yN \) and \( y = \lim_{\alpha \to 1} u_\alpha(\lambda t_\alpha, \theta) \). Therefore, we have

\[
\dot{\omega}_\alpha(\lambda t_\alpha) = \frac{\sqrt{\alpha - 1}}{\lambda t_\alpha}(\tilde{a} + o(1)), \quad h_\alpha \left( \frac{d}{dr}, \frac{d}{dr} \right) = |\dot{\omega}_\alpha|^2 = \frac{\alpha - 1}{\lambda t_\alpha}(\tilde{a}^2 + o(1)),
\]

where \( o(1) \to 0 \) as \( \alpha \to 1 \). Moreover, we have

\[
G_\alpha(\lambda t_\alpha) = \frac{1}{2\pi} \int_0^{2\pi} A(u_\alpha)(du_\alpha, du_\alpha) d\theta + \frac{\alpha - 1}{\lambda t_\alpha} \left( \frac{1}{2\pi} \int_0^{2\pi} A(y)(\tilde{a}, \tilde{a}) d\theta + o(1) \right) + \alpha - 1 \int_0^{2\pi} O\left( \frac{|\nabla^2 u_\alpha|}{\sqrt{\alpha - 1}} \right) d\theta
\]

\[
= \frac{\alpha - 1}{\lambda t_\alpha} \left( \frac{1}{2\pi} \int_0^{2\pi} A(y)(\tilde{a}, \tilde{a}) d\theta + o(1) \right) + \sqrt{\alpha - 1} \int_0^{2\pi} O\left( \frac{|\nabla^2 u_\alpha|}{\sqrt{\alpha - 1}} \right) d\theta
\]

\[
= \frac{\alpha - 1}{\lambda t_\alpha} \left( A(y)(\tilde{a}, \tilde{a}) + o(1) + O(\sqrt{\alpha - 1}) \right)
\]

\[
= \frac{\alpha - 1}{\lambda t_\alpha} \left( A(y)(\tilde{a}, \tilde{a}) + o(1) \right),
\]

where we have used estimate (4.17) on \( v_\alpha \) defined in the proof of Proposition 4.3, and have also made use of the fact \( \nu > 1 \) which implies that for any \( m > 0 \)

\[
\lambda_{2\alpha}^{2m} = o((\alpha - 1)^m).
\]

Noting that \( \langle A(y)(\tilde{a}, \tilde{a}), \tilde{a} \rangle = 0 \), we get

\[
-A_{\Gamma_\alpha}(d\omega_\alpha, d\omega_\alpha) = \dot{\omega}_\alpha - \frac{\langle \dot{\omega}_\alpha, \dot{\omega}_\alpha \rangle}{|\dot{\omega}_\alpha|^2} \dot{\omega}_\alpha = -G_\alpha + \frac{\langle G_\alpha, \dot{\omega}_\alpha \rangle}{|\dot{\omega}_\alpha|^2} \\
= -\frac{\alpha - 1}{\lambda t_\alpha} \left( A(y)(\tilde{a}, \tilde{a}) + o(1) \right).
\]

Hence, we conclude \( \| A_{\Gamma_\alpha} \|_{L_\infty}(\lambda t_\alpha) < C \). Similar to the proof of Corollary 4.4, after passing to a subsequence we have that for any \( t \in [t_2, t_1] \),

\[
\| A_{\Gamma_\alpha} \|_{L_\infty}(\lambda t_\alpha) < C.
\]

