Topology and its Applications 54 (1993) 47–64
North-Holland

The creation of homoclinic points of C^1-maps

Kazumine Moriyasu

Department of Mathematics, Tokushima University, Minamijosanjima 1-1, Tokushima 770, Japan

Masatoshi Oka

Department of Mathematics, Science University of Tokyo, Noda, Chiba 278, Japan

Received 27 December 1991

Abstract

Moriyasu, K. and M. Oka, The creation of homoclinic points of C^1-maps, Topology and its Applications 54 (1993) 47–64.

We create homoclinic points for C^1-maps on closed manifolds. Under supplementary hypotheses of probabilities Mañé constructed homoclinic points of isolated hyperbolic sets for C^r-diffeomorphisms, $r = 1, 2$. We extend the result to C^1-maps.

Keywords: Homoclinic point; Isolated hyperbolic set.

AMS (MOS) Subj. Class.: Primary 58F10.

Let M be a closed C^∞-manifold and $f: M \to M$ be a C^r-diffeomorphism, $r \geqslant 1$. Let $p \in M$ be a hyperbolic fixed point of f. The stable and unstable sets of p are denoted respectively by

$$W^s(p, f) = \left\{x \in M:\ \lim_{n\to\infty} d(f^n(x), p) = 0\right\},$$

$$W^u(p, f) = \left\{x \in M:\ \lim_{n\to\infty} d(f^{-n}(x), p) = 0\right\}.$$

Then it is well known that $W^\sigma(p, f)$ $(\sigma = s, u)$ is a C^r injectively immersed submanifold of M. The points of intersection of $W^s(p, f)$ with $W^u(p, f)$, different from p, are called homoclinic points associated to p. The points of intersection of the closure of $W^s(p, f)$ with $W^u(p, f)$ or the closure of $W^u(p, f)$ with $W^s(p, f)$, different from p, will be called almost homoclinic points associated to p.

Correspondence to: K. Moriyasu, Department of Mathematics, Tokushima University, Minamijosanjima 1-1, Tokushima 770, Japan.

We know the problem of whether it is possible to create homoclinic points by a small perturbation of diffeomorphisms when there exist almost homoclinic points.

For diffeomorphisms of the two-dimensional sphere Robinson [9] solved affirmatively the problem in the C^r-topology ($r \geqslant 1$). Pixton [6] extended the result of Robinson to a separable C^∞ two-dimensional planar manifold. After that Oliveira [5] proved the same results for area preserving diffeomorphisms of compact orientable surfaces. Takens [10] solved the problem for Hamiltonian diffeomorphisms, but in the case $r = 1$.

Mañé [3] solved the problem for diffeomorphisms under supplementary hypotheses of probabilities for the cases $r = 1$ or 2. The theorems of Mañé play an important role to solve the Stability Conjecture [4].

The purpose of this paper is to show that the theorems of Mañé are extended for differentiable maps. However our proof does not unfortunately work for the C^2-topology.

Let M be a closed C^∞-manifold and $C^1(M)$ be the set of all C^1-maps from M into itself endowed with the C^1-topology. For $f \in C^1(M)$ a point $x \in M$ is said to be *singular* if the differential $D_x f : T_x M \to T_{f(x)} M$ is not surjective. Denote as $S(f)$ the set of all singular points of f. Obviously $S(f)$ is closed in M.

For $f \in C^1(M)$ denote a closed set $A(f)$ by $A(f) = \bigcap_{n \geqslant 0} f^n(M)$. Then $A(f)$ is the maximal f-invariant subset of M. Define as $\boldsymbol{M}_f$ the set $\{(x_i)\colon x_i \in A(f)$ and $f(x_i) = x_{i+1},\ i \in \mathbb{Z}\}$. Then $\boldsymbol{M}_f$ is a closed subset of the product topological space $\prod_{i=-\infty}^{\infty} M_i$ (each M_i is a replica of M). For a subset W of M denote as $\mathrm{Cl}\, W$ the closure of W in M.

Theorem A. *Let M be a closed C^∞-manifold and $f : M \to M$ be a C^1-map with an isolated hyperbolic set Λ. Suppose $x \notin \overline{\Lambda}$. If there are a sequence $\{\boldsymbol{x}^k\} \subset \boldsymbol{M}_f$ with $x_0^k \to \bar{x}$ as $k \to \infty$ and a strictly increasing sequence $\{m_k\} \subset \mathbb{Z}^+$ such that $\mathrm{Cl}\{x_i^k\colon k \geqslant 0$ and $0 \leqslant i \leqslant m_k\} \cap S(f) = \emptyset$ ($\mathrm{Cl}\{x_{-i}^k\colon k \geqslant 0$ and $0 \leqslant i \leqslant m_k\} \cap S(f) = \emptyset$) and $\mu_k^+ = 1/m_k \sum_{i=1}^{m_k} \delta_{x_i^k}$ ($\mu_k^- = 1/m_k \sum_{i=1}^{m_k} \delta_{x_{-i}^k}$) converges to an f-invariant Borel probability measure μ and $\mu(\Lambda) > 0$, then given a neighborhood $\mathscr{U}(f)$ of f in $C^1(M)$ there is $g \in \mathscr{U}(f)$ such that $g = f$ on some neighborhood of Λ and one of the following properties holds*:

(I) $W^s(\Lambda, g) \cap W^u(\Lambda, g) \setminus \Lambda \neq \emptyset$,

(II) *there is $k > 0$ such that $x_0^k \in W^s(\Lambda, g)$ ($x_0^k \in W^u(\Lambda, g)$).*

As a corollary we have the following

Corollary B. *Under the assumptions of Theorem* A, *if $\{x_0^k\} \subset W^u(\Lambda, f)$ ($\{x_0^k\} \subset W^s(\Lambda, f)$), then given a neighborhood $\mathscr{U}(f)$ of f in $C^1(M)$ there is $g \in \mathscr{U}(f)$ with $g = f$ on some neighborhood of Λ such that $W^s(\Lambda, g) \cap W^u(\Lambda, g) \setminus \Lambda \neq \emptyset$.*

For $\boldsymbol{x} \in \boldsymbol{M}_f$ we denote by $\mathscr{M}^+(\boldsymbol{x})$ ($\mathscr{M}^-(\boldsymbol{x})$) the set of all f-invariant Borel probability measures to which $1/m_k \sum_{i=1}^{m_k} \delta_{x_i}$ ($1/m_k \sum_{i=1}^{m_k} \delta_{x_{-i}}$) converges for some strictly increasing sequence $\{m_k\} \subset \mathbb{Z}^+$.

Theorem C. *Let Λ be an isolated hyperbolic set for a C^1-map $f: M \to M$ satisfying $\Omega(f \mid_\Lambda) = \Lambda$ and denote as $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_m$ the spectral decomposition of Λ. If there are $x \notin W^s(\Lambda, f)$ ($x \notin W^u(\Lambda, f)$) and an orbit $\boldsymbol{x} \in \boldsymbol{M}_f$ with $x_0 = x$ such that $\mathrm{Cl}\{x_i: i \geqslant 0\} \cap S(f) = \emptyset$ ($\mathrm{Cl}\{x_{-i}: i \geqslant 0\} \cap S(f) = \emptyset$) and $\mu(\Lambda) > 0$ for all $\mu \in \mathscr{M}^+(\boldsymbol{x})$ ($\mu \in \mathscr{M}^-(\boldsymbol{x})$) then there exists a basic set Λ_i such that given a neighborhood $\mathscr{U}(f)$ of f in $C^1(M)$ there is $g \in \mathscr{U}(f)$ satisfying $g = f$ on some neighborhood of Λ_i and $W^s(\Lambda_i, g) \cap W^u(\Lambda_i, g) \setminus \Lambda_i \neq \emptyset$.*

Before starting the proof we recall some definitions and notations. Let $f \in C^1(M)$. For a subset $\Lambda \subset A(f)$ write $\Lambda_f = \{(x_i) \in \boldsymbol{M}_f: x_i \in \Lambda, i \in \mathbb{Z}\}$. If Λ is a closed f-invariant subset ($f(\Lambda) = \Lambda$) of $A(f)$, then we say that Λ is *hyperbolic* if $\Lambda \cap S(f) = \emptyset$ and there exist a Riemannian metric $\|\cdot\|$ on TM and $c > 0$, $0 < \lambda < 1$ such that for every $\boldsymbol{x} = (x_i) \in \Lambda_f$ there is a splitting $T_{\boldsymbol{x}}M = \bigcup_{i \in \mathbb{Z}} T_{x_i}M = \bigcup_{i \in \mathbb{Z}} (E^s(x_i, \boldsymbol{x}) \oplus E^u(x_i, \boldsymbol{x}))$ such that for every $i \in \mathbb{Z}$

(a) $D_{x_i}f(E^\sigma(x_i, \boldsymbol{x})) = E^\sigma(x_{i+1}, \boldsymbol{x})$ ($\sigma = s, u$),

(b) for every $n \geqslant 0$ and $v \in E^s(x_i, \boldsymbol{x})$, $\| D_{x_i} f^n(v) \| \leqslant c\lambda^n \| v \|$,

(c) for every $n \geqslant 0$ and $v \in E^u(x_i, \boldsymbol{x})$, $\| D_{x_i} f^n(v) \| \geqslant c^{-1}\lambda^{-n} \| v \|$.

