Multifractal Properties of Interest Rates in Bond Market

Zhongxing Wang\textsuperscript{a,b}, Yan Yan\textsuperscript{a,b,*}, Xiaosong Chen\textsuperscript{c}

\textsuperscript{a}Research Center on Fictitious Economy and Data Sciences, Chinese Academy of Sciences, Beijing, 100190, PR China
\textsuperscript{b}School of Management in Graduate University, University of Chinese Academy of Sciences, Beijing, 100080, PR China
\textsuperscript{c}Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, 100190, PR China

Abstract

In the article, we investigated the multifractal properties of interest rates, which are the core variables in bond market. In a large sample including nearly all the interest rates in China bond market, we found a clear empirical evidence of long-range correlations and multifractality. Furthermore, by tracking the shape of multifractal spectra, we found the dynamics of large price fluctuation is significantly different from that of the small ones, and the spectrum widths of interest rates are related the maturity terms and market development stage. Finally, we destroyed the long-range memories by shuffle the data to detect the underlying mechanisms of multifractality and identified the non-linear temporal correlation to be the major cause.

1. Introduction

As the measure of cost of money in modern financial market, interest rates are crucially important to nearly all countries in the world. Interest rates have become a hot research issue not only for its nature of value-measurement in bond market, but for its importance in monetary policy conduction. The change of interest rates is often cited as the signal of monetary policies tightening or easing. In this way, interest rates significantly influence money supply, lending, stock market, and real economy in the end [1].

In early, interest rates (prices of bonds) and prices of other financial properties are believed to follow random walk, thus price changes are assumed to obey Gaussian distributions. However, recent researches prove that price changes follow a complex distribution with a more obvious peak and fatter tails than Gaussian’s. These properties cannot be described by normal economic methods, and recent literature focus on some mathematical or physics methods [2-3]. Some economists and physicists found that interest rates reveal some complex properties, such as long range correlation or memory [4], [6-7], fractals/multifractals [4-5], chaos [8], and so on. Thereby, the interest rates dynamics may be better described by fractals first proposed by Mandelbrot who applied it to agricultural

* Corresponding author. Tel.: +0-108-268-0923
E-mail address: wzx9963@pku.edu.cn
commodity spot prices [9]. And fractal analysis has been widely used ever since. Literature proved returns of stock markets have monofractal properties or multifractal properties [10]. He and Chen [11] found that crude oil price also exhibit multifractal properties.

As monofractals cannot describe the multi-scale and subtle substructures of fractals in complex systems, many measures are applied to investigate the multifractality, such as height-height correlation function [12], partition function [13], multifractal detrended fluctuation analysis (MF-DMA, [14]), etc. What’s more, many empirical and theoretical researches seek to find out the cause of market multifractality. Usually, there are two major sources of multifractality which can be found in various time series: one is nonlinear temporal correlation for small and large fluctuations; the other is fat-tailed probability distribution of increments.

Although some pieces of empirical literature proved the existence of long-range correlation or monofractal properties in bond market, few results reveal interest rates may embody multifractal properties, not to mention the underlying mechanisms of multifractality or offer plausible explanation to this styled fact in bond market.

In our work, multifractal properties of interest rates are analyzed in a relatively large sample, which includes nearly all the important rates in China bond market. Instead of simply providing empirical evidence, we will investigate the underlying mechanisms of multifractality formation by group the interest rates, and we also proved plausible explanation of multifractality by shuffling procedure (the underlying long-range correlation is destroyed in this way). Our contribution to current literature can be summarized as follows: firstly, we provided a clear evidence of the existence of long-range correlation and multifractality in interest rates; secondly, we reveal the underlying information in multifractal properties by means of statistic; thirdly, we proposed a plausible explanation of multifractality by shuffling the data.

Our paper is organized as follows: the first part is an introduction of background of our research. The MF-DMA method we use is briefly introduced in part two. The monofractal and multifractal analysis are present in part three, and followed part is to study the underlying mechanisms of multifractal. The paper end with a conclusion in section 5.

2. Data and Methodology

Our sample is composed of more than 500 interest rates published by China Central Depository and Clearing company (CCDC) everyday, which includes nearly all the kinds of bonds traded in the interbank market, including not only short-term monetary rates, but also long-term rates, like corporate bonds rates and policy financial bonds rates, and so on. Since we study the fractal property in time scale, the timeline is as long as possible. The study period is from January 1, 2002 to January 11, 2016. The basic data is from Wind Co..

