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The Locating-Chromatic Number of Binary Trees

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Abstract

Let $G = (V, E)$ be a connected graph. The locating-chromatic number of $G$, denoted by $\chi_L(G)$, is the cardinality of a minimum locating coloring of the vertex set $V(G)$ such that all vertices have distinct coordinates. The results on locating-chromatic number of graphs are still limited. In particular, the locating-chromatic number of trees is not completely solved. Therefore, in this paper, we study the locating-chromatic number of any binary tree.

1. Introduction

The concept of locating-chromatic number for graphs was introduced by Chartrand et al. in 2002\cite{4} as a special case of the partition dimension notion. Let $G = (V, E)$ be a connected graph. Let $f$ be a vertex-coloring of connected graph $G$ and induces the color partition $\Pi = \{C_1, C_2, \ldots, C_k\}$ for some $k$. For a vertex $v$ of $G$, the color code of $v$ with respect to $\Pi$, denoted by $f_\Pi(v)$, is defined as the $k$-vector $(d(v, C_1), d(v, C_2), \ldots, d(v, C_k))$ where $d(v, C_i) = \min\{d(v, x) | x \in C_i\}$ for $1 \leq i \leq k$. If all vertices have distinct color codes, then $f$ is called a locating $k$-coloring of $G$. The minimum numbers of colors needed in a locating $k$-coloring of $G$ is called the locating-chromatic number of $G$, denoted by $\chi_L(G)$.

The locating-chromatic numbers of some well-known classes of graphs have been studied; however the results have not been complete. Chartrand et al. in\cite{4} determined the locating-chromatic number of some well-known classes of trees, namely paths and double stars. In 2003\cite{5}, they showed that for any integer $k \in [3, n]$ and $k \neq n - 1$, there is a tree of order $n$ with locating-chromatic number $k$. They also showed that no tree on $n$ vertices with locating-chromatic number $n - 1$. Further results were also obtained by Asmiati et al.\cite{1,2} in amalgamation of stars and firecrackers. Recently, Syofyan et al.\cite{6} and Welyyanti et al.\cite{7} have determined the locating-chromatic numbers of lobsters and...
2. Main Results

A binary tree is defined as a tree in which there is exactly one vertex of degree two, namely root vertex $x_0$, and each of the remaining vertices is of degree one or three. A complete binary tree is a binary tree which all leaves are on the same level or all leaves have same distance to the root vertex $x_0$. We denote a complete binary tree with diameter $2k$ by $BT(k)$, where $k \geq 1$. A non complete binary tree is a binary tree which some of its leaves have different distance to root vertex $x_0$. We denote a non complete binary tree with diameter $2k$ by $BT_{nc}(k)$, where $k \geq 1$.

A graph $BT(k)$ can be constructed recursively from two copies of $BT(k-1)$ by connecting their root vertices to a new vertex $x_0$. We denote the two copies of $BT(k-1)$ by $BT_1(k-1)$ and $BT_2(k-1)$. Next, for any integer $p$ and $q$, we denote $BT_p(k-q)$ as a subgraph $BT(k-q)$ of $BT_{\frac{k+q}{2}}(k+q+1)$. In this paper, every subgraph $BT(l)$ of $BT(k)$, where $l < k$, contains all leaves of $BT(k)$.

Let $N(u)$ be the set of neighbors of vertex $u$. For a coloring $f$ of $V(BT(k))$, define $f(N(u)) = \{f(v)|v \in N(u)\}$. Its easy to show that for $k = 1$ and $2$, $\chi_L(BT(k)) = 3$. Now we show the locating-chromatic number of $BT(k)$ for $k \geq 3$.

![Fig. 1. A locating 4-coloring of $BT(3)$](image)

**Theorem 1.** Let $BT(k)$ be a complete binary tree with $k = 3, 4$, and $5$. Then, $\chi_L(BT(k)) = 4$.

**Proof.** All trees with locating-chromatic number $3$ have characterized by Baskoro et al.\cite{3}. Since $BT(k)$ is not a graph with locating-chromatic number $3$ for $k = 3, 4$, and $5$, then $\chi_L(BT(k)) \geq 4$.

