On the Shimura Lift, après Selberg

BARRY A. CIPRA*

St. Olaf College, Department of Mathematics, Northfield, Minnesota 55057

Communicated by H. Zassenhaus
Received June 11, 1987

INTRODUCTION

The “Shimura lift” is a family of maps which, by design, send modular forms of half-integral weight to forms of integral weight. The lifts are defined via Fourier/Mellin series. They were defined and first studied by Shimura [1], who proved the main result: cusp forms lift to cusp forms (except for weight $\frac{3}{2}$; a more precise version is stated below).

Shimura’s results have been expanded upon by many researchers, including Niwa, Shintani, Flicker, Kohnen, and Waldspurger [2–6]. Waldspurger’s deep and definitive results played a key role in Tunnell’s remarkable solution to the “congruent number” problem [7]. Shimura has also investigated the subject further, most recently with an article on the lift for Hilbert modular forms [8].

Interestingly, a version of the Shimura lift was discovered much earlier by Selberg, but never published. Briefly stated, Selberg found that an eigenform times a theta function “lifts” to, essentially, the square of the eigenform. Thus not every cusp form gets lifted, but those that do are actually identified.

Selberg’s result is especially interesting because, in contrast to the difficult analytic proof in Shimura’s paper, its proof is elementary; aside from certain standard facts for modular forms, the proof is a simple combinatoric derivation.

In this paper we generalize Selberg’s version in two respects: we treat arbitrary level and character, and we use a large class of theta functions. Our version may be paraphrased as follows: a newform times a theta function with character “lifts” to, essentially, a twist (by the character) of the “square” of a newform. By “square” of $f$, we mean either $f^2$ or $ff'$. The proof remains elementary.

Notation. $q = e(z) = \exp(2\pi iz)$.

* Present address: 305 Oxford St., Northfield, MN 55057.
1. Preliminaries

**Definition.** Let $F(z) = \sum_{n=1}^{\infty} b(n)q^n$. Let $\chi$ be a character mod $4N$, let $t$ be a square-free positive integer, and let $\lambda \geq 1$ be an integer. We define

$$S_\lambda(F)(z) = \sum_{n=1}^{\infty} A_\lambda(n)q^n$$  \hspace{1cm} (1.1)

by

$$\sum_{n=1}^{\infty} A_\lambda(n)n^{-s} = L(s - \lambda + 1, \chi_{t}^{(\lambda)}) \sum_{m=1}^{\infty} b(tm^2)m^{-s},$$  \hspace{1cm} (1.2)

where $\chi_{t}^{(\lambda)}(m) = \chi(m)(-1/m)^{\lambda}(t/m)$ is a character mod $4Nt$. (When $t = 1$, we write $\chi_{1}^{(\lambda)} = \chi^{(\lambda)}$.)

**Shimura's result.** Suppose $F \in S_{\lambda+1/2}(4N, \chi)$ (where $\chi(-1) = 1$). Then

$$S_\lambda(F) \in \begin{cases} G_{2\lambda}(2N, \chi^2) & \text{if } \lambda = 1, \\ S_{2\lambda}(2N, \chi^2) & \text{if } \lambda > 1. \end{cases}$$

Moreover, $S_\lambda$ "commutes" with the Hecke algebra:

$$S_\lambda(T(p^2)F) = T(p)S_\lambda(F).$$  \hspace{1cm} (1.3)

Furthermore, if $F$ is an eigenform for all $T(p^2)$ (including those dividing the level), then for any square-free positive integers $\tau$ and $t$,

$$b(\tau)S_\lambda(F) = b(t)S_\lambda(F).$$  \hspace{1cm} (1.4)

**Remarks.** Shimura's result was in fact slightly weaker than our statement of it. Shimura did not prove that $S_\lambda(F)$ transformed at level $2N$. Also, he had to place a very weak condition on $F$, namely that it be a Hecke eigenform for a very few operators $T(p^2)$. However, subsequent work of Niwa [2], Cipra [9], and others established the level as $2N$ and removed the condition entirely.

For complete definitions of $S_{\lambda+1/2}$ and $T(p^\lambda)$, see Shimura [1].

**Selberg's Version.** Suppose $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda}(1, 1)$ is a normalized eigenform for all Hecke operators $T(p)$ (i.e., $T(p)f = a(p)f$). Define

$$F(z) = f(4z)\theta(z) \in S_{\lambda+1/2}(4, 1),$$  \hspace{1cm} (1.5)
where

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^n \in G_{1/2}(4, 1). \quad (1.6)$$

Then

$$S_1(F)(z) = f^2(z) - 2^k f^2(2z) \in S_{2,k}(2, 1). \quad (1.7)$$

Remarks. Selberg's version is stronger in that it identifies the lift. It is weaker in that it does not lift (in general) everything in the space $S_{k,1/2}$, but only those forms obtainable by letting the Hecke algebra act on linear combinations of $f\theta$'s for newforms $f$. Since $f\theta$ is not likely to be an eigenform, this could fill out the entire space, but not necessarily. There are non-empty spaces $S_{\lambda,1/2}$ for which $S_{\lambda}$ is empty (low values of $\lambda$, for example).

