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On Extreme Points of Convex Sets

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SUMMARY

A convex subset K of a vector space E over the field of real numbers is *linearly bounded (linearly closed)* if every line intersects K in a bounded (closed) subset of the line. A *hyperplane* is the set of $x \in E$ that satisfy a linear equation $f(x) = c$, where f is a linear functional and c is a real number.

A main, but not the only, purpose of this note is to establish the following simple theorem, inspired in part by an interesting observation of Karlin's [1], and, in part, by certain anticipated applications.

MAIN THEOREM. *Let L be the intersection of a linearly closed and linearly bounded convex set K with n hyperplanes. Then every extreme point of L is a convex combination of at most $n + 1$ extreme points of K.*

This theorem often simplifies the problem of finding the minimum or maximum of a linear functional restricted to the intersection of a convex set with those vectors that satisfy a given finite number of linear equalities. Such problems, in various guises, arise in many investigations; sometimes the convex set is a set of (probability) measures, sometimes a set of matrices.

Though it costs us something in simplicity and directness of argument, we present the proof as a sequence of lemmas, some of which we label as theorems and corollaries, in order also to make a small contribution to the study of general faces of convex sets.

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I. Internal Sets

Let us recall that if x and y are two distinct points in a real vector space, E, then the set of points, $\lambda x + \mu y$, with $\lambda > 0, \mu > 0$, and $\lambda + \mu = 1$, is called the open interval determined by the *end points* x and y. Following Bourbaki, [2, p. 66], a point, z, in a convex set, K, is said to be an *internal point* of K, if, for every $x \in K$ with $x \neq z$, there is a $y \in K$, such that z is in the open interval determined by x and y. We say that a convex set, S, is an *internal* set if every point of S is an internal point of S. It is easy to see that S is an internal set if, and only if, for every pair of distinct points z_1 and z_2 that are elements of S, there is an open interval, I, such that $z_1 \in I$, $z_2 \in I$, and $I \subset S$.

It is not true that the convex hull of the union of two internal sets is an internal set. But, it is true in the special case that the two sets are two intersecting open intervals. From this it easily follows that the convex hull of the union of any two intersecting internal sets is an internal set. This in turn quickly implies:

(1.1) Sublemma. *Let $S_1, \cdots, S_n$ be a finite number of internal sets with the property that for each i, $2 \leq i \leq n$, S_i has a non empty intersection with the union of the S_j for $j < i$, then the convex hull of the union of all the S_i is an internal set.*

The proof of the following is now immediate.

(1.2) Sublemma. *Let S_α be a collection of sets with the property that for every finite collection of subscripts $\alpha_1, \cdots, \alpha_n$, the convex hull of the union of $S_{\alpha_1}, \cdots, S_{\alpha_n}$, is an internal set. Then the convex hull of the union of all the sets, S_α, is an internal set.*

As a corollary one has:

(1.3) Lemma. *Given any collection of internal sets with a non empty intersection, the convex hull of their union is an internal set.*

Suppose that S is an internal set, K is convex and $S \subset K$. Then the collection of all internal subsets A, of K, such that $S \subset A$ has a union, whose convex hull is, in view of (1.3), also an internal set.

Therefore, we have established:

(1.4) Proposition. *For each convex set K and each internal set, S, such that $S \subset K$, there exists a unique internal set, F, such that (1), $F \subset K$; and (2), if G is any internal set such that $S \subset G \subset K$, then $G \subset F$.*

If in (1.4) one lets S consist of a single point one obtains:

(1.5) *For each element, x, of a convex set, K, there is a unique largest internal subset, F, of K such that $x \in F$.*

This set will be called the *internal component* of x, or *the internal component of x in K*, and will be designated by $F(x)$, or by $F(x, K)$, when greater precision of notation is needed. It is trivial to see that if $y \in F(x)$, then $F(y) = F(x)$. This implies that

(1.6) *For every x and y, the internal component of x and the internal component of y, are disjoint, or else they are identical.*

II. Faces of a Convex Set

Let K be convex. A subset, F, of K is said to be a *face* of K if (i) F is convex, and (ii) for every $x \in F$, and every open interval I, if $x \in I \subset K$, then $I \subset F$.

Two kinds of faces of K are particularly interesting: (1) those obtained by intersecting K with an affine variety of support to K; these have been called *facettes* of K by Bourbaki [2, p. 86]; (2) the *elementary faces* of K, where a face, F, of K will be said to be elementary provided that it is an internal set. The word "elementary" is used because it will soon be shown that every face of K is the union of a unique disjoint collection of elementary faces of K.

Since the intersection of any collection of faces of K is again a face of K, it follows that for each subset, S, of K, among the faces of K that contain S, there is a smallest.