Now, we will show that $\chi_L(BT(k)) \leq 4$ for $k = 3, 4$, and $5$. We show that there is a locating $4$–coloring of $BT(k)$ for $k = 3, 4$, and $5$. For $k = 3$, we define a coloring $f_3 : V(BT(3)) \rightarrow \{1, 2, 3, 4\}$ as depicted in Fig. 1. We need only to consider every two vertices $u$ and $v$ where $f_3(u) = f_3(v)$ and $f_3(N(u)) = f_3(N(v))$. By the coloring $f_3$, there are only two vertices satisfying this condition (two vertices with bold labels). Their color codes are distinguished by the unique color used only once in $BT(3)$. Therefore, all vertices in $BT(3)$ have different color codes. Thus, $f_3$ is a locating $4$–coloring of $BT(3)$.

Now, for $k = 4$, we define a coloring $f_4 : V(BT(4)) \rightarrow \{1, 2, 3, 4\}$ as follows.

$$f_4(u) = \begin{cases} 
1, & \text{if } u = x_0, \\
f_3(u), & \text{if } u \in V(BT_1(3)), \\
(f_3(u) + 2) \mod 4, & \text{if } u \in V(BT_2(3)). 
\end{cases}$$

Next, we show that $f_4$ is a locating $4$–coloring. Again, we need only to consider every two vertices $u$ and $v$ where $f_4(u) = f_4(v)$ and $f_4(N(u)) = f_4(N(v))$. Observe three cases below.

**Case 1.** $u \in V(BT_i(2))$ and $v \in V(BT_j(2))$ for $i \neq j$.

By the coloring $f_4$, the possibilities for $u$ and $v$ are occurred for every pairs $i$ and $j$, where $i = 1$, $j = 2$ and $i = 3$, $j = 4$. The color codes of vertices $u$ and $v$ are different since they are distinguished by the unique color used only once in $BT_1(3)$ and $BT_2(3)$.
Case 2. $u \in V(BT_j(2))$ and $v \in V(BT_j(2))$ for some $i, j$.

By the coloring $f_4$, the possibilities for $u$ is a stem vertex in $BT_3(2)$ or $BT_4(2)$ and the possibilities for $v$ is the root vertex of $BT_2(3)$. The vertices $u$ and $v$ have different color codes distinguished by the unique color in $BT_2(3)$.

Case 3. $u \notin V(BT_j(2))$ and $v \notin V(BT_j(2))$ for some $i, j$.

By the coloring $f_4$, there is no possibilities for vertices $u$ and $v$ satisfying this condition as depicted in Fig. 2.

By three cases above, all vertices in $BT(4)$ have different color codes. Thus, $f_4$ is a locating 4–coloring of $BT(4)$.

Next, for $k = 5$, we define a coloring $f_5 : V(BT(5)) \to \{1, 2, 3, 4\}$ as follows.

$$f_5(u) = \begin{cases} 
3, & \text{if } u = x_0, \\
4, & \text{if } u \in V(BT_1(4)), \\
(f_4(u) + 1) \mod 4, & \text{if } u \in V(BT_2(4)). 
\end{cases}$$

Next, we show that $f_5$ is a locating 4–coloring. Similarly with the previous cases, we need only to consider every two vertices $u$ and $v$ where $f_5(u) = f_5(v)$ and $f_5(N(u)) = f_5(N(v))$. Observe three cases below.

Case 1. $u \in V(BT_j(2))$ and $v \in V(BT_j(2))$ for $i \neq j$.

By the coloring $f_5$, if their occurs then $i$ and $j$ are in different paritie. Observe that, the coloring $f_5$ implies the vertices in $BT_j(2)$ are colored by three colors for odd $i$ and the vertices in $BT_j(2)$ are colored by four colors for even $j$. By this condition, the vertices $u$ and $v$ have different color codes distinguished by the unique color used only once in $BT_j(3)$ for $j$ is odd.