**Proposition 4.6.** Assume that \( \{ u_\alpha \} \) be the sequence stated in Theorem 1.3. Then, after passing to a subsequence, the curve sequence \( \Gamma_\alpha \) in \( \mathbb{R}^K \), each of which is defined by \( \omega_\alpha(r) \) and parameterized by its arc length, converges to a geodesic on \( (N, h) \) as \( \alpha \) tends to 1.
Proof. For every $\omega_\alpha$ we fix

$$y_\alpha = \omega_\alpha(\lambda^{t_\alpha}_\alpha) = \frac{1}{2\pi} \int_0^{2\pi} u_\alpha(\lambda^{t_\alpha}_\alpha, \theta) \, d\theta,$$

and let $s$ be the arc length parameter of $\omega_\alpha(t)$ with $s(\lambda^{t_\alpha}_\alpha) = 0$. We know that the sequence of points $\{\omega_\alpha(\lambda^{t_\alpha}_\alpha)\}$ is convergent as $\alpha \to 1$. Denote $y = \lim_{\alpha \to 1} y_\alpha$. It is well known that the norm of the second fundamental form $\|A_{\Gamma_\alpha}\|_{\h_\alpha}^2$ does not depend on the choice of parameter, and

$$\frac{d^2\omega_\alpha}{ds^2} = -A'_{\Gamma_\alpha}(\omega_\alpha) \left( \frac{d\omega_\alpha}{ds}, \frac{d\omega_\alpha}{ds} \right),$$

then $\omega_\alpha(s)$ converges locally to a smooth vector value function from $[0, s_1]$ into $\mathbb{R}^K$, denoted by $\omega(s)$, in $C^1$, where $s_1$ is sufficiently small and $s$ is still the arc length parameter. This implies that $\Gamma_\alpha|_{[\lambda^{t_1}_\alpha, \lambda^{t_2}_\alpha]}$ converges locally to a curve $\Gamma$.

The remaining part of proof is to prove $\Gamma$ is a geodesic on $N$. To verify $\Gamma$ satisfies geodesic equation, it is crucial to show

$$A_{\Gamma_\alpha}(\omega_\alpha) \left( \frac{d\omega_\alpha}{ds}, \frac{d\omega_\alpha}{ds} \right) \to A(\omega) \left( \frac{d\omega}{ds}, \frac{d\omega}{ds} \right)$$

strongly in $C^0([0, s_1], \mathbb{R}^K)$, where $A(\cdot)(\cdot)$ is the second fundamental form of $N$ as a submanifold in $\mathbb{R}^K$. We claim that the above convergence is true. If this was not true, then for any small $s_1$ we could find a subsequence of $\{u_\alpha\}$, still denoted by $\{u_\alpha\}$, such that

$$s'_\alpha = \int_{\lambda^{t_1}_\alpha}^{\lambda^{t_2}_\alpha} |\dot{\omega}_\alpha| \, dr \to s' \in (0, s_1),$$

and

$$\left| A'_{\Gamma_\alpha}(\omega_\alpha) \left( \frac{d\omega_\alpha}{ds}, \frac{d\omega_\alpha}{ds} \right) - A(\omega) \left( \frac{d\omega}{ds}, \frac{d\omega}{ds} \right) \right|_{s=s'_\alpha} > \epsilon.$$

We can always choose a small enough $s_1$ such that, when $s'_\alpha < s_1$ and $\alpha - 1$ is small enough, the corresponding $t'_\alpha \in [\frac{P}{2}, t_1]$. This is implied by Corollary 4.4. For simplicity, we may assume

$$\lambda^{t_2}_\alpha = 2P\lambda^{t_\alpha}_\alpha$$

where $P$ is an integer. Indeed, applying Corollary 4.4 we have

$$\int_{2^{i+1}\lambda^{t_\alpha}_\alpha}^{2^i\lambda^{t_\alpha}_\alpha} |\dot{\omega}_\alpha| \, dr = \sqrt{\alpha - 1} \mu^{1-\log_\alpha 2} \left( \log 2 \sqrt{\frac{E(v)}{\pi}} + o_\alpha(1) \right).$$
Therefore, as $\alpha$ is close to 1 enough,
\[ \int \limits_{\lambda_{a}^{\alpha}}^{\lambda_{a}^{2\alpha}} |\dot{\omega}_{\alpha}| dr = \sum_{i=0}^{p-1} \int \limits_{2^{i+1}\lambda_{a}^{\alpha}}^{2^{i}\lambda_{a}^{\alpha}} |\dot{\omega}_{\alpha}| dr \geq P \sqrt{\alpha - 1} \left( \sqrt{\frac{E(v)}{\pi}} \log 2 + o_{\alpha}(1) \right) \]
\[ \geq C \left( t_{\alpha} - \frac{t_{2}}{2} \right) \log \lambda_{\alpha}^{-1} \sqrt{\alpha - 1} \]
\[ \geq C \frac{t_{2}}{2} \log v > 0. \]