Let us suppose R(i) (i=1,2,...,T) to be the time series of interest rates, where T is the length of the series. The fluctuation is defined as

\[ X(t) = |R(t) - R(t-1)| \]

The “profile” is given by

\[ Y(t) = \sum_{i=1}^{t} (X(i) - \bar{X}) \]

Divide the profile (length of series is N) Y(t) into \( N_s = \lceil N/s \rceil \) non-overlapping segments of equal length s. Since the series length N may not be a multiple of the time scale s, a proper way is to repeat the method from the opposite end. Thereby, \( 2N_s \) segments are obtained altogether. And then the local trends \( p_v(i) \) are calculated by polynomial fit or moving average method. Some literature claimed that the moving average method is better than polynomial fit (), so the moving average method is applied in this paper. In every segment v, we use the original data minus the local trend and get the detrended time series.

\[ Y_s(i) = Y(i) - p_v(i) \]
Then the variance is given by

$$F^2_s(v) = \langle Y^2_s(i) \rangle = \frac{1}{s} \sum_{i=1}^{s} Y^2_s[(v-1)s+i],$$

The qth order detrended variance is calculated as follows

$$F(q,s) = \left[ \frac{1}{2N_s} \sum_{v=1}^{2N_s} F^2_s(v)^{q/2} \right]^{1/q}$$

When $q \neq 0$ and

$$F(0,s) = \exp[\frac{1}{4N_s} \sum_{v=1}^{m} \ln F^2(s,v)]$$

We then expect the following scaling relation

$$F(q,s) \propto s^{h(q)}$$

Through the least-squares fit, the slope of $\ln F(q,s)$ and $\ln s$ is the generalized Hurst exponent $h(q)$, if $h(q)$ is a constant which is independent on $q$, then the series is monofractal; or it is multifractal. When $q=2$, $h(2)$ is the well-known Hurst exponent. The variance order $q$ can range from $-\infty$ to $+\infty$. According to the standard multifractal formalism, when $q$ is positive the partition function is mainly influenced by large fluctuation $X$; when $q$ is negative, the partition function is mainly influenced by small value of $X$.

The multifractal scaling exponent $\sigma(q)$ can be used to characterize the multifractal nature, which reads

$$\sigma(q) = q h(q) - D_f,$$

where $D_f$ is the fractal dimension of the geometric support of the multifractal measure. For time series analysis, we have $D_f = 1$. It is easy to obtain the singularity strength function $\alpha(q)$ and the multifractal spectrum $f(\alpha)$ via the Legendre transform

$$\left\{ \begin{array}{l} \alpha(q) = d \sigma(q)/dq \\ f(\alpha) = q \alpha - \sigma(q) \end{array} \right.$$

3. Empirical results

With the MF-DMA method mentioned above, we detect the fractal properties of interest rates in China bond market. The variance order $q$ is range from -50 to +50 in our study, and the window size $s$ ranges from 20 to 200 with the computation interval 10.

3.1. Long range correlation

When $q=2$, the MF-DMA method is turn to be DMA method, and $h(2)$ turn to be Hurst exponent. Through previous part, when the time series have long range self-correlation, $\ln F(2,s)$ and $\ln s$ will have a linear relationship. We detected the power law correlation in the data sample and got the following result shown in Fig.1.

From the figures we can find the relationships are linear, which implies that the bond market does not obey random walk and there exist power-law relationships in self-correlation of all interest rates.
Fig. 2 shows the distribution of Hurst exponent in our study sample. The Hurst exponents are range from 0.63 to 0.95, larger than 0.5, which implies that the long-range correlation is positive. That is, a sudden shock in interest rates may have a long positive influence on itself in the future.

3.2. Multifractal properties

It is not enough to describe a time series in monofractal properties if they are actually multifractal. Thus we will detect the multifractal properties in this section. The variance order $q$ ranges from -50 to +50 with the interval 1. We calculate the generalized Hurst exponent $h(q)$ for each interest rate, and get the result as shown in Fig. 3. Obviously, $h(q)$ is not a constant according to $q$ for all the interest rates. This means that interest rates are
multifractal. We also calculate the multifractal spectrum $f(\alpha)$ according to the second part of our paper. The result is shown in Fig.4, which reflects that the relationship between $f(\alpha)$ and $\alpha$ is in an inverted U shape. And different interest rates have different multifractal properties.

![Figure 3](image3.png)  
Figure 3 $h(q)$ is not a constant according to $q$

![Figure 4](image4.png)  
Figure 4 spectrum of interest rates

According to the Legendre transform, we can obtain $q = df(\alpha)/d\alpha$. We thereby divided the multifractal spectrum into left and right-half part by maximum extreme value of $f(\alpha)$. The left-half corresponds to the section where $q>0$, which reflects the interest rates behavior influenced by the large fluctuation, while the right
one corresponding to the section where \( q < 0 \), influenced by the small one. We calculate the left- and right-half of spectrum for our sample and calculate the difference between left and right (left-right), the distribution of the differences is shown in Fig. 5. Most interest rates have a larger left-half than right, that is, there exist an unlinear response between large and small fluctuation for bond price behavior.