Case 2. $u \in V(BT_j(2))$ and $v \notin V(BT_j(2))$ for some $i, j$.

By the coloring $f_5$, the minimum distance between the vertex $v$ to a vertex with the unique color in $BT_j(3)$ is smaller than the minimum distance between the vertex $u$ to a vertex with the unique color in $BT_1(3)$. Therefore, the color codes of vertices $u$ and $v$ are different.

Case 3. $u \notin V(BT_j(2))$ and $v \notin V(BT_j(2))$ for some $i, j$.

Since the coloring $f_5$ is a permutation of $f_4$, there is no possibilities for vertices $u$ and $v$ satisfying this condition.

By three cases above, all vertices in $BT(5)$ have different color codes. Thus, $f_5$ is a locating 4–coloring of $BT(5)$.

Theorem 2. If $a \geq 5$ and $k$ such that $3 + \sum_{i=1}^{k} (2 + \lfloor \frac{a-i}{4} \rfloor) \leq k \leq (2 + \sum_{i=0}^{a-4} (2 + \lfloor \frac{a-i}{4} \rfloor))$, then $\chi_L(BT(k)) \leq a$.

Proof. Let $f_5$ be a locating 4–coloring for a $BT(5)$ as defined in the proof of Theorem 1. We define a coloring for $BT(k)$ by the recursive definition for $k \geq 6$ as follows.
1. For $k = 3 + \sum_{i=1}^{a-4}(2 + \lceil \frac{a-i}{4} \rceil)$ define $f_k : (V(BT(k))) \rightarrow \{1, 2, \ldots, a\}$ as follows.

$$f_k(u) = \begin{cases} 
1, & \text{if } u = x_0, \\
f_{k-1}(u), & \text{if } u \in V(BT_1(k-1)), \\
f_{k-1}(u), & \text{if } u \text{ is the root vertex of } BT_2(k-1), \\
f_{k-2}(u), & \text{if } u \in V(BT_3(k-2)), \\
f_{k-2}(u), & \text{if } u \text{ is the root vertex of } BT_4(k-2), \\
f_{k-3}(u), & \text{if } u \in V(BT_7(k-3)), \\
(f_{k-3}(u) + (a - f_{k-3}(r))) \mod a, & \text{if } u \in V(BT_8(k-3)) \text{ and } r \text{ is the root vertex of } BT(k-3). 
\end{cases}$$

2. For $(4 + \sum_{i=1}^{a-4}(2 + \lceil \frac{a-i}{4} \rceil)) \leq k \leq (2 + \sum_{i=0}^{a-4}(2 + \lceil \frac{a-i}{4} \rceil))$, let $j = k - 3 + \sum_{i=1}^{a-4}(2 + \lceil \frac{a-i}{4} \rceil)$. Define a coloring $f_k : (V(BT(k))) \rightarrow \{1, 2, \ldots, a\}$ as follows.

$$f_k(u) = \begin{cases} 
 j + 2, & \text{if } u = x_0, \\
f_{k-1}(u), & \text{if } u \in V(BT_1(k-1)), \\
(f_{k-1}(u) + a - f_{k-1}(r)) \mod a, & \text{if } u \in V(BT_2(k-1)) \text{ and } r \text{ is the root vertex of } BT_1(k-1). 
\end{cases}$$

Next, we will show that the coloring $f_k$ is a locating $a$–coloring of $BT(k)$. For $k = 3 + \sum_{i=1}^{a-4}(2 + \lceil \frac{a-i}{4} \rceil)$, by the coloring $f_k$, every vertices in each $BT_i(k-2)$ colored by a locating $(a-1)$–coloring for $i = 1, 2, 3$. Now, we need only to consider every two vertices $u$ and $v$, where $u \in V(B_i(k-2))$ and $v \in V(B_i(k-2))$ for $i \neq j$ such that $f_k(u) = f_k(v)$ and $f_k(N(u)) = f_k(N(v))$. Note that the color $a$ is only used in $BT_4(k-2)$. Hence, the color codes of $u$ and $v$ are distinct since their distances to a vertex with color $a$. Therefore, all vertices of $BT(k)$ have different color codes and $f_k$ is a locating $a$–coloring of $BT(k)$ for $k = 3 + \sum_{i=1}^{a-4}(2 + \lceil \frac{a-i}{4} \rceil)$.