Selberg's version is also weaker in that it only treats the lift for $t = 1$.

2. NEWFORMS AND THETA FUNCTIONS

Newforms. Let $f(z) = \sum_{n=-\infty}^{\infty} a(n)q^n \in S_{k}(N, \chi)$ (where $\chi$ is a character mod $N$ with $\chi(-1) = (-1)^k$) be a normalized Hecke eigenform for all $T(p)$. We call $f$ a newform if it is not "descended" from forms of lower level, i.e., if $f$ cannot be written as a linear combination of cusp forms $g(d')$ where $g \in S_{k}(d', \chi)$ with $dd' \mid N$ and $\chi \mod d'$ inducing $\chi \mod N$.

What is important here for us are the following multiplicative relations for the coefficients of newforms.

**Proposition.** Let $f(z) = \sum_{n=-\infty}^{\infty} a(n)q^n \in S_{k}(N, \chi)$ (where $\chi$ is a character mod $N$ with $\chi(-1) = (-1)^k$) be a newform. Then

$$a(m)a(n) = \sum_{d \mid (m,n)} \chi(d)d^{k-1}a(mn/d^2) \quad (2.1)$$

and

$$a(mn) = \sum_{d \mid (m,n)} \mu(d)\chi(d)d^{k-1}a(m/d)a(n/d), \quad (2.2)$$

where $\mu$ is the Möbius function.

**Proof.** Equation (2.1) follows from the usual definitions of $T(n)$ (see [10, p. 329], for instance). Equation (2.2)—which we shall call Selberg's inversion—is easily derived from (2.1):
ON THE SHIMURA LIFT, APRÈS SELBERG

\[
\sum_{d|\gcd(m,n)} \mu(d) \chi(d) d^{k-1} a(m/d) a(n/d)
\]

\[
= \sum_{d|\gcd(m,n)} \mu(d) \chi(d) d^{k-1} \sum_{\delta|\gcd(m/d,n/d)} \chi(\delta) \delta^{k-1} a(mn/d^2 \delta^2)
\]

\[
= \sum_{d|\gcd(m,n)} \mu(d) \chi(d\delta) (d\delta)^{k-1} a(mn/(d\delta)^2)
\]

\[
= \sum_{D|\gcd(m,n)} \left( \sum_{d|D} \mu(d) \right) \chi(D) D^{k-1} a(mn/D^2)
\]

\[
= a(mn),
\]

since \( \sum_{d|D} \mu(d) = 0 \) if \( D > 1 \).

**Theta Functions.** Let \( \psi \) be a primitive character mod \( r \), and let \( v = 0 \) or \( 1 \) so that \( \psi(-1) = (-1)^v \). Define a theta function

\[
h_\psi(z) = \frac{1}{2} \sum_{-\infty}^{\infty} \psi(m) m^v q^{m^2}.
\]

(2.3)

The basic result is that \( h_\psi \in G_{v+1/2}(4r^2, \psi^{(v)}) \) (see [1] for the complete statements and proofs).

It will be convenient to alter the definition of \( h_\psi \) slightly when \( \psi \) is the trivial character mod 1, by deleting the factor \( \frac{1}{2} \): \( h_\psi(z) = \sum_{-\infty}^{\infty} q^{m^2} = \theta(z) \) in that case.

### 3. Main Theorem

We now give our generalization of Selberg's result.

**Theorem.** Let \( f(z) = \sum_{n=1}^{\infty} a(n) q^n \in S_k(N, \chi) \) be a newform, where \( \chi \) is a character mod \( N \) with \( \chi(-1) = (-1)^k \). Let \( \psi \) be a primitive character mod \( r \), where \( r \) is a prime power: \( r = p^m \) (with \( p = 1 \) if \( r = 1 \)). Take \( \mu \geq m \) and define

\[
F(z) = f(4p^m z) h_\psi(z).
\]

(3.1)

Define also

\[
g(z) = \sum_{n=1}^{\infty} c(n) q^n = \begin{cases} f(z) f(p^m z) & \text{if } v = 0, \\
(1/2\pi i) [f'(z) f(p^m z) - p^m f(z) f'(p^m z)] & \text{if } v = 1,
\end{cases}
\]

(3.2)
and

\[ G(z) = \sum_{n=1}^{\infty} \psi(n) c(n) q^n. \]  

(3.3)

Let \( N' = \text{lcm}(Np^\mu, r^2) \). Then \( F \in S_{k+v+1/2}(4N', \chi^{(k)} \psi^{(\nu)}) \), \( g \in S_{2(k+v)}(N', \chi^2) \), \( G \in S_{2(k+v)}(N'r, \chi^2 \psi^2) \), and

\[ S_1(F) = G(z) - 2^{k+v+1} \chi(2) \psi(2) G(2z) \]  

(3.4)

belongs to \( S_{2(k+v)}(2N'r, \chi^2 \psi^2) \).