(2.1) Theorem. *Let x be an element of a convex set, K. Then the smallest face of K that contains x is identical with the largest internal subset of K that contains x.*

Proof. Let $x \in K$, and let F be the largest internal subset of K that contains x. We first show that F is a face of K. Let $y \in F$, and let I be an open interval such that $y \in I$ and $I \subset K$. We must show that $I \subset F$. If $y = x$, then I is some internal subset of K that contains x. Therefore $I \subset F$. If $y \neq x$, then there exists an open interval I_1 such that $x \in I_1$, $y \in I_1$, and $I_1 \subset K$. Moreover I_1 and I have a point in common, namely y. Therefore the convex hull of I_1 and I is an internal set that contains x. Therefore it is a subset of F. Hence $I \subset F$. Thus F is a face of K. Now let G be any face of K that contains x. We want to show that $F \subset G$. Let $y \in F$. Then there is an open interval I such that $x \in I$, $y \in I$, and $I \subset K$. Since G is a face of K, $I \subset G$. Therefore $y \in G$. This completes the proof.

For *an example of interest*, observe that if K is the set of all countably additive probability measures, x, defined on some σ-field, $\mathcal{U}$, of subsets of a set, U, then, for every x, the smallest face of K that contains x is the set of all probability measures, y, such that (i) y and x are absolutely continuous with respect to each other, and (ii) there is a positive constant k, such that for every $A \in \mathscr{U}$, $x(A) \leq ky(A) \leq k^2x(A)$. (Of course (i) is redundant in the light of (ii)).

The following three rather similar propositions are easy corollaries to Theorem (2.1).

(2.2) *Every internal subset, S, of K is a subset of one and only one elementary face of K. This face is the smallest face of K that contains S. In particular every point, x, of K, is an element of one and only one elementary face of K, namely the smallest face of K that contains x.*

(2.3) *Every face of K is a union of a unique collection of disjoint elementary faces of K. In particular, the elementary faces of K form a partition of K.*

(2.4) *A nonempty convex set, K, has precisely one face, if and only if, it is an internal set.*

III. The Smallest Face of K Generated by a Subset, S, of K

(3.1) *For any convex subset, S, of K, $\bigcup_{x \in S} F(x)$ is a convex set.*

Proof. Let $y_1 \in F(x_1)$, and let $y_2 \in F(x_2)$, with $y_1 \neq y_2$, and $x_1 \in S$, and $x_2 \in S$. Let $y = \lambda y_1 + (1 - \lambda) y_2$, with $\lambda > 0$. It is necessary only to see that there exists an $x \in S$ such that $y \in F(x)$. There are four cases depending on whether or not $x_i = y_i$, $i = 1, 2$. The trivial case is $x_i = y_i$, $i = 1, 2$. In this event $y = \lambda x_1 + (1 - \lambda) x_2$. Since S is convex, $y \in S$, and, certainly $y \in F(y)$. We now outline the proof for the case $x_1 \neq y_1$ and $x_2 \neq y_2$, the other two cases being similar and even easier. There exist open intervals I_1 and I_2 with $x_j \in I_j$, $y_j \in I_j$ and $I_j \subset K$, $j = 1, 2$. Let L be the set of internal points of the convex hull of $I_1 \cup I_2$. Then, for any x in the open interval determined by x_1 and x_2, $x \in L$. Moreover $y \in L$. Since x and y are members of some internal set, $y \in F(x)$. This completes the proof.

From (3.1) easily follows

(3.2) *For any convex subset S of K, $\bigcup_{x \in S} F(x)$ is the smallest face of K that contains S.*

For any set S, *let $C(S)$ be the convex hull of S*, that is, the smallest convex set that contains S as a subset. Of course, if $S \subset K$, the smallest face of K that contains S is the same as the smallest face of K that contains $C(S)$. Therefore one obtains:

(3.3) *For any subset S of K, $\bigcup_{x \in C(S)} F(x)$ is the smallest face of K that contains S.*

IV. Faces of the Intersection of Two Convex Sets

(4.1) *Let K and L be convex sets, $K \subset L$. Let $S \subset K$. Then the smallest face of K that contains S is a subset of the smallest face of L that contains S.*

Proof. Every internal subset of K is an internal subset of L. Therefore, for each x, each internal subset of K that contains x is an internal subset of L that contains x. This implies that the largest internal subset of K that contains x is some internal subset of L that contains x. Hence for every $x \in K$, $F(x, K) \subset F(x, L)$. In particular, this is true for every x in the convex hull of S. Now apply (3.3) to complete the proof.