Therefore, we may always pick $s_{1}$ to be very small, for example $s_{1} < C \frac{t_{2}}{2} \log v$, such that $t_{\alpha}^{'} \in [\frac{t_{2}}{2}, t_{1}]$ as $\alpha - 1$ is small enough.

Since $t_{\alpha}^{'} \in [\frac{t_{2}}{2}, t_{1}]$, by applying Proposition 4.3 we have
\[ \frac{u_{\alpha}(\lambda_{a}^{\alpha} r, \theta) - u_{\alpha}(\lambda_{a}^{\alpha} r, 0)}{\sqrt{\alpha - 1}} \to \vec{a}^{'\prime} \log r. \]

Obviously,
\[ \frac{\dot{\omega}_{\alpha}(\lambda_{a}^{\alpha})}{|\dot{\omega}_{\alpha}(\lambda_{a}^{\alpha})|} = \frac{d\omega_{\alpha}}{ds}(s_{\alpha}) \to \frac{d\omega}{ds}(s). \]

Applying (4.18) and (4.19) in Lemma 4.5 with $t_{\alpha}^{'}$ instead of $t_{\alpha}$, we know that, after passing to a subsequence, the following holds
\[ A_{\Gamma_{\alpha}}(\omega_{\alpha}) \left( \frac{d\omega_{\alpha}}{ds}, \frac{d\omega_{\alpha}}{ds} \right) \bigg|_{s=s_{\alpha}} = \frac{1}{|\dot{\omega}_{\alpha}(\lambda_{a}^{\alpha})|^{2}} A_{\Gamma_{\alpha}}(\dot{\omega}_{\alpha}, \dot{\omega}_{\alpha}) \bigg|_{r=r_{\alpha}} \to A(\omega) \left( \frac{d\omega}{ds}, \frac{d\omega}{ds} \right) \bigg|_{s=s_{\alpha}^{\prime}}, \]
which contradicts the choice of $s_{\alpha}^{\prime}$. Thus, the assertion (4.20) is proved.

So, we infer from the curve equation of $\omega_{\alpha}$ and the fact (4.20) that the following
\[ \frac{d\omega}{ds}(s) - \frac{d\omega}{ds}(0) = - \int \limits_{0}^{s} A(\omega) \left( \frac{d\omega}{ds}, \frac{d\omega}{ds} \right) ds \]
holds true in interval $[0, s_{1}]$. This shows $\omega$ is smooth near 0 and satisfies in $[0, s_{1}]$
\[ \frac{d^{2}\omega}{ds^{2}} = -A(\omega) \left( \frac{d\omega}{ds}, \frac{d\omega}{ds} \right). \]

Therefore, we obtain
\[ \nabla_{\omega}^{N} \frac{d\omega}{ds} = \frac{d^{2}\omega}{ds^{2}} + A(\omega) \left( \frac{d\omega}{ds}, \frac{d\omega}{ds} \right) = 0, \]
which means that $\Gamma$ is a geodesic. Thus we complete the proof of this proposition.
4.2.3. The proofs of Theorem 1.3 for the case $\nu > 1$ and Corollary 1.5

Next, we give the length formula of the neck, i.e. calculate the length of the geodesic $\Gamma$.