Remark that if Λ is hyperbolic and $(x_i), (y_i) \in \Lambda_f$ with $x_0 = y_0$, then $E^s(x_0, (x_i)) = E^s(y_0, (y_i))$, but this is not the case for $E^u(x_0, (x_i))$ (c.f. [7]). Thus we write simply $E^s(x_0) = E^s(x_0, (x_i))$. We say that a hyperbolic set Λ for $f \in C^1(M)$ is *isolated* if there is a compact neighborhood U of Λ such that $U_f = \Lambda_f$. Such a neighborhood U is called an *isolating block* of Λ. Note that if Λ is an isolated hyperbolic set with $\Omega(f \mid_\Lambda) = \Lambda$, where $\Omega(f \mid_\Lambda)$ is the nonwandering set of $f \mid_\Lambda$, then Λ splits into a finite disjoint union $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_m$ of basic sets Λ_i (i.e., Λ_i is a closed f-invariant set and there is $x \in \Lambda_i$ such that $\mathrm{Cl}\{f^n(x): n \geqslant 0\} = \Lambda_i$) (see [7,8]). Such a decomposition is called the *spectral decomposition* of Λ.

For $x \in \Lambda$ and $(x_i) \in \Lambda_f$ the *stable* and *unstable sets* are denoted respectively by

$$W^s(x, f) = \{y \in M: d(f^i(y), f^i(x)) \to 0 \text{ as } i \to \infty\},$$

$$W^u((x_i), f) = \{y \in A(f): \text{there is } (y_i) \in \boldsymbol{M}_f \text{ such that } y_0 = y \text{ and } d(y_{-i}, x_{-i}) \to 0 \text{ as } i \to \infty\}.$$

The *stable* and *unstable sets* for Λ are defined by

$$W^s(\Lambda, f) = \{y \in M: d(f^i(y), \Lambda) \to 0 \text{ as } i \to \infty\},$$

$$W^u(\Lambda, f) = \{y \in A(f): \text{there is } (y_i) \in \boldsymbol{M}_f \text{ such that } y_0 = y \text{ and } d(y_{-i}, \Lambda) \to 0 \text{ as } i \to \infty\}.$$

If Λ is isolated, then we have $W^s(\Lambda, f) = \bigcup_{x \in \Lambda} W^s(x, f)$ and $W^u(\Lambda, f) = \bigcup_{\boldsymbol{x} \in \Lambda_f} W^u(\boldsymbol{x}, f)$. The points of $W^s(\Lambda, f) \cap W^u(\Lambda, f) \setminus \Lambda$ are called *homoclinic points* associated to Λ.

To obtain Theorem A we shall give the proof for the case when $\bar{x}$, $\{\boldsymbol{x}^k\}$ and $\{m_k\}$ are chosen such that

$$\mu_k^- = \frac{1}{m_k} \sum_{i=1}^{m_k} \delta_{x^k_{-i}}$$

converges to μ and $\mu(\Lambda) > 0$. Another case will be obtained by the same way and so we omit the proof.

In order to prove Theorem A for diffeomorphisms Mañé [3] prepared several lemmas which describe the orbit behaviour nearby isolated hyperbolic sets. Our proof is in the framework of that of Mañé. Thus we need to extend his lemmas for C^1-maps.

Let D^m be an m-dimensional disk of $\mathbb{R}^m$ and $\mathrm{Emb}^1(D^m, M)$ be the set of all embeddings of D^m into M with the C^1-topology. Let $\{D_x^m\}_{x \in \Lambda}$ ($\{D_{\boldsymbol{x}}^m\}_{\boldsymbol{x} \in \Lambda_f}$) be a family of m-dimensional C^1-disks with $x \in D_x^m$ for $x \in \Lambda$ ($x_0 \in D_{\boldsymbol{x}}^m$ for $\boldsymbol{x} \in \Lambda_f$). Then we say that $\{D_x^m\}_{x \in \Lambda}$ ($\{D_{\boldsymbol{x}}^m\}_{\boldsymbol{x} \in \Lambda_f}$) is continuous if for $x \in \Lambda$ ($\boldsymbol{x} \in \Lambda_f$) there are a neighborhood U of x in Λ ($\boldsymbol{x}$ in Λ_f) and a continuous map $\phi: U \to \mathrm{Emb}^1(D^m, M)$ such that $\phi(y)(D^m) = D_y^m$ for $y \in U$ ($\phi(\boldsymbol{y})(D^m) = D_{\boldsymbol{y}}^m$ for $\boldsymbol{y} \in U$).

For $\varepsilon > 0$, $x \in M$ and $\boldsymbol{x} \in \boldsymbol{M}_f$ denote the *local stable* and *local unstable sets* by

$$W_\varepsilon^s(x, f) = \{ y \in M\colon d(f^n(x), f^n(y)) \leqslant \varepsilon \text{ for } n \geqslant 0 \},$$

$$W_\varepsilon^u(\boldsymbol{x}, f) = \{ y \in M\colon \text{there is } \boldsymbol{y} \in \boldsymbol{M}_f \text{ such that } y_0 = y \text{ and } d(x_{-n}, y_{-n}) \leqslant \varepsilon \text{ for } n \geqslant 0 \}.$$

Let Λ be a hyperbolic set. By [7, Proposition 1.4] we may assume that $\|\cdot\|$ is adapted to Λ, that is there exists $0 < \nu < 1$ such that $\| D_p f(v) \| \leqslant \nu \| v \|$ for $p \in \Lambda$ and $v \in E^s(p)$, and $\| D_{p_0} f(v) \| \geqslant \nu^{-1} \| v \|$ for $\boldsymbol{p} \in \Lambda_f$ and $v \in E^u(p_0, \boldsymbol{p})$. Then for $\varepsilon > 0$ sufficiently small we have the following (1) and (2):

(a) $\{W_\varepsilon^s(x, f)\}_{x \in \Lambda}$ is a continuous family of C^1-disks with $T_x W_\varepsilon^s(x, f) = E^s(x)$,

(b) $\{W_\varepsilon^u(\boldsymbol{x}, f)\}_{\boldsymbol{x} \in \Lambda_f}$ is a continuous family of C^1-disks with $T_{x_0} W_\varepsilon^u(\boldsymbol{x}, f) = E^u(x_0, \boldsymbol{x})$. (1)

There exists λ_0 with $0 < \nu < \lambda_0 < 1$ such that

(a) if $y, z \in W_\varepsilon^s(x, f)$ $(x \in \Lambda)$, then $d(f^n(y), f^n(z)) \leqslant \lambda_0^n d(y, z)$ for every $n \geqslant 0$,

(b) if $y, z \in W_\varepsilon^u(\boldsymbol{x}, f)$ $(\boldsymbol{x} \in \Lambda_f)$ and if $\boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{M}_f$ with $y_0 = y$ and $z_0 = z$ satisfy $d(x_{-n}, y_{-n}) \leqslant \varepsilon$ and $d(x_{-n}, z_{-n}) \leqslant \varepsilon$ for every $n \geqslant 0$, then we have $d(y_{-n}, z_{-n}) \leqslant \lambda_0^n d(y, z)$ for every $n \geqslant 0$. (2)

The following is a result described in Mañé [3] for diffeomorphisms.