![Figure 5 the differences between left- and right-half of spectrum](image)

In order to describe the strength of multifractality more specifically, we define the width of the multifractal spectrum by

\[
\Delta \alpha = \alpha_{\text{max}} - \alpha_{\text{min}}
\]

The larger spectrum width, the stronger multifractality is. If a time series is monofractal, the width will be 0. And then study the spectrum widths of interest rates of different terms, as well as in different time.

Fig.6(a) shows the distribution of interest rates spectrum width for short-term (less than 1 year, red line), mid-term (1 to 5 years, green line), and long-term (longer than 5 years, blue line). The width of short-term is slightly larger than mid- and long-term, which implies that short-term bonds present stronger multifractal properties. Fig.6(b) discusses multifractality in other perspective. The interest rates sample is divided into two group according to its time: group 1 includes data from 2007 to 2012 (red line), and the other group is from 2011 to 2016 (blue line). The full sample is represent by green line. We can obviously find that multifractality is stronger in recent years compared with earlier years.
3.3. Sources of multifractality

In general, there are two major sources of multifractality which can be found in various time series. One is nonlinear temporal correlation for small and large fluctuations; the other is fat-tailed probability distribution of increments. To identify which causes constitute the major contributions of multifractality, we shuffled the original data to destroy the temporal correlation, or long-range memory in interest rates. If the generalized Hurst exponent \( h(q) \) or spectrum width \( \Delta \alpha \) does not change after we shuffle the series, it implies that there is no influence by nonlinear temporal correlation, and the multifractality mainly comes from non-Gaussian distribution.

The shuffling procedure consists of following steps: firstly, we generate pairs \((m, n)\) of random integer numbers, which satisfies \( m, n < N \), where \( N \) is the length of the time series to be shuffled; then, we interchange entries \( m \) and \( n \) of the time series. To ensure that ordering of entries in the time series is fully shuffled, we repeat the first and second steps for \( 20N \) times so that long-range or short-range memories, if any, will be destroyed.

The following figures show the self-correlation functions of one series of original data (Repo 7d) and its shuffled version. It is obvious that the shuffling procedure we mentioned above have destroyed the long-range correlation completely.

Together we compare the Hurst exponent \( h(2) \) of the two. The Hurst exponent has change from 0.76 to 0.51, that is, the shuffled series is close to random walk.

Also, we have compared the generalized Hurst exponent \( h(q) \) and spectrum width \( \Delta \alpha \) for the original data and shuffled one. Take Repo 7d for example, the result is shown in Fig.8. compared to original data, Repo 7d rate exhibits different properties of multifractality. From the point of spectrum width, the spectrum width become narrower for the shuffled data, which means that multifractality become much weaker after shuffling produce.

To get a more general result, we calculate the spectrum width of original and shuffled data for all the interest rates. The following figure shows difference of some main rates. For nearly all the rates, spectrum widths become narrower but not zero after shuffling procedure, which means the multifractality persists but much weaker after we shuffled the series. We can summarize that the major cause of multifractality in the bond market is due to long-range correlation, but is also influenced by non-Gaussian distribution.
Figure 7(a) absolute fluctuation of repo 7d and its autocorrelation function

Figure 7(b) shuffled data of repo 7d and its autocorrelation function
Conclusions

In this article, we investigated the multifractal properties in interest rates, which is an important but receive scant attention variable in financial market, and got the following findings:

Firstly, in a relatively large study sample, we found a clear evidence of fractal and multifractal features in bond market. There are long-range and positive correlations in interest rates.

Secondly, we found that the multifractal properties of interest rates might relate to their maturity term or market development stage.

Thirdly, by tracking the evolution of left- and right-half spectra, the dynamics of large fluctuations is significantly different from that of small ones.

Finally, we studied the underlying mechanisms of multifractality with a shuffling procedure. We found that multifractal strength is significantly weaken but remains, that is, we identified long-range correlation is the major cause of multifractality in bond market.

What we get from our study has significantly meaning to describe the dynamic features of interest rates, and it can also help to discover the potential motion of bond market. Of course, there are still some questions waiting to be answered. What can the multifractal properties do for our bond investment or monetary policy making is still need a lot of to do.
Figure 9 difference between spectrum width of original data and shuffled

References