**Proof of Theorem 1.3 for the case $\nu > 1$.** By Proposition 4.6 we have known that the neck converges to a geodesic. We only need to calculate the length of the geodesic. Without loss of generality, as before we may assume $\lambda_{\alpha}^{t_2} = 2^P \lambda_{\alpha}^{t_1}$ for some integer $P$. Then we have

$$P = \frac{t_2 - t_1}{\log 2} \log \lambda_{\alpha}.$$  

When $\nu = +\infty$, by Corollary 4.4, we have

$$L(\Gamma_{\alpha}|_{B_{2^{k+1} \lambda_{\alpha}^{t_1}, 2^k \lambda_{\alpha}^{t_1}}(x_\alpha)}) \geq \sqrt{\alpha - 1} \left( \sqrt{\frac{E(v)}{\pi}} \log 2 + o(1) \right).$$

Then,

$$L(\Gamma_{\alpha}) \geq C P \sqrt{\alpha - 1} \geq C \log \lambda^{\sqrt{\alpha - 1}} \to +\infty.$$  

This implies

$$L(\Gamma) = +\infty.$$

Now, we assume $\nu < +\infty$. By Corollary 4.4,

$$L(\Gamma_{\alpha}|_{B_{2^{k+1} \lambda_{\alpha}^{t_1}, 2^k \lambda_{\alpha}^{t_1}}(x_\alpha)}) = \sqrt{\alpha - 1} \left( \sqrt{\frac{E(v)}{\pi}} \log 2 + o(1) \right),$$

where $o(1) \to 0$ as $\alpha \to 1$ uniformly. Hence

$$L(\Gamma) = \lim_{\alpha \to 1} \sqrt{\alpha - 1} P \sqrt{\frac{E(v)}{\pi}} \log 2 = (t_1 - t_2) \sqrt{\frac{E(v)}{\pi}} \log \nu.$$  

Now, it is easy to see that to complete the proof of Theorem 1.3 we only need to prove the following:

$$\text{osc}_{B_{\lambda_{\alpha}} \setminus B_{R \lambda_{\alpha}}(x_\alpha)} u_{\alpha} \to 0, \quad \text{as } \alpha \to 1, \text{ then } R \to +\infty \text{ and } t \to 1; \quad (4.21)$$

and

$$\text{osc}_{B_{\lambda_{\alpha}} \setminus B_{\lambda_{\alpha}}(x_\alpha)} u_{\alpha} \to 0, \quad \text{as } \alpha \to 1, \text{ then } \delta \to 0 \text{ and } t \to 0. \quad (4.22)$$

Since $\nu < +\infty$ implies $\mu = 1$, from Theorem 1.1 we know

$$\lim_{t \to 1} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_{\lambda_{\alpha}} \setminus B_{R \lambda_{\alpha}}(x_\alpha)} |\nabla u_{\alpha}|^2 = 0.$$
Therefore, we can use the same method as in Section 4.1 (we replace $\delta$ with $\lambda t$) to deduce
\[
\text{osc}_{B_{\lambda t} \setminus B_{R\lambda t}} u_\alpha \leq C \sqrt{E(u_\alpha, B_{\lambda t} \setminus B_{R\lambda t}(x_\alpha))} + C(1 - t) \log v + C\sqrt{\alpha - 1},
\]
then (4.21) follows. Similarly, we can prove (4.22). Hence, we derive the length formula of the geodesic $\Gamma$
\[
L = \sqrt{\frac{E(v)}{\pi}} \log v.
\]
Thus, we finish the proof of Theorem 1.3.