$$\begin{array}{ll} & \text{There exist } 0<\gamma<\lambda<1 \text{ such that for } \varepsilon>0 \\ & \text{sufficiently small there is } \delta>0 \text{ satisfying} \\ \text{(a)} & \text{if } p\in\Lambda \text{ and } x\in M \text{ with } d(x,p)\leqslant\delta, \text{ then } \gamma d(f(x), \\ & W_\varepsilon^s(f(p),f))\leqslant d(x,W_\varepsilon^s(p,f))\leqslant\lambda d(f(x),W_\varepsilon^s(f(p),f)), \\ \text{(b)} & \text{if } \boldsymbol{p}\in\Lambda_f \text{ and } x\in M \text{ with } d(x,p_0)\leqslant\delta, \text{ then } \gamma d(x,W_\varepsilon^u(\boldsymbol{p},f)) \\ & \leqslant d\big(f(x),W_\varepsilon^u(\tilde f(\boldsymbol{p}),f)\big)\leqslant\lambda d(x,W_\varepsilon^u(\boldsymbol{p},f)). \end{array} \tag{3}$$

Here $\tilde{f}: \boldsymbol{M}_f \to \boldsymbol{M}_f$ is a homeomorphism defined by $\tilde{f}((x_i)) = (f(x_i))$ for $(x_i) \in \boldsymbol{M}_f$.

(3) is checked as follows. Take $y \in W_\varepsilon^u(\boldsymbol{p}, f)$ with $d(x, y) = d(x, W_\varepsilon^u(\boldsymbol{p}, f))$ and put $v = \exp_x^{-1} y$. Let $\eta > 0$ be a small number. Since $\| v \| = d(x, y) \leqslant d(x, p)$, if the distance between x and p is small then $f(y) \in W_\varepsilon^u(\tilde{f}(\boldsymbol{p}), f)$ and $\| D_x f(v) - \exp_{f(x)}^{-1} \circ f \circ \exp_x(v) \| \leqslant \eta \| v \|$ and so

$$\begin{aligned} d\big(f(x), W_\varepsilon^u(\tilde{f}(\boldsymbol{p}), f)\big) &\leqslant d(f(x), f(y)) \\ &= \big\| \exp_{f(x)}^{-1} f(y) \big\| \\ &= \big\| \exp_{f(x)}^{-1} \circ f \circ \exp_x(v) \big\| \\ &\leqslant \| D_x f(v) \| + \eta \| v \|. \end{aligned}$$

Let $\theta(p, x)$ be the parallel translation of tangent vectors along the minimal geodesic from p to x and put

$$v = v_1 + v_2 \in \theta(p, x)(E^s(p)) \oplus \theta(p, x)(E^u(p, \boldsymbol{p}))$$

where $p = p_0$ and $T_{\boldsymbol{p}} M = \bigcup_{i \in \mathbb{Z}} (E^s(p_i) \oplus E^u(p_i, \boldsymbol{p}))$ is the hyperbolic splitting. Since $T_p W_\varepsilon^u(\boldsymbol{p}, f) = E^u(p, \boldsymbol{p})$, if the distance between x and p is small then so is $\| v_2 \| / \| v_1 \|$. Thus we can find $\delta > 0$ such that $\| v_2 \| \leqslant \eta \| v_1 \|$ when $d(x, p) < \delta$. Take ν' with $\nu < \nu' < 1$ where ν is as before. Then we have $\| D_x f(v_1) \| \leqslant \nu' \| v_1 \|$ if $\delta > 0$ is small. Thus

$$\begin{aligned} d\big(f(x), W_\varepsilon^u(\tilde{f}(\boldsymbol{p}), f)\big) &\leqslant \| D_x f(v) \| + \eta \| v \| \\ &\leqslant \| D_x f(v_1) \| + \| D_x f(v_2) \| + \eta \| v \| \\ &\leqslant \nu' \| v_1 \| + K\eta \| v_1 \| + \eta \| v \| \\ &\leqslant \{ (\nu' + K\eta + \eta(1 - \eta)) / (1 - \eta) \} \| v \| \\ &= \{ (\nu' + K\eta + \eta(1 - \eta)) / (1 - \eta) \} d(x, y) \end{aligned}$$

where $K = \sup_{x \in M} \| D_x f \|$. Taking $\eta > 0$ small we have $\{\nu' + K\eta + \eta(1 - \eta)\}/(1 - \eta) = \lambda < 1$, which ensures that $d(f(x), W_\varepsilon^u(\tilde{f}(\boldsymbol{p}), f)) \leqslant \lambda d(x, y) = \lambda d(x, W_\varepsilon^u(\boldsymbol{p}, f))$.

To show another inequality in (3) (b) we need the following

Take a closed neighborhood $B(\Lambda)$ of Λ in M with $B(\Lambda) \cap S(f) = \emptyset$. Then there are positive numbers α_0 and α_1 such that

(a) $f | U_{\alpha_0}(x) : U_{\alpha_0}(x) \to f(U_{\alpha_0}(x))$ is a diffeomorphism and $f(U_{\alpha_0}(x)) \supset U_{\alpha_1}(f(x))$ for $x \in B(\Lambda)$ where $U_\alpha(x) = \{ y \in M: d(x, y) < \alpha \}$, (4)

(b) for $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x, y) \leqslant \delta$ then for $x' \in f^{-1}(x) \cap B(\Lambda)$ there is a unique $y' \in f^{-1}(y)$ with $d(x', y') \leqslant \varepsilon$.

We may suppose that $\bigcup_{p \in \Lambda_f} W_\varepsilon^u(p, f) \subset B(\Lambda)$. Take $y \in W_\varepsilon^u(\tilde{f}(p), f)$ with $d(f(x), y) = d(f(x), W_\varepsilon^u(\tilde{f}(p), f))$. Since $d(f(x), y) \leqslant d(f(x), f(p))$, we have $d(y, f(p)) \leqslant d(y, f(x)) + d(f(x), f(p)) \leqslant 2d(f(x), f(p))$. If the distance between x and p is small, by (4)(b) there exists a unique $y_{-1} \in f^{-1}(y)$ such that $y_{-1} \in W_\varepsilon^u(p, f)$. Put $v = \exp_{f(x)}^{-1} y$. Then, by the same method as above and by (4)(a)

$$\begin{aligned} d(x, W_\varepsilon^u(p, f)) &\leqslant d(x, y_{-1}) \\ &\leqslant \left\| (D_x f)^{-1}(v) \right\| + \eta \| v \| \\ &\leqslant (K' + \eta) \| v \| \\ &\leqslant (K' + \eta) d\big(f(x), W_\varepsilon^u(\tilde{f}(p), f)\big) \end{aligned}$$

where $K' = \max\{\sup_{x \in B(\Lambda)} \|(D_x f)^{-1}\|, 1\}$. Therefore, put $\gamma = 1/(K' + \eta)$ then we have the conclusion. Similarly we obtain (3)(a).

For $f \in C^1(M)$ the following Proposition 1 shall be proven by the same method as in [3].

Proposition 1. *Let $f \in C^1(M)$ and Λ be an isolated hyperbolic set for f. Let $0 < \gamma < \lambda < 1$ be as in (3). Then for $\varepsilon_0 > 0$ sufficiently small there exists $r_0 > 0$ such that if $d(x, V^+) \leqslant r_0$ and $d(x, V^-) \leqslant r_0$, where $V^+ = \bigcup_{x \in \Lambda} W_{\varepsilon_0}^s(x, f)$ and $V^- = \bigcup_{x \in \Lambda_f} W_{\varepsilon_0}^u(x, f)$, then*

(a) (i) $\gamma d(f(x), V^+) \leqslant d(x, V^+)$,

(ii) *there is* $y \in f^{-1}(x)$ *such that* $d(y, V^+) \leqslant \lambda d(f(y), V^+) = \lambda d(x, V^+)$,

(b) (i) *there is* $y \in f^{-1}(x)$ *such that* $\gamma d(y, V^-) \leqslant d(f(y), V^-) = d(x, V^-)$,

(ii) $d(f(x), V^-) \leqslant \lambda d(x, V^-)$.