Next, we prove that $f_k$ is a locating $a$–coloring of $BT(k)$ for $(4 + \sum_{i=1}^{a-4}(2 + \lceil \frac{a-i}{4} \rceil)) \leq k \leq (2 + \sum_{i=0}^{a-4}(2 + \lceil \frac{a-i}{4} \rceil))$. The proof is similar with the proof of Theorem 1 for case $k = 4$ and $5$. By the coloring $f_{k-1}$, we have all vertices in each $BT(k-2)$ have different color codes. We need only to consider every two vertices $u$ and $v$ satisfying $f_k(u) = f_k(v)$ and $f_k(N(u)) = f_k(N(v))$. Observe three cases below.

Case 1. $u \in V(BT_i(k- \lfloor \frac{k}{2} \rfloor))$ and $v \notin V(BT_i(k- \lfloor \frac{k}{2} \rfloor))$ for $i \neq j$.

Observe that, the coloring $f_k$ implies the vertices in $BT_i(k- \lfloor \frac{k}{2} \rfloor)$ are colored by $a-1$ colors for odd $i$ and the vertices in $BT_i(k- \lfloor \frac{k}{2} \rfloor)$ are colored by $a$ colors for even $j$. By the coloring $f_k$, their color codes are distinguished by the unique color used only in $BT_{(i+1)}(k- \lfloor \frac{k}{2} \rfloor)$ if $i$ is odd and $BT_{(i)}(k- \lfloor \frac{k}{2} \rfloor)$ if $i$ is even.

Case 2. $u \in V(BT_i(k- \lfloor \frac{k}{2} \rfloor))$ and $v \notin V(BT_i(k- \lfloor \frac{k}{2} \rfloor))$ for some $i, j$.

By the coloring $f_k$, the minimum distance between the vertex $v$ to a vertex with the unique color in used only in $BT_{(i+1)}(k+1- \lfloor \frac{k}{2} \rfloor)$ is smaller than the minimum distance between the vertex $u$ to a vertex with the unique color in $BT_{(i)}(k+1- \lfloor \frac{k}{2} \rfloor)$. Therefore, the color codes of vertices $u$ and $v$ are different.

Case 3. $u \notin V(BT_i(k- \lfloor \frac{k}{2} \rfloor))$ and $v \notin V(BT_i(k- \lfloor \frac{k}{2} \rfloor))$ for some $i, j$.

By the coloring $f_k$, there is no possibilities for vertices $u$ and $v$ satisfying this condition.

By three cases above, all vertices in $BT(k)$ have different color codes. Thus, $f_k$ is a locating $a$–coloring of $BT(k)$.

For a non complete binary tree, we have upper bound for its locating-chromatic number in the theorem below.

![Fig. 3. A locating 3-coloring of $BT(2)$](image-url)
Theorem 3. Let $BT_{nc}(k) \subset BT(k)$ be a non complete binary tree with $k \geq 3$, where the root of $BT_{nc}(k)$ is the only vertex of degree 2. Then $\chi_L(BT_{nc}(k)) \leq 3 + \lfloor \frac{k}{2} \rfloor$.