\textbf{Remark 4.7.} Although we state and prove Theorem 1.3 only for one bubble case, it is not difficult to follow the steps in Section 3.2 to prove the general case. However, the general case is quite complicated, for example, if we have 2 bubbles: $u_\alpha(\lambda_1 x + x_1) \to v^1$, and $u_\alpha(\lambda_2 x + x_1) \to v^2$ which satisfy: $\lambda_1 / \lambda_2 \to 0$ and $v^1, v^2 < \infty$, then $u_0(B_{\delta}(x_1)), v^2(S^2)$ are connected by a geodesic with length
\[
L = \sqrt{\frac{E(v^1) + E(v^2)}{\pi}} \log v^2,
\]
$v^1(S^2)$ and $v^2(S^2)$ are connected by a geodesic of length
\[
L = \sqrt{\frac{E(v^1)}{\pi}} \log \frac{v^1}{v^2}.
\]

\textbf{Proposition 4.8.} If $v < +\infty$, then when $(\alpha - 1)$ is sufficiently small, all the $u_\alpha$ belong to a homotopy class.

\textbf{Proof.} When $\alpha_i - 1$ and $\alpha_j - 1$ are sufficiently small, we have $\|u_i - u_j\|_{C^0} \leq i(N)$ where $i(N)$ is the injective radius of $N$. Hence, by using exponential map we know that $u_{\alpha_i}$ and $u_{\alpha_j}$ are homotopic in $M \setminus B_{\delta}, B_{\lambda_2} \setminus B_{t_2}$ and $B_{R\lambda_2}$ respectively.

Let $p = u_0(0)$ and $q = v(\pm \infty)$. By (4.21), we know that $u_i(B_{\delta}(x_\alpha) \setminus B_{\lambda_2}(x_\alpha))$ is contained in a simply connected ball centered at $p$ when $\alpha$ is close to 1 enough, $\delta$ and $t_2$ are small enough. Similarly, by (4.22) we also have $u_j(B_{\lambda_2} \setminus B_{R\lambda_2}(x_\alpha))$ is contained in a small simply connected ball in $N$ with center $q$ when $\alpha - 1, \delta$ and $1 - t_2$ are sufficiently small. Hence $u_i$ and $u_j$ are homotopic in $B_{\delta} \setminus B_{\lambda_2}$ and $B_{\lambda_2} \setminus B_{R\lambda_2}$ respectively. So $u_i$ and $u_j$ are homotopic.

\textbf{Proof of Corollary 1.5.} Corollary 1.5 follows directly from the above proposition.

5. Some comments and an example

In this paper we only consider the case $\{u_\alpha\} (\alpha \to 1)$ is a sequence of $\alpha$-harmonic maps from a closed compact Riemann surface the conformal structure of which is fixed. Naturally, one will ask the following problems: (i) What can we say in the case $u_\alpha$ is a sequence of $\alpha$-harmonic maps
and the conformal structure of $M$ varies with $\alpha$? (ii) Can the methods in this paper be extended to a class of variational problems which are more general than $\alpha$-energy? In a forthcoming paper we will further develop some tools to discuss some issues which relate to the above problems.

On the other hand, one want to know whether one can give an example of a sequence of $\alpha$-harmonic maps to show there is a neck joining the bubbles which is of infinite length or not. However, if we can construct a manifold $N$ and find a minimizing $\alpha$-harmonic map sequence which satisfies the conditions of Corollary 1.5, then the corollary tells us that, indeed, there exists a neck which is of infinite length. By modifying the example given by Duzaar and Kuwert (see p. 304 of [8]) we can construct such example as follows:

**Example.** Let $\mathbb{Z}^3$ act on $\mathbb{R}^3$ by $\tau_\kappa(x, y, z) = (x + 4k_1, y + 4k_2, z + 4k_3)$, where $\kappa = (k_1, k_2, k_3) \in \mathbb{Z}^3$. Consider

$$\tilde{X} = \mathbb{R}^3 \setminus \bigcup_\kappa \tau_\kappa(B_1(0)).$$

Let $X$ be the quotient space of $\tilde{X}$ modulo $\mathbb{Z}^3$. Obviously, $X$ is a compact manifold with boundary. Topologically, $X$ is $T^3$ minus a small ball.