(4.2) *The intersection of two internal sets is an internal set.*

Proof. Let $x_1 \neq x_2$, $x_j \in X_1 \cap X_2$ where X_j are internal sets, $j = 1, 2$. There exists an open interval I_1 such that $x_j \in I_1 \subset X_1$, $j = 1, 2$; and there exists an open interval I_2 such that $x_j \in I_2 \subset X_2$, $j = 1, 2$. Let $I = I_1 \cap I_2$. Then $x_j \in I \subset X_1 \cap X_2$, $j = 1, 2$. This completes the proof.

(4.3) Theorem. *Let K be the intersection of two convex sets K_1 and K_2, and suppose that $x \in K$. For $i = 1$ and $i = 2$, let F_i be the smallest face of K_i that contains x. Similarly, let F be the smallest face of K that contains x. Then $F = F_1 \cap F_2$.*

Proof. (4.1) implies that $F \subset F_1 \cap F_2$. (4.2) implies that $F_1 \cap F_2$ is some internal set that contains x. Since $F_1 \cap F_2 \subset K$, it is a subset of the largest internal subset of K that contains x. That is, in view of (2.1), $F_1 \cap F_2 \subset F$. This completes the proof.

An easy corollary of (4.3), essentially a reformulation of it, is:

(4.4) *Let $F \subset K = K_1 \cap K_2$, where K_1 and K_2 are convex. Then F is an elementary face of K, if and only if, F is the intersection of an elementary face, F_1, of K_1 with an elementary face, F_2, of K_2. Moreover F_i is unique, and is the smallest face of K_i that contains F, $i = 1$, and $i = 2$.*

Clearly F_i would not necessarily be unique were it not assumed to be elementary.

Of course it is trivial that if F_i is *any* face of K_i, then $F_1 \cap F_2$ is a face of $K_1 \cap K_2$. However, the converse is not true. That is,

(4.5) *There exist convex sets K_1 and K_2 and a face F of $K_1 \cap K_2$ such that for no faces F_i of K_i does F equal the intersection of F_1 with F_2.* This phenomenon can occur even in a vector space of two real dimensions, as the interested reader may enjoy verifying for himself.

The special case of (4.4) in which one of the K_i is an internal set is simpler to formulate, and does arise, for example when one of the K_i is a hyperplane.

(4.6) *Let K be a convex set, suppose that H is an internal set and that F is a subset of $H \cap K$. Then F is an elementary face of $H \cap K$, if and only if, F is the intersection of H with an elementary face, G, of K. Moreover G is unique, and is the smallest face of K that contains F.*

As remarked in (4.5), (4.4) cannot be improved so as to be applicable to faces that are not necessarily elementary. However, (4.6) does yield:

(4.7) *Let K be convex, and let H be an internal set. Then F is a face of $H \cap K$, if and only if, F is the intersection of H with a face, G, of K.*

PROOF. The "if" part is trivial. So suppose that F is a face of $H \cap K$. Let F_α be the collection of elementary faces of $H \cap K$ that are subsets of F. By (2.3), $F = \bigcup F_\alpha$. By (4.6) there exist elementary faces, G_α, of K such that $F_\alpha = H \cap G_\alpha$. Let $G = \bigcup G_\alpha$. Then

$$F = \bigcup F_\alpha = \bigcup (H \cap G_\alpha) = H \cap (\bigcup G_\alpha) = H \cap G.$$

To verify that G is a face of K, first observe that $G = \bigcup F(x, K)$ as x ranges over F. Then apply (3.2) to see that G is a face of K, and thereby complete the proof.

V. THE SKELETONS OF CONVEX SETS

Let α be a cardinal number. A vector subspace, S, of a real vector space, E, has (Hamel) dimension, α, if α is the cardinality of a basis for S. Any translate of the subspace S is called an affine variety and has (affine) dimension α. A convex set will be said to have *(affine) dimension* α if the smallest affine variety containing it has dimension α. In particular, every face of a convex set K has a dimension. Modifying a notion borrowed from combinatorial topology, we define for each cardinal α, the *α-skeleton* of K as the union of all faces of K that have dimension less than or equal to α. The 0-dimensional skeleton of K is, of course, the set of *extreme points* of K.

For another example, if K is a convex cone, then a point is in the one dimensional skeleton of K, if and only if, it is on some *extreme generator* as defined by Bourbaki [2, p. 82].

(5.1) THEOREM. *Let K be the intersection of two convex sets K_1 and K_2 and let $x \in K$. Then x is an extreme point of K, if and only if, x is the only point in both $F_1(x)$ and $F_2(x)$, when $F_i(x)$ is the smallest face of K_i that contains x.*

PROOF. Immediate from (4.3).