If f has homoclinic points associated to Λ, then it satisfies (I) of Theorem A. Therefore, to complete Theorem A it suffices to give the proof for the following case

f has no homoclinic points associated to Λ. (5)

Proposition 2. *Under the notations of Proposition* 1, *if* Λ *satisfies* (5), *then for* $\varepsilon_0 > 0$ *sufficiently small there exists* $r_0 > 0$ *such that if* $d(x, V^+) \leqslant r_0$ *and* $d(x, V^-) \leqslant r_0$ *then* $d(x, V^+) \leqslant \lambda d(f(x), V^+)$.

For the proof we need some notations. Write $W_\varepsilon^s(\Lambda, f) = \bigcup_{x\in\Lambda} W_\varepsilon^s(x, f)$ and $W_\varepsilon^u(\Lambda, f) = \bigcup_{x\in\Lambda_f} W_\varepsilon^u(x, f)$ for $\varepsilon > 0$. Then it is easily checked that for sufficiently small $\varepsilon > 0$ and $0 < \delta < \varepsilon$ we have $\mathrm{Cl}[W_\varepsilon^s(\Lambda, f)\setminus W_\delta^s(\Lambda, f)] \cap \Lambda = \emptyset$ and $\mathrm{Cl}[W_\varepsilon^u(\Lambda, f)\setminus W_\delta^u(\Lambda, f)] \cap \Lambda = \emptyset$.

For $\varepsilon > 0$ small enough define a map $f_\Lambda : W_\varepsilon^u(\Lambda, f) \to W^u(\Lambda, f)$ by $f_\Lambda = f \mid W_\varepsilon^u(\Lambda, f)$. Then $f_\Lambda(W_\varepsilon^u(\Lambda, f)) \supset W_\varepsilon^u(\Lambda, f)$ and for every $0 < \delta \leqslant \varepsilon$ there exists $k \geqslant 1$ such that $f_\Lambda^{-k}(W_\varepsilon^u(\Lambda, f)) \subset W_\delta^u(\Lambda, f)$. For $k \geqslant 1$ define $D_k^s = \mathrm{Cl}[W_\varepsilon^s(\Lambda, f)\setminus f^k(W_\varepsilon^s(\Lambda, f))]$ and $D_k^u = \mathrm{Cl}[W_\varepsilon^u(\Lambda, f)\setminus f_\Lambda^{-k}(W_\varepsilon^u(\Lambda, f))]$. Clearly D_k^σ is compact ($\sigma = s, u$) and satisfies $\bigcup_{n\geqslant 0} f^n(D_k^s) \supset W_\varepsilon^s(\Lambda, f)\setminus\Lambda$, $\bigcup_{n\geqslant 0} f_\Lambda^{-n}(D_k^u) \supset W_\varepsilon^u(\Lambda, f)\setminus\Lambda$, $D_k^s \cap \Lambda = \emptyset$ and $D_k^u \cap \Lambda = \emptyset$. D_1^s and D_1^u are called proper fundamental domains for $W_\varepsilon^s(\Lambda, f)$ and $W_\varepsilon^u(\Lambda, f)$ respectively.

Making use of the above notations the following lemma is obtained as a slight extension of [3, Lemma 6].

Lemma 3. *For* $\varepsilon > 0$ *small enough and* $N > 0$ *there is* $c = c(\varepsilon, N) > 0$ *such that*

(a) *if* $d(x, \Lambda) \leqslant c$ *and* $p \in W_\varepsilon^s(\Lambda, f)$ *satisfies* $d(x, p) = d(x, W_\varepsilon^s(\Lambda, f))$, *then* $p \in f^N(W_\varepsilon^s(\Lambda, f))$,

(b) *if* $d(x, \Lambda) \leqslant c$ *and* $p \in W_\varepsilon^u(\Lambda, f)$ *satisfies* $d(x, p) = d(x, W_\varepsilon^u(\Lambda, f))$, *then* $p \in f_\Lambda^{-N}(W_\varepsilon^u(\Lambda, f))$.

Now we give the proof of Proposition 2. Let $0 < \delta_0 \leqslant \varepsilon_0/2$ be as in (3) for ε_0 and $B(\Lambda)$ be as in (4). By (4)(b) we can find $0 < \delta_1 < \delta_0$ such that if $d(x, y) \leqslant \delta_1$ then for $x_{-1} \in f^{-1}(x) \cap B(\Lambda)$ there is a unique $y_{-1} \in f^{-1}(y)$ with $d(x_{-1}, y_{-1}) \leqslant 2\delta_0$. Choose $0 < \delta_2 < \delta_1$ such that if $d(x, y) \leqslant \delta_2$ then for $x_{-1} \in f^{-1}(x) \cap B(\Lambda)$ there is $y_{-1} \in f^{-1}(y)$ satisfying $d(x_{-1}, y_{-1}) \leqslant \delta_0$. By [8] there is $0 < \delta_3 \leqslant \delta_2$ such that if $d(x, y) \leqslant \delta_3$ $(x, y \in \Lambda)$, then $W_{\varepsilon_0}^s(x, f) \cap W_{\varepsilon_0}^u(y, f)$ consists of one point for $y \in \Lambda_f$ with $y_0 = y$. Since $\{W_{\varepsilon_0}^s(x, f)\}_{x\in\Lambda}$ and $\{W_{\varepsilon_0}^u(x, f)\}_{x\in\Lambda_f}$ are continuous families, for a sufficiently small δ_3 if $d(x, y) \leqslant \delta_3$ and $\{z\} = W_{\varepsilon_0}^s(x, f) \cap W_{\varepsilon_0}^u(y, f)$ for $y \in M_f$ with $y_0 = y$, then we have that $d(x, z) \leqslant \delta_2/3$ and $d(y, z) \leqslant \delta_2/3$. Take $0 < \delta_4 \leqslant \delta_3/2$ such that if $d(x, y) \leqslant \delta_4$, then $d(f(x), f(y)) \leqslant \delta_3/2$. Let N_1 be a number such that $\lambda^{N_1}\varepsilon_0 < \delta_4/2$. By Lemma 3 we can take $0 < c = c(\varepsilon_0, N_1) < \delta_4/2$ such that

$$\text{if } d(x, \Lambda) \leqslant c \text{ and } p \in V^+ \text{ satisfies } d(x, p) = d(x, V^+),$$
$$\text{then } p \in f^{N_1}(V^+) \subset W_{\delta_4/2}^s(\Lambda, f). \tag{6}$$

Choose $0 < c' < c$ such that $d(x, y) \leqslant c'$ implies $d(f(x), f(y)) \leqslant c$. Then there exists $0 < r_0 < \varepsilon_0$ such that if $d(x, V^+) \leqslant r_0$ and $d(x, V^-) \leqslant r_0$ then $d(x, \Lambda) \leqslant c'$ and $x \in B(\Lambda)$, which is our requirement.

In fact, if $d(x, V^+) \leqslant r_0$ and $d(x, V^-) \leqslant r_0$, then $d(x, \Lambda) \leqslant c'$ and so $d(f(x), \Lambda) \leqslant c$. Thus there is $p \in V^+$ such that $d(f(x), p) = d(f(x), V^+) \leqslant c \leqslant \delta_4/2$. By (6) we have that $p \in W^s_{\delta_4/2}(y, f)$ for some $y \in \Lambda$. Since $d(f(x), y) \leqslant d(f(x), p) + d(p, y) \leqslant \delta_4 < \delta_2$, we can take $y_{-1} \in f^{-1}(y)$ such that $d(y_{-1}, x) < \delta_0$.