Let $\Phi$ be a smooth map from $\mathbb{R}^2$ to $\partial B_1(0)$ satisfying

$$\Phi(x) = \begin{cases} (1, 0, 0), & |x| > 2, \\ (-1, 0, 0), & |x| < 1; \end{cases}$$

and the degree of map $\Phi$ denoted by $\deg(\Phi)$ is $1$ if we consider $\Phi$ as a map from $S^2$ to $S^2$, i.e. $\deg(\Phi) = 1$. Moreover, we let $\gamma_k : [0, 1] \to \tilde{X}$ be a curve which connects $(4k_1 + 1, 0, 0)$ and $(-1, 0, 0)$ with $\gamma(1) = (-1, 0, 0)$ and $\|\dot{\gamma}_k\|_{L^\infty} \leq 8k$. For $\delta \leq 1$ and $R \geq 2$ we define

$$v_k = \begin{cases} \Phi(x), & |x| \geq \delta, \\ \gamma_k(\log |x| - \log R \epsilon_k), & R \epsilon_k < |x| < \delta, \\ \tau(k, 0, 0)(\Phi(\frac{x}{\epsilon_k})), & |x| \leq R \epsilon_k. \end{cases}$$

Denote the projection from $\tilde{X}$ to $X$ by $\Pi$, then $\Pi(v_k) \in \pi_2(X)$. It is easy to verify

$$\int_{B_\delta \setminus B_{R \epsilon_k}} |\nabla v_k|^2 = 2\pi \int_{R \epsilon_k}^{\delta} \left| \frac{\partial \gamma_k}{\partial r} \right|^2 r dr$$

$$< \frac{c \|\dot{\gamma}_k\|_{L^\infty}^2}{(\log \delta - \log(R \epsilon_k))^2} \int_{R \epsilon_k}^{\delta} \frac{dr}{r} = \frac{c \|\dot{\gamma}_k\|_{L^\infty}^2}{\log \delta - \log(R \epsilon_k)},$$

$$\int_{\mathbb{R}^2 \setminus B_\delta} |\nabla v_k|^2 \leq E(\Phi), \quad \text{and} \quad \int_{B_{R \epsilon_k}} |\nabla v_k|^2 \leq E(\Phi).$$
So, we can choose suitably $\epsilon_k$ such that

$$E(\Pi(v_k)) = E(v_k) \leq 2E(\Phi) + 1.$$  

We claim that $[\Pi(v_k)]$ are some homotopy classes which are different from each other. If this was not true, we could find a continuous map

$$H(x, t): S^2 \times [0, 1] \to X$$  

such that

$$H(x, 0) = \Pi(v_i) \quad \text{and} \quad H(x, 1) = \Pi(v_j).$$

Since $S^2 \times [0, 1]$ is simply connected, we are able to lift $H$ to $\tilde{H}$ which is a map from $S^2 \times [0, 1]$ into $\tilde{X}$ with $\tilde{H}(x, 0) = v_i$. We assume that $\tilde{H}(x, 1) = \tau_k(v_j)$. Hence $[v_i] = [\tau_k(v_j)]$. Therefore,

$$[\partial B_1(0) + \gamma_1 + \partial \tau_{(i,0,0)}(B_1(0))] = [\partial \tau_k(B_1(0)) + \tau_k \gamma_j + \partial \tau_{(j,0,0)}(B_1(0))] \quad \text{in} \quad \pi_2(\tilde{X}),$$

where $\pi_2(\tilde{X})$ is the second homotopy group of $\tilde{X}$. However, it is easy to check that $\pi_2(X) = \{1\}$, then by Hurewicz Theorem, the above identity is not true.