**Proof.** The assertions regarding which spaces the functions belong to all follow from definitions and elementary properties of modular forms. The only thing of substance to prove is (3.4). For simplicity, we shall follow the proof when \( r > 1 \). One easily sees that \( F(z) = \sum b(n) q^n \) with

\[ b(n) = \frac{1}{2} \sum_{m=-\infty}^{\infty} a \left( \frac{n-m^2}{4p^\mu} \right) \psi(m) m^\nu. \]  

(3.5)

It is also easy to check that

\[ c(n) = \sum_{m=-\infty}^{\infty} a(m) a(n-p^\mu m)(n-2p^\mu m)^\nu. \]  

(3.6)

Now from (3.5), we have

\[ b(n^2) = \frac{1}{2} \sum_{m=-\infty}^{\infty} a \left( \frac{(n-m)(n+m)}{4p^\mu} \right) \psi(m) m^\nu. \]  

(3.7)

Since \( \psi(p) = 0 \), it is easily seen that the sum splits into two pieces: those terms with \( m \equiv n \mod 2p^\mu \), and those with \( m \equiv -n \mod 2p^\mu \). However, since \( \psi(-1) = (-1)^\nu \), each piece is the same. Thus

\[ b(n^2) = \sum_{m=-\infty}^{\infty} a(m(n-p^\mu m)) \psi(n)(n-2p^\mu m)^\nu. \]  

(3.8)

(It is here we use the assumption \( r \mid p^\mu \); otherwise we get \( \psi(n-2p^\mu m) \). Also, (3.8) is now correct even for \( r = p = 1 \); the sum (3.7) does not split into two pieces in that case, but there is not a \( \frac{1}{2} \) out front either.)
We now apply Selberg’s inversion to get

\[
b(n^2) = \sum_{m = -\infty}^{\infty} \sum_{d \mid (m,n)} \mu(d) \chi(d) d^{k-1} a(m/d) a \left( \frac{n - p^m m}{d} \right) \psi(n)(n - 2p^m m)^v
\]

\[
= \sum_{d \mid n} \mu(d) \chi(d) \psi(d) d^{k + v - 1} \psi(n/d) \sum_{m = -\infty}^{\infty} a(n) a \left( \frac{n}{d} - p^m m \right) \left( \frac{n}{d} - 2p^m m \right)^v
\]

\[
= \sum_{d \mid n} \mu(d) \chi \psi(d) d^{k + v - 1} \psi(n/d)(n/d), \quad (3.9)
\]

using (3.7). Thus

\[
\sum_{n = 1}^{\infty} b(n^2) n^{-s} = \sum_{n = 1}^{\infty} \sum_{d \mid n} \mu(d) \chi \psi(d) d^{k + v - 1} \psi(n/d) c(n/d)n^{-s}
\]

\[
= \sum_{d = 1}^{\infty} \mu(d) \chi \psi(d) d^{k + v - 1 - s} \sum_{\delta = 1}^{\infty} \psi(\delta) c(\delta) \delta^{-s}
\]

\[
= L(s - (k + v) + 1, \chi \psi)^{1} \sum_{\delta = 1}^{\infty} \psi(\delta) c(\delta) \delta^{-s}. \quad (3.10)
\]

Therefore

\[
\sum_{n = 1}^{\infty} A_1(n) n^{-s} = L(s - (k + v) + 1, (\chi^{(k)} \psi^{(v)})^{(k + v)}) \sum_{n = 1}^{\infty} b(n^2) n^{-s}
\]

\[
= \frac{L(s - (k + v) + 1, (\chi^{(k)} \psi^{(v)})^{(k + v)})}{L(s - (k + v) + 1, \chi \psi)} \sum_{n = 1}^{\infty} \psi(n) c(n)n^{-s}. \quad (3.11)
\]

Now \((\chi^{(k)} \psi^{(v)})^{(k + v)} = \chi \psi\), but the modulus is \(4N\), not just \(N\). Thus the quotient of the \(L\)-series leaves a factor \((1 - \chi(2) \psi(2) 2^{k + v - 1 - s})\), and we get

\[
\sum_{n = 1}^{\infty} A_1(n) n^{-s} = (1 - \chi(2) \psi(2) 2^{k + v - 1 - s}) \sum_{n = 1}^{\infty} \psi(n) c(n)n^{-s}, \quad (3.12)
\]

so that

\[
S_1(F) = \sum_{n = 1}^{\infty} \psi(n) c(n) q^n - \chi(2) \psi(2) 2^{k + v - 1} \sum_{n = 1}^{\infty} \psi(n) c(n) q^{2n}, \quad (3.13)
\]

which is the desired result.
REFERENCES