If $y_{-1} \in \Lambda$, then by (3) we obtain

$$d(x, V^+) \leqslant d\big(x, W^s_{\varepsilon_0}(y_{-1}, f)\big) \leqslant \lambda d\big(f(x), W^s_{\varepsilon_0}(y, f)\big)$$
$$= \lambda d(f(x), V^+).$$

It remains to show that Proposition 2 holds for $y_{-1} \notin \Lambda$. Since $d(x, \Lambda) \leqslant c$ and $c < \delta_4/2$, there is $y' \in \Lambda$ such that $d(x, y') \leqslant \delta_4$. Hence $d(f(x), f(y')) \leqslant \delta_3/2$ and so $d(y, f(y')) \leqslant d(y, f(x)) + d(f(x), f(y')) \leqslant \delta_4 + \delta_3/2 \leqslant \delta_3$. Take $y' \in \Lambda_f$ with $y'_0 = f(y')$ and $y'_{-1} = y'$. Then we can find $z \in W^u_{\varepsilon_0}(y', f) \cap W^s_{\varepsilon_0}(y, f) \subset \Lambda$ such that $d(z, f(y')) \leqslant \delta_2/3$ and $d(z, y) \leqslant \delta_2/3$. Since $d(z, p) \leqslant d(z, y) + d(y, p) \leqslant \delta_2/3 + \delta_4/2 < \delta_2 < \varepsilon_0$ and $z, p \in W^s_{\varepsilon_0}(y, f)$, by (2) we have $p \in W^s_{\varepsilon_0}(z, f)$. Since $d(z, f(x)) \leqslant d(z, p) + d(p, f(x)) \leqslant \delta_2/3 + \delta_4/2 + c < \delta_2$ and $x \in B(\Lambda)$, there exists $z_{-1} \in f^{-1}(z)$ satisfying $d(z_{-1}, x) \leqslant \delta_0$. Notice that $z_{-1} \in W^u_{\varepsilon_0}(\tilde{f}^{-1}(y'), f)$. Indeed, $d(z, f(y')) \leqslant \delta_2/3 < \delta_2 < \delta_1$ and $d(z_{-1}, y') \leqslant d(z_{-1}, x) + d(x, y') < \delta_0 + \delta_4 < 2\delta_0$. The choice of δ_0 implies $z_{-1} \in W^u_{\varepsilon_0}(\tilde{f}^{-1}(y'), f)$. From this

$$z_{-1} \in W^u_{\varepsilon_0}\big(\tilde{f}^{-1}(y'), f\big) \cap W^s(\Lambda, f) \subset W^u(\Lambda, f) \cap W^s(\Lambda, f),$$

and so $z_{-1} \in \Lambda$ by (5). Since $d(z_{-1}, x) \leqslant \delta_0$, by (3) we obtain

$$d(x, V^+) \leqslant d\big(x, W^s_{\varepsilon_0}(z_{-1}, f)\big)$$
$$\leqslant \lambda d\big(f(x), W^s_{\varepsilon_0}(z, f)\big)$$
$$\leqslant \lambda d(f(x), p) = \lambda d(f(x), V^+).$$

The proof of Proposition 2 is completed.

Let $\varepsilon_0 > 0$ be sufficiently small and $r_0 > 0$ as in Propositions 1 and 2. Take $0 < \delta < 1$ and a sequence $\{r_n\}_{n=1}^{\infty}$ with $r_{n+1} = r_n^{1+\delta}$ $(n \geqslant 0)$. Put

$$V_n = \{x \in M\colon d(x, V^+) \leqslant r_n \text{ and } d(x, V^-) \leqslant r_n\}$$

where $V^+ = \bigcup_{x \in \Lambda} W^s_{\varepsilon_0}(x, f)$ and $V^- = \bigcup_{x \in \Lambda_f} W^u_{\varepsilon_0}(\boldsymbol{x}, f)$. Let $\bar{x} \notin \Lambda$ and take a sequence $\{\boldsymbol{x}^k\} \subset \boldsymbol{M}_f$ such that $x_0^k \to \bar{x}$ as $k \to \infty$. Let $\{m_k\}$ be a strictly increasing sequence of positive integers. For $\boldsymbol{x}^k = (x_i^k)_{i \in \mathbb{Z}}$ and $n \geqslant 0$, call an $(\boldsymbol{x}^k, n)$-*string* a finite sequence $\sigma = \{x_l^k, x_{l-1}^k, \ldots, x_{m+1}^k, x_m^k\} \subset V_0$ $(-m_k \leqslant m < l < 0)$ satisfying

(i) $\sigma \cap V_n \neq \emptyset$,

(ii) $x_{l+1}^k \notin V_0$ and $x_{m-1}^k \notin V_0 \cap \{x_l^k, x_{l-1}^k, \ldots, x_{-m_k}^k\}$.

Let $\sigma_1 = \{x_{l_1}^k, \ldots, x_{m_1}^k\}$ and $\sigma_2 = \{x_{l_2}^k, \ldots, x_{m_2}^k\}$ be $(\boldsymbol{x}^k, 0)$-strings. Define an ordered relation between σ_1 and σ_2 by $\sigma_1 < \sigma_2$ if $m_1 > l_2$.

As mentioned before we define a probability $\mu_k = 1/m_k \sum_{i=1}^{m_k} \delta_{x_{-i}^k}$. Without loss of generality we assume that μ_k converges to an f-invariant Borel probability measure μ.

Proposition 4. *Let $f \in C^1(M)$ and Λ be an isolated hyperbolic set satisfying (5). Under the above notations suppose $\mu(\Lambda) > 0$. Then for every $n_1 > 0$ one of the following properties holds*:

(a) *there are $n \geqslant n_1$, $k > 0$ and $(\boldsymbol{x}^k, n+1)$-strings $\sigma_1 < \sigma_2$ such that $\sigma \cap V_n = \emptyset$ for every $(\boldsymbol{x}^k, 0)$-string σ with $\sigma_1 < \sigma < \sigma_2$,*

(b) *there are $n \geqslant n_1$, $k > 0$ and an $(\boldsymbol{x}^k, n+1)$-string σ_1 such that $\sigma \cap V_n = \emptyset$ for every $(\boldsymbol{x}^k, 0)$-string σ with $\sigma \neq \sigma_1$.*

For the proof we need the following lemma.

Lemma 5. *There are constants C_1 and C_2 with $C_2 > C_1 > 0$ such that for every k*

(a) *if an $(\boldsymbol{x}^k, 0)$-string σ is not an $(\boldsymbol{x}^k, n)$-string, then $\#\sigma \leqslant C_2(1+\delta)^n$,*

(b) *there is $N_1 > 0$ such that if $n \geqslant N_1$ and σ is an $(\boldsymbol{x}^k, n)$-string, then $\#\sigma \geqslant C_1(1+\delta)^n$.*

First we prove (a). Let $\sigma = \{x_l^k, \dots, x_m^k\}$ be an $(\boldsymbol{x}^k, 0)$-string and not an $(\boldsymbol{x}^k, n)$-string. Then we can find $t \in \mathbb{Z}$ and the maximal integer $s \geqslant 0$ such that

(i) $m \leqslant -s+t \leqslant s+t \leqslant l$,

(ii) $m = -s+t$ or $l = s+t$.

By Propositions 1 and 2 we have

$$r_0 \geqslant d\left(x_{t-s}^k, V^-\right) \geqslant \lambda^{-s} d\left(x_t^k, V^-\right),$$

$$r_0 \geqslant d\left(x_{t+s}^k, V^+\right) \geqslant \lambda^{-s} d\left(x_t^k, V^+\right).$$

Since σ is not an $(\boldsymbol{x}^k, n)$-string, we have $x_t^k \notin V_n$, which implies that $d(x_t^k, V^+) > r_n$ or $d(x_t^k, V^-) > r_n$, and so $r_n < \lambda^s r_0$. Thus we have

$$\#\sigma \leqslant 2s+1 < 2(\log r_0/\log \lambda)(1+\delta)^n.$$

Put $C_2 = 2\log r_0/\log \lambda$, then $\#\sigma \leqslant C_2(1+\delta)^n$ when $s \geqslant 1$. Since $0 < r_0 < \gamma < \lambda < 1$ and $C_2 \geqslant 2$, (a) holds for $s = 0$. (a) was proved.

If σ is an $(\boldsymbol{x}^k, n)$-string, then $x_t^k \in \sigma \cap V_n$ for some $m \leqslant t \leqslant l$. Since $x_{l+1}^k \notin V_0$, by Proposition 1

$$\gamma^{l+1-t} r_0 \leqslant \gamma^{l+1-t} d\left(x_{l+1}^k, V^+\right) \leqslant d\left(x_t^k, V^+\right) \leqslant r_n$$

and hence

$$(l+1-t) \geqslant (\log r_0/\log \gamma)(1+\delta)^n - \log r_0/\log \gamma.$$

Take C_1 with $0 < C_1 < \log r_0/\log \gamma < C_2$. Then we can find $N_1 > 0$ such that $(\log r_0/\log \gamma - C_1)(1+\delta)^n \geqslant \log r_0/\log \gamma$ for $n \geqslant N_1$, and so $(\log r_0/\log \gamma)(1+\delta)^n - \log r_0/\log \delta \geqslant C_1(1+\delta)^n$. Therefore $\#\sigma \geqslant l+1-t \geqslant C_1(1+\delta)^n$.