Now, we construct the target manifold $N$ in the following way: Let $f$ be a homeomorphism from $X$ to $Y = X$. We consider the quotient space of $X \cup Y$, obtained by gluing every point $x \in \partial X$ with $f(x) \in \partial Y$ together. Thus, we get a closed compact manifold $N$ and a projection $\phi: N \to X$. One is easy to check that $\Pi(v_k)$ can be also considered as a map from $S^2$ to $N$ with $E(\Pi(v_k)) < C$. It is easy to check that $[\Pi(v_k)]$ are some homotopy classes which are different from each other in $\pi_2(N)$. Indeed, if it was not true. Then, we could find a continuous map $H(x, t): S^2 \times [0, 1] \to N$ such that $H(x, 0) = \Pi(v_i)$ and $H(x, 1) = \Pi(v_j)$. Hence, $\phi(H(x, t))$ was just a homotopic map of $\Pi(v_i)$ and $\Pi(v_j)$ in $X$. This is a contradiction.

Let $u_{\alpha k}: S^2 \to N$ be the minimizer of $E_{\alpha k}(u)$ in $[\Pi(v_k)]$, where $\alpha_k \to 1$. It is easy to see that $\{u_{\alpha k}\}$ is a sequence of $\alpha$-harmonic maps which satisfies the assumptions in Corollary 1.5. We conclude that $\{u_{\alpha_k}\}$ blows up certainly at some points in $S^2$ and there exists at least a geodesic of infinite length in the limiting set of $\{u_{\alpha_k}\}$.

Finally, we would like to ask the following problems:

**Problem 1.** Suppose all $\alpha$-harmonic maps $u_{\alpha}$ belong to the same homotopy class and satisfy the energy identity as $\alpha \to 1$. Do the necks consist of some geodesics of finite length?

**Problem 2.** Could we find a sequence of $\alpha_k$-harmonic maps $u_{\alpha_k}$ ($\alpha_k \to 1$) such that (1) the Morse index tends to infinite; (2) $\sup_k E_{\alpha_k}(u_{\alpha_k}) < \infty$; (3) for any $i \neq j$, $u_{\alpha_i}$ and $u_{\alpha_j}$ are not homotopic to each other.

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Appendix A

In this section we demonstrate how to construct the bubble trees at every energy concentration point of a sequence of $\alpha$-harmonic maps.

Assume that $u_\alpha \in C^\infty(B_\sigma, N)$ is the sequence stated in Theorem 1.1. Denote by $S(u_\alpha, B)$ the set

$$\left\{ x \in B_\sigma : \lim_{\alpha \to 1} \liminf_{r \to 0} \int_{B_r(x)} |\nabla_0 u_\alpha|^2 \, dx > \epsilon_0 \right\},$$

where $\epsilon_0$ is the constant defined in Theorem 2.1. Usually, $S(u_\alpha, B_\sigma)$ is called the set of energy concentration points (or simply concentration points) for $\{u_\alpha\}$ on $B_\sigma$. Obviously, $S(u_\alpha, B_\sigma)$ is a finite set. One is easy to see that for any $x \in S(u_\alpha, B_\sigma)$, there holds

$$\int_{B_r(x)} |\nabla_0 u_\alpha|^2 \, dx < \epsilon_0,$$

for sufficiently small $r$ and $\alpha - 1$. So, for any $\Omega \subset B_\sigma \setminus S(u_\alpha, B_\sigma)$, there exists a subsequence of $\{u_\alpha\}$ which converges in $\Omega$.

Now, we show how to construct the bubble trees at each concentration point of $S(u_\alpha, B_\sigma)$. For simplicity we assume 0 is the only concentration point for $\{u_\alpha\}$.

Step 1. Set $r_1^1 = \lambda_1^1 = \frac{1}{\max_{B_{\sigma/2}} |u_\alpha|}$. Let $\max_{B_{\sigma/2}} |u_\alpha|$ be attained at the point $x_1^1$. Since $u_\alpha$ converges smoothly away the concentration point, we have $x_1^1 \to 0$. Let $v_1^1 = u_\alpha(x_1^1 + \lambda_1^1 x)$. Obviously, $\|\nabla v_1^1\|_{L^\infty} = 1$. Hence, $v_1^1$ converges to a harmonic map $v^1$ from $\mathbb{R}^2$ to $N$ locally and smoothly. As $|\nabla v^1(0)| = 1$, we know that $v^1$ is not trivial. We call $v^1$ is the first bubble.