Proof. For $k = 2$ define a locating 3–coloring $g_2 : V(BT(2)) \rightarrow \{1, 2, 3\}$ as depicted in Fig. 3. We define a function $g_k : V(BT(k)) \rightarrow \{1, 2, \ldots, 3 + \lfloor \frac{k}{2} \rfloor\}$, for $k \geq 3$ by recursive definition as follows.

$$
g_k(u) = \begin{cases} 
1, & \text{if } u = x_0 \text{ and } k \text{ is even}, \\
2, & \text{if } u = x_0 \text{ and } k \text{ is odd}, \\
(g_{k-1}(u) + 2) \mod (3 + \lfloor \frac{k}{2} \rfloor), & \text{if } u \in V(BT_1(k-1)), \\
(g_{k-1}(u) + 3) \mod (3 + \lfloor \frac{k}{2} \rfloor), & \text{if } k \text{ is odd and } u \in V(BT_2(k-1)).
\end{cases}
$$

Next, we show that $g_k$ is a locating $(3 + \lfloor \frac{k}{2} \rfloor)$–coloring for $BT(k)$. For odd $k$, observe that the coloring $g_k$ is a locating $(2 + \lfloor \frac{k}{2} \rfloor)$–coloring for $BT_1(k-1)$ and $BT_2(k-1)$. There is no color $(3 + \lfloor \frac{k}{2} \rfloor)$ in $BT_1(k-1)$ and there is no color $(2 + \lfloor \frac{k}{2} \rfloor)$ in $BT_2(k-1)$. Therefore, the vertices in $BT_1(k-1)$ and $BT_2(k-1)$ have different color codes. They are distinguished by their neighbor colors. While the root vertex has different color code with others vertices; it is differed by the colors $(3 + \lfloor \frac{k}{2} \rfloor)$ and $(2 + \lfloor \frac{k}{2} \rfloor)$. Then, $g_k$ is a locating $(3 + \lfloor \frac{k}{2} \rfloor)$–coloring for $BT(k)$.

For even $k$, we consider four distinct $BT(k-2)$, each is colored by distinct locating $(2 + \lfloor \frac{k}{2} \rfloor)$–coloring. All vertices in every $BT(k-2)$ have different color codes. They are distinguished by their neighbor colors. Next, we check for the root vertex $x_0$ and vertices adjacent to $x_0$. These three vertices have different color codes with vertices in every $BT(k-2)$. They are distinguished by the color $2 + \lfloor \frac{k}{2} \rfloor$. Since all vertices have different color codes, $g_k$ is a locating $(3 + \lfloor \frac{k}{2} \rfloor)$–coloring for $BT(k)$.

Now, we define $g^*_k : V(BT_{nc}(k)) \rightarrow \{1, 2, \ldots, 3 + \lfloor \frac{k}{2} \rfloor\}$, which is a restriction of $g_k$ on $V(BT_{nc}(k)) \subset V(BT(k))$. We will show that $g_k^*$ is a locating coloring of $BT_{nc}(k)$. We observe for two vertices $u$ and $v$, where $g_k^*(u) = g_k^*(v)$ and $g_k^*(N(u)) = g_k^*(N(v))$. For odd $k$, let $u \in V(BT_1(k-1))$ and $v \in V(BT_2(k-1))$, their color codes are differed by colors $(3 + \lfloor \frac{k}{2} \rfloor)$ and $(2 + \lfloor \frac{k}{2} \rfloor)$. While the root vertex has different color code with others vertices, it is differed by the colors $(3 + \lfloor \frac{k}{2} \rfloor)$ and $(2 + \lfloor \frac{k}{2} \rfloor)$. For even $k$, let $u \in V(BT_1(k-2))$ and $v \in V(BT_2(k-2))$, for $i \neq j$, their color codes are differed by a color which missing in $BT_i(k-2)$ or $BT_j(k-2)$. Next, we check for the root vertex $x_0$ and vertices adjacent to $x_0$. These three vertices have different color codes with vertices in every $BT(k-2)$. They are distinguished by the color $2 + \lfloor \frac{k}{2} \rfloor$. Since all vertices have different color codes, $g_k^*$ is a locating coloring of $BT_{nc}(k)$.

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