Of Course, (5.1) can trivially be generalized to:

(5.2) *Let K be the intersection of n convex sets K_i, $i = 1, \cdots, n$. Then x is an extreme point of K if, and only if, x is the only point in all $F_i(x)$.*

Let S be a vector subspace of E. The quotient space, E/S, has a dimension called the codimension of S in E. Given two affine varieties A_1 and A_2 with $A_1 \subset A_2$, *the codimension of* A_1 *in* A_2 is the codimension of S_1 in S_2 where S_i is the unique subspace of which A_i is a translate. Finally, given two convex sets K_1 and K_2 with $K_1 \subset K_2$, the *codimension of* K_1 *in* K_2 is the codimension of A_1 in A_2 where A_i is the affine variety generated by K_i.

It is well known that if S_1 and S_2 are subspaces, then the codimension of $S_1 \cap S_2$ in S_2 equals the codimension of S_1 in $S_1 \vee S_2$, where $S_1 \vee S_2$ is the smallest convex set containing both S_1 and S_2. This easily implies that if A_1 and A_2 are affine varieties with a nonempty intersection, then the codimension of $A_1 \cap A_2$ in A_2 equals that of A_1 in $A_1 \vee A_2$. This is not true for arbitrary convex sets A_1 and A_2. However, for internal sets this fact does hold. Namely, it is easy to prove:

(5.3) *Let* K_1 *and* K_2 *be internal sets with a nonempty intersection. Then the codimension of* $K_1 \cap K_2$ *in* K_2 *equals that of* K_1 *in* $K_1 \vee K_2$.

(5.4) *If a hyperplane,* H, *has a nonempty intersection with an internal set,* F, *and if* $H \cap F$ *is* k*-dimensional, then* F *is at most* $(k + 1)$*-dimensional.*

(5.5) THEOREM. *Let* K *be a convex set and let* H *be an internal set of codimension* α *in the real vector space* E. *Then, for all* β, *the* β*-skeleton of* $K \cap H$ *is a subset of the* $(\alpha + \beta)$*-skeleton of* K.

PROOF. Let F be an elementary face of $K \cap H$ whose dimension does not exceed β, and, let G be the smallest face of K such that $F \subset G$. Then, according to (4.6), G is an internal set, and $G \cap H = F$. Thus, the codimension of F in $G =$ the codimension of $G \cap H$ in $G =$ the codimension of H in $G \vee H \leq$ the codimension of H in $E = \alpha$. The dimension of $G =$ the dimension of F plus the codimension of F in $G \leq \beta + \alpha$. That is, G is part of the $(\alpha + \beta)$-skeleton of K, and, consequently so is F. This completes the proof.

If $\beta = 0$, the β-skeleton is the same as the set of extreme points. In this case (5.5) can be reformulated as

(5.6) *Let* K *be convex and let* H *be an internal set of codimension* α *in* E. *Then the extreme points of* $K \cap H$ *are a subset of the* α*-skeleton of* K.

Some special cases of (5.6) are

(5.7) *If* H *is an open halfspace, or the intersection of a finite number of open halfspaces, then every extreme point of* $K \cap H$ *is an extreme point of* K.

(5.8) *If H is the intersection of finite number, n, of hyperplanes, then every extreme point of $H \cap K$ is an element of the n-dimensional skeleton of K.*

If in (5.5) one lets $\alpha = 1$, one obtains

(5.9) *Let H be a hyperplane. Then, for every β, the β-skeleton of $K \cap H$ is a subset of the $(\beta + 1)$-skeleton of K.*

VI. Linearly Closed and Linearly Bounded Sets

A set K is linearly closed if every line, l, intersects K in a closed subset of l. The intersection of all linearly closed sets that contain a set, K, is said to be *the linear closure of* K.

(6.1) Theorem. *The linear closure of a face of a linearly closed convex set, K, is a face of K.*

We leave the proof of (6.1) to the interested reader.

(6.2) *Let K be a linearly closed and linearly bounded convex set. Then for each integer $n > 1$, and every x in the n-dimensional skeleton of K, x is a convex combination of at most $n + 1$ extreme points of K.*

Proof. Let x be in the n-skeleton of K. Then for some face, F, of K of dimension not exceeding n, $x \in F$. The linear closure, $\bar{F}$, of F can be seen to be a compact convex set of dimension not exceeding n, and $x \in \bar{F}$. As shown in [1], x is a convex combination of at most $n + 1$ extreme points of $\bar{F}$. By (6.1), $\bar{F}$ is a face of K. Therefore every extreme point of $\bar{F}$ is an extreme point of K. This completes the proof.

The main theorem is an immediate corollary to (5.8) and (6.2).

References

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