Step 2. To find the second bubble, we need to consider the following two cases.

Case 1. If for any $\epsilon > 0$, we can find a positive number $\delta_0$ and a positive $R$ such that

$$\int_{B_{\rho}(x_1^1)} |\nabla u_\alpha|^2 < \epsilon$$

holds for any $\rho \in [Rr_1^1, \delta_0]$, then energy concentration phenomena disappears. This means that there is only one bubble for the sequence $\{u_\alpha\}$.

Case 2. If (A.1) is not true, then we can find $r_2^2 \to 0$ such that $\lim_{\alpha \to 1} \frac{r_2^2}{r_1^1} = +\infty$, and

$$\int_{B_{2r_2^2}(x_1^1)} |\nabla u_\alpha|^2 > \epsilon_1 > 0,$$

i.e. there are still some points where energy concentration happens. We set $u_2^2 = u_\alpha(x_1^1 + r_2^2 x)$, and denote the set of energy concentration points of $\{u_2^2\}$ on $\mathbb{R}^2$ by $S(u_2^2, \mathbb{R}^2)$. Obviously,
0 ∈ S(u^2_α, R^2), and v^1 is also a bubble for the sequence \{u^2_α\}. Moreover, passing to a subsequence we have u^2_α converges to a harmonic map u^0_2 away from S^2(u^2_α, R^2), where u^0_2 is a harmonic map from R^2 into N. Then we have two cases: (a) u^2_0 is trivial, i.e. a constant map; (b) u^2_0 is a non-trivial harmonic map from R^2 to N.

If u^2_0 is trivial, (A.2) implies that S^2(u^2_α, R^2) ∩ (B_3 \ B_{1/2}) ≠ ∅, i.e. there must exist another concentration point p for u^2_α lies in B_3 \ B_{1/2}. At this new point of energy concentration for \{u^2_α\}, we can get a bubble for \{u^2_α\} by the same program as in Step 1. That is to say, we may find x'_α → p and \lambda'_α → 0 such that

\[ u^2_α(x'_α + \lambda'_α x) \rightarrow v^2, \]

where v^2 is a harmonic map from R^2 to N. If we set x^2_α = x'_α + \lambda'_α x^1_α, \lambda^2_α = \lambda'_α \lambda^1_α, we get

\[ u^2_α(x^2_α + \lambda^2_α x) \rightarrow v^2, \]

therefore, v^2 is also a bubble for the sequence u_α. Next, we repeat Step 1 and then repeat Step 2 at each concentration point for \{u^2_α\}.

When u^2_0 is non-trivial, v^2 = u^2_0 is a harmonic map from R^2 to N. v^2 is regarded as the second bubble. At each point of S^2(u^2_α, R^2), we repeat Step 1, and then repeat Step 2.

**Step 3.** We replace r^1_α in Step 1 with r^2_α and continue Step 2 again.

In this way, we can get x^k_α → 0, \lambda^k_α → 0 such that

\[ u_α(x^k_α + \lambda^k_α x) \rightarrow v^k, \quad k = 1, 2, \ldots, m, \]

where v^k are harmonic maps from R^2 to N. One is easy to check that for any i ≠ j one of the (H1) and (H2) always hold. Moreover, since

\[ \int_{B_σ} (\epsilon u_α + |\nabla u_α|^2)^α dV_g \geq \sum_{i=1}^m \int_{S^2} |\nabla v^i|^2, \]

our procedure of construction must stop after finite steps, i.e. we can find a sequence of r^k_0 such that for any \epsilon > 0, there exist \delta_0 and R such that for any \lambda ∈ [Rr^k_0, \delta_0]

\[ \int_{B_{2R} \setminus B_\lambda} |\nabla u_α|^2 < \epsilon. \quad (A.3) \]

**References**