Next we prove Proposition 4. Suppose that there is $n_1 > 0$ such that both (a) and (b) do not hold. Then for every $n \geqslant n_1$ and every $k > 0$

(a′) if $(\boldsymbol{x}^k, n+1)$-strings σ_1 and σ_2 satisfy $\sigma_1 < \sigma_2$, then there is an $(\boldsymbol{x}^k, n)$-string σ with $\sigma_1 < \sigma < \sigma_2$, (b$'$) if σ_1 is an $(\boldsymbol{x}^k, n+1)$-string, then there is an $(\boldsymbol{x}^k, n)$-string σ with $\sigma \neq \sigma_1$.

Let $0 < \delta < 1$ and V_n, S_n be as above. Take ξ with $1+\delta < \xi < 2$. Then we can find integer s_0 such that $2s - 1 > \xi s$ for every $s \geqslant s_0$. We denote as $\nu_k(V_n)$ the number of the set of all $(\boldsymbol{x}^k, n)$-strings. For $k > 0$ and $n \geqslant n_1$ with $\nu_k(V_{n+1}) > s_0$ we have by (a$'$)

$$\nu_k(V_{n+1}) \leqslant \nu_k(V_n)/\xi. \tag{7}$$

Denote as $\sigma(k, n)$ the set of all $(\boldsymbol{x}^k, 0)$-strings which are not $(\boldsymbol{x}^k, n)$-strings. S_n is the set of all points $x \in V_0$ satisfying that there is $\boldsymbol{x} \in \boldsymbol{M}_f$ with $x_0 = x$ such that $x_m \in V_n$ for some $m \in \mathbb{Z}$ and $x_i \in V_0$ for $0 \leqslant i \leqslant m$ if $m \geqslant 0$ and $x_i \in V_0$ for $m \leqslant i \leqslant 0$ if $m < 0$. Put $l(k, n) = \sum_{\sigma \in \sigma(k, n)} \#(\sigma \cap S_n)$. Then we have

$$\mu_k(S_n - S_{n+1}) < C_2\{(1+\delta)/\xi\}^n (1+\delta)\xi^{n_1} + \{l(k, n) - l(k, n+1)\}/m_k \tag{8}$$

for $k > 0$ and $n \geqslant n_1$ with $\nu_k(V_n) > s_0$.

In fact, from the definition of μ_k

$$\begin{aligned}\mu_k(S_n - S_{n+1}) &= \#\{1 \leqslant j \leqslant m_k \colon x^k_{-j} \in S_n - S_{n+1}\}/m_k \\ &\leqslant \{T(\nu_k(V_n) - \nu_k(V_{n+1})) + l(k, n) - l(k, n+1)\}/m_k \\ &\leqslant T\nu_k(V_n)/m_k + \{l(k, n) - l(k, n+1)\}/m_k\end{aligned}$$

where T is the maximal number of all cardinalities of $(\boldsymbol{x}^k, n)$-strings but not $(\boldsymbol{x}^k, n+1)$-strings. Since $T \leqslant C_2(1+\delta)^{n+1}$ by Lemma 5 and $\nu_k(V_n) \leqslant (1/\xi)^{n-n_1}\nu_k(V_{n_1})$ by (7), we have

$$\begin{aligned}\mu_k(S_n - S_{n+1}) &\leqslant C_2(1+\delta)^{n+1}(1/m_k)\nu_k(V_n) \\ &\quad + \{l(k, n) - l(k, n+1)\}/m_k \\ &\leqslant C_2(1+\delta)^{n+1}(1/m_k)(1/\xi)^{n-n_1}\nu_k(V_{n_1}) \\ &\quad + \{l(k, n) - l(k, n+1)\}/m_k \\ &\leqslant C_2\{(1+\delta)/\xi\}^n(1+\delta)\xi^{n_1} \\ &\quad + \{l(k, n) - l(k, n+1)\}/m_k.\end{aligned}$$

(8) was proved.

Similarly we have

$$\mu_k(V_n - V_{n+1}) \leqslant C_2\{(1+\delta)/\xi\}^n(1+\delta)\xi^{n_1} \tag{9}$$

for $n \geqslant n_1$ with $\nu_k(V_n) > s_0$.

Define $r(k) = \min\{j\colon \nu_k(V_j) \leqslant s_0\}$. Obviously $r(k) \to \infty$ as $k \to \infty$, and $\nu_k(V_{r(k)-1}) - \nu_k(V_{r(k)}) \geqslant 1$. Thus

$$\begin{aligned}&\mu_k(S_{r(k)-1} - S_{r(k)}) \\ &\quad \geqslant \{C_1(1+\delta)^{r(k)-1} + l(k, r(k)-1) - l(k, r(k))\}/m_k.\end{aligned} \tag{10}$$

Since $\nu_k(V_{r(k)-1}) > s_0$, by (8) and (10)

$$\begin{aligned}&\left\{C_1(1+\delta)^{r(k)-1} + l(k, r(k)-1) - l(k, r(k))\right\}/m_k \\ &\leqslant \mu_k(S_{r(k)-1} - S_{r(k)}) \\ &< C_2((1+\delta)/\xi)^{r(k)-1}(1+\delta)\xi^{n_1} \\ &\quad + \{l(k, r(k)-1) - l(k, r(k))\}/m_k\end{aligned}$$

and so

$$m_k^{-1} < C_1^{-1}C_2(1/\xi)^{r(k)-1}(1+\delta)\xi^{n_1}. \tag{11}$$

Denote as T' the maximal number of all cardinalities of $(x^k, r(k))$-strings. Then $\mu_k(V_{r(k)}) \leqslant (1/m_k)T'\nu_k(V_{r(k)})$. Since $\nu_k(V_{r(k)}) \leqslant s_0$, by (b′) we have $\sigma \cap V_{r(k)+s_0} = \emptyset$ for every $(x^k, 0)$-string σ. By Lemma 5 we have $T' \leqslant C_2(1+\delta)^{r(k)+s_0}$ and so $\mu_k(V_{r(k)}) \leqslant m_k^{-1}C_2 s_0(1+\delta)^{r(k)+s_0}$. By (11) we have $\mu_k(V_{r(k)}) < C_3((1+\delta)/\xi)^{r(k)-1}$ where $C_3 = C_1^{-1}C_2^2\xi^{n_1}s_0(1+\delta)^{s_0+1}$. Thus (9) implies

$$\begin{aligned}\mu_k(V_n) &= \mu_k(V_{r(k)}) + \sum_{n \leqslant j < r(k)} \mu_k(V_j - V_{j+1}) \\ &< C_3\{(1+\delta)/\xi\}^{r(k)-1} + C_4 \sum_{n \leqslant j < r(k)} \{(1+\delta)/\xi\}^j\end{aligned}$$

where $C_4 = C_2(1+\delta)\xi^{n_1}$. Therefore

$$\mu(\Lambda) \leqslant \lim_{n\to\infty} \mu(\operatorname{int} V_n) \leqslant \lim_{n\to\infty} \varliminf_{k\to\infty} \mu_k(\operatorname{int} V_n) = 0$$

where $\operatorname{int} V_n$ denotes the interior of V_n, thus contradicting.

We are in a position to give the proof of Theorem A. As mentioned before we suppose that Λ satisfies the condition (5). Thus by Proposition 4 there exist $n > 0$, arbitrarily large, and $k > 0$ satisfying one of the following properties:

(a) there exist $(x^k, n+1)$-strings $\sigma_1 < \sigma_2$ such that $\sigma \cap V_n = \emptyset$ for every $(x^k, 0)$-string σ with $\sigma_1 < \sigma < \sigma_2$,

(b) there exists an $(x^k, n+1)$-string σ_1 such that $\sigma \cap V_n = \emptyset$ for every $(x^k, 0)$-string σ with $\sigma \neq \sigma_1$.

First we check that Theorem A holds for the case (a). Let q^1 be the last point of $\sigma_1 \cap V_n$ and q^2 be the first point of $\sigma_2 \cap V_n$. Then we can write $q^1 = x_m^k$ and $q^2 = x_l^k$ for some $-m_k < l < m < 0$. Since σ_1 is an $(x^k, n+1)$-string, there exist $p^1 \in \sigma_1 \cap V_{n+1}$ and $a \geqslant 0$ such that $f^a(q^1) = p^1$ and $f^t(q^1) \in V_n$ for every $0 \leqslant t \leqslant a$. By Proposition 2 we have

$$d(q^1, V^+) \leqslant \lambda^a d(p^1, V^+) \leqslant \lambda^a r_{n+1} \leqslant r_{n+1} = r_n^{1+\delta}.$$

Thus there is $y_0^1 \in V^+$ such that $y_0^1 \in B(r_n^{1+\delta}, q^1)$, where $B(r, q) = \{y \in M\colon d(y, q) \leqslant r\}$.

To create a homoclinic point associated to Λ the proof is divided into four claims. Take and fix α with $0 < \alpha < \delta$.

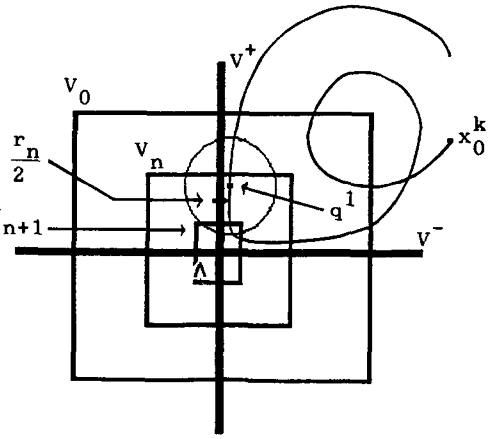

Fig. 1.

Claim 1. *If $d(q^1, V^-) > r_n/2$ and n is large enough, then we have*

(i) $x^k_{m-i} \notin B(r_n^{1+\alpha}, q^1)\ (1 \leqslant i \leqslant m-l)$,

(ii) $d(B(r_n^{1+\alpha}, q^1), V^-) > r_n/4$,

(iii) $f^i(y_0^1) \notin B(r_n^{1+\alpha}, q^1)\ (i \geqslant 1)$.

(*See Fig.* 1.)

To see (i) suppose $x^k_{m-1} \in B(r_n^{1+\alpha}, q^1)$. Since $d(q^1, V^-) \leqslant \lambda d(x^k_{m-1}, V^-)$ by Proposition 1, for n large enough

$$d(x^k_{m-1}, q^1) \geqslant d(x^k_{m-1}, V^-) - d(q^1, V^-) > (1/\lambda - 1) r_n/2 > r_n^{1+\alpha}$$

which is a contradiction. Thus we have (i) for $i = 1$.

If $d(x^k_{m-i}, q^1) \leqslant r_n^{1+\alpha}$ for some $2 \leqslant i \leqslant m-l$, then

$$\begin{aligned} d(x^k_{m-i+1}, V^+) &\leqslant d(x^k_{m-i+1}, f(y_0^1)) \leqslant A d(x^k_{m-i}, y_0^1) \\ &\leqslant A d(x^k_{m-i}, q^1) + A d(q^1, y_0^1) \\ &\leqslant A(r_n^{1+\alpha} + r_n^{1+\delta}) \leqslant r_n \quad (\text{if } n \text{ is large}), \end{aligned}$$

where $A > 0$ is a number such that $d(f(z), f(w)) \leqslant A d(z, w)$ for $z, w \in M$. Since

$$\begin{aligned} d(x^k_{m-i+1}, V^-) &\leqslant d(x^k_{m-i+1}, f(q^1)) + d(f(q^1), V^-) \\ &\leqslant A r_n^{1+\alpha} + \lambda r_n \leqslant r_n \quad (\text{if } n \text{ is large}), \end{aligned}$$

we have $x^k_{m-i+1} \in V_n$, which contradicts that $x^k_{m-i} \notin V_n$ for $1 \leqslant i \leqslant m-l-1$. (i) was proved.

(ii) follows from the fact that

$$d(x, V^-) \geqslant d(q^1, V^-) - d(x, q^1) > r_n/2 - r_n^{1+\alpha} > r_n/4$$

for every $x \in B(r_n^{1+\alpha}, q^1)$.

Finally, to check (iii) we use Proposition 1. Then

$$d(f^i(y_0^1), V^-) \leqslant \lambda^i d(y_0^1, V^-) \leqslant \lambda d(y_0^1, V^-)$$

for every $i \geqslant 1$. Since $d(y_0^i, V^-) \geqslant d(q^1, V^-) - d(q^1, y_0^1) > r_n/2 - r_n^{1+\delta}$, we have

$$d(f^i(y_0^1), q^1) \geqslant d(q^1, V^-) - d(f^i(y_0^1), V^-)$$
$$\geqslant r_n/2 - \lambda(r_n/2 - r_n^{1+\delta}) > r_n^{1+\alpha}$$

for sufficiently large n. Therefore we obtain (iii).

Set $W = \mathrm{Cl}\{x_{-i}^k: k \geqslant 0 \text{ and } 0 \leqslant i \leqslant m_k\} \cup B(\Lambda)$ where $B(\Lambda)$ is as in (4). Then $W \cap S(f) = \emptyset$ by the assumption of Theorem A. Thus there is $K > 0$ such that if the distance between x and y is sufficiently small then for every $x_{-1} \in f^{-1}(x) \cap W$ there exists $y_{-1} \in f^{-1}(y)$ such that $d(x_{-1}, y_{-1}) \leqslant Kd(x, y)$. This ensures the existence of $y_{-1}^1 \in f^{-1}(y_0^1)$ such that $d(x_{m-1}^k, y_{-1}^1) \leqslant Kd(x_m^k, y_0^1) \leqslant K(n)$ where $K(n) = Kr_n^{1+\delta}$ for large $n > 0$.

Claim 2. *If $d(q^1, V^-) \leqslant r_n/2$ and n is sufficiently large then*

(i) $x_{m-i}^k \notin B(K(n)^{1/(1+\alpha)}, x_{m-1}^k)$ *for* $2 \leqslant i \leqslant m - l$,

(ii) *either* $d(B(K(n)^{1/(1+\alpha)}, x_{m-1}^k), V^-) > 2r_n/3$ *or* $B(K(n)^{1/(1+\alpha)}, x_{m-1}^k) \cap V_n = \emptyset$,

(iii) $f^i(y_0^1) \notin B(K(n)^{1/(1+\alpha)}, x_{m-1}^k)$ $(i \geqslant 0)$.

(*See Fig.* 2.)

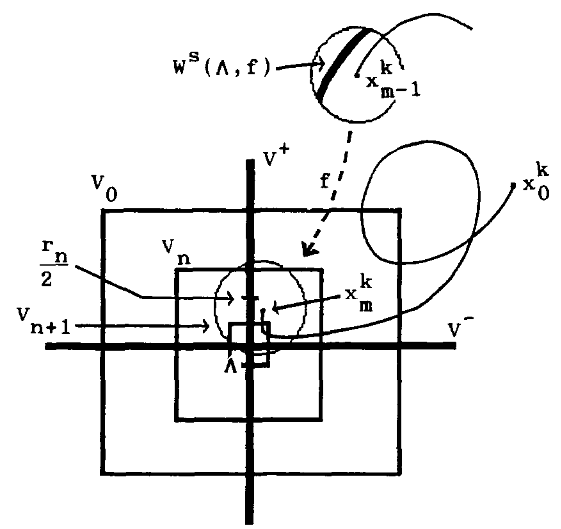

Fig. 2.

To show (i) suppose that $x^k_{m-i} \in B(K(n)^{1/(1+\alpha)}, x^k_{m-1})$ for some $2 \leqslant i \leqslant m-l$, then

$$d(x^k_{m-i+1}, x^k_m) \leqslant A d(x^k_{m-i}, x^k_{m-1}) \leqslant AK(n)^{1/(1+\alpha)}.$$

Thus

$$d(x^k_{m-i+1}, V^+) \leqslant AK(n)^{1/(1+\alpha)} + r_n^{1+\delta} \leqslant r_n,$$

$$d(x^k_{m-i+1}, V^-) \leqslant AK(n)^{1/(1+\alpha)} + r_n/2 \leqslant r_n,$$

from which we have $x^k_{m-i+1} \in V_n$, thus contradicting.

If $x^k_{m-1} \in V_{n-1}$, by Proposition 2 we have $d(x^k_{m-1}, V^+) \leqslant \lambda d(x^k_m, V^+) < r_n$, which implies that $d(x^k_{m-1}, V^-) > r_n$ since $x^k_{m-1} \notin V_n$. Thus $d(x, V^-) \geqslant r_n - K(n)^{1/(1+\alpha)} > 2r_n/3$ for every $x \in B(K(n)^{1/(1+\alpha)}, x^k_{m-1})$ if n is large. When $x^k_{m-1} \notin V_{n-1}$, we have either $d(x^k_{m-1}, V^+) > r_{n-1}$ or $d(x^k_{m-1}, V^-) > r_{n-1}$. This implies that either $d(x, V^+) > r_n$ or $d(x, V^-) > r_n$ for $x \in B(K(n)^{1/(1+\alpha)}, x^k_{m-1})$. Therefore $x \notin V_n$ and so we obtain (ii).

By Proposition 1 we have

$$d(f^i(y^1_0), V^-) \leqslant \lambda^i d(y^1_0, V^-) \leqslant \lambda^i (d(y^1_0, q^1) + d(q^1, V^-))$$
$$< \lambda^i (r_n^{1+\delta} + r_n/2) < 2r_n/3 < r_n.$$

Moreover $f^i(y^1_0) \in V^+$ for every $i \geqslant 0$ since $y^1_0 \in V^+$. Thus we have (iii) from (ii).

Since q^2 is the first point of $\sigma_2 \cap V_n$, we have $f(q^2) \notin V_n$, which implies that $d(f(q^2), V^+) > r_n$ or $d(f(q^2), V^-) > r_n$. From Proposition 1

$$d(f(q^2), V^-) \leqslant \lambda d(q^2, V^-) \leqslant \lambda r_n < r_n$$

and hence $d(f(q^2), V^+) > r_n$. Since σ_2 is an $(x^k, n+1)$-string, we can find $p^2 \in \sigma_2 \cap V_{n+1}$ and $a \geqslant 0$ such that $f^a(p^2) = q^2$. Using Proposition 1 again

$$r_n < d(f(q^2), V^+) = d(f^{a+1}(p^2), V^+)$$
$$\leqslant \gamma^{-(a+1)} d(p^2, V^+) \leqslant \gamma^{-(a+1)} r_n^{1+\delta},$$

from which $r_n^\delta/\gamma > \gamma^a$. Since $d(q^2, V^-) = d(f^a(p^2), V^-) \leqslant \lambda^a d(p^2, V^-) \leqslant \lambda^a r_n^{1+\delta}$, we have

$$d(q^2, V^-) \leqslant \gamma^{\beta a} r_n^{1+\delta} \leqslant \gamma^{-\beta} r_n^{1+\delta+\beta\delta} \leqslant \gamma^{-\beta} r_n^{1+\delta}$$

where $\lambda = \gamma^\beta$ with $0 < \beta < 1$.

Take $t > 0$ such that $\lambda^t < 1/2$. Then we have the following

Claim 3. *For n sufficiently large, there are points $q^2_0, q^2_{-1}, \ldots, q^2_{-t} \in V_n$ such that*

(i) $q^2_0 = q^2$,

(ii) $f(q^2_{-i}) = q^2_{-i+1}$ $(1 \leqslant i \leqslant t)$,

(iii) $\gamma d(q^2_{-i}, V^-) \leqslant d(q^2_{-i+1}, V^-)$ $(1 \leqslant i \leqslant t)$,

(iv) $d(q^2_{-i}, V^+) \leqslant \lambda d(q^2_{-i+1}, V^+)$ $(1 \leqslant i \leqslant t)$.

(*See Fig.* 3.)

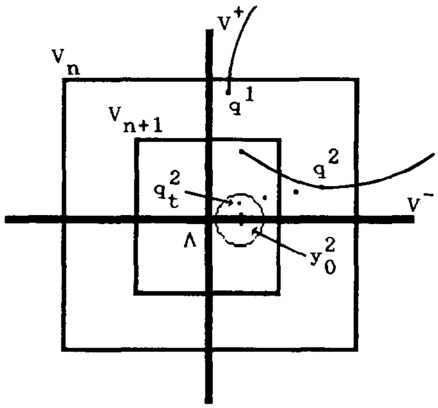

Fig. 3.

To check Claim 3 let $r_0 > 0$ and $\varepsilon_0 > 0$ be as before. Take $0 < \delta_0 < r_0$ as in (3) for $\varepsilon = \varepsilon_0$. Then there exists $0 < \delta_1 < \delta_0$ such that if $d(x, y) \leqslant \delta_1$ then for $x_{-1} \in f^{-1}(x) \cap B(\Lambda)$ there is a unique $y_{-1} \in f^{-1}(y)$ satisfying $d(x_{-1}, y_{-1}) \leqslant \delta_0$. If n is sufficiently large, then V_n is contained in the δ_1-neighborhood $B_{\delta_1}(\Lambda)$ of Λ and $\gamma^{-t-\beta} r_n^{1+\delta} < r_n$. Since $q^2 \in V_n$, there exists $z \in \Lambda$ such that $d(z, q^2) < \delta_1$. For $z_{-1} \in f^{-1}(z) \cap \Lambda$ we can choose $q_{-1}^2 \in f^{-1}(q^2)$ as in (b)(i) of Proposition 1 such that

$$d(q_{-1}^2, z_{-1}) < \delta_0 \quad \text{and} \quad \gamma d(q_{-1}^2, V^-) \leqslant d(q^2, V^-).$$

Thus we have

$$d(q_{-1}^2, V^-) \leqslant \gamma^{-1} d(q^2, V^-) \leqslant \gamma^{-1-\beta} r_n^{1+\delta} < r_n < r_0.$$

Moreover $d(q_{-1}^2, V^+) \leqslant d(q_{-1}^2, z_{-1}) \leqslant \delta_0$ and so $q_{-1}^2 \in V_0$. Thus, by Proposition 2

$$d(q_{-1}^2, V^+) \leqslant \lambda d(q^2, V^+) \leqslant \lambda r_n < r_n.$$

Since $q_{-1}^2 \in V_n$, we repeat this process and then we have Claim 3.

From Claim 3(i) and the fact that $\lambda^t < 1/2$ we have

$$d(q_{-t}^2, V^+) \leqslant \lambda^t d(q_0^2, V^+) \leqslant r_n/2,$$

$$d(q_{-t}^2, V^-) \leqslant \gamma^{-(t+\beta)} r_n^{1+\delta} = C r_n^{1+\delta}$$

where $C = \gamma^{-(t+\beta)}$. Write $C(n) = C r_n^{1+\delta}$ for simplicity. Then there is $y_0^2 \in V^-$ such that $q_{-t}^2 \in B(C(n), y_0^2)$. By Proposition 2 it is easily checked that there exists a sequence $\{y_{-i}^2\}_{i \geqslant 0} \subset V_0 \cap V^-$ such that

(i) $f(y_{-i}^2) = y_{-i+1}^2$ $(i \geqslant 1)$,

(ii) $d(y_{-i}^2, V^+) \leqslant \lambda d(y_{-i+1}^2, V^+)$ $(i \geqslant 1)$.

Claim 4. *For n sufficiently large,*

(i) $y_{-i}^2 \notin B(C(n)^{1/(1+\alpha)}, y_0^2)$ $(i \geqslant 1)$,

(ii) $f^s(q_{-t}^2) = q_{-t+s}^2 \notin B(C(n)^{1/(1+\alpha)}, y_0^2)$ $(1 \leqslant s \leqslant t)$,

(iii) $B(C(n)^{1/(1+\alpha)}, y_0^2) \subset V_n$,
(iv) $x_{m-i}^k \notin B(C(n)^{1/(1+\alpha)}, y_0^2)$ $(1 \leqslant i \leqslant m-l-1)$,
(v) $d(z, V^-) < r_n/4$ *for every* $z \in B(C(n)^{1/(1+\alpha)}, y_0^2)$.

First we check (i). By Proposition 1

$$d(q_{-t}^2, V^+) \geqslant \gamma^{t+1} d(f^{t+1}(q_{-t}^2), V^+)$$
$$= \gamma^{t+1} d(f(q_0^2), V^+) > \gamma^{t+1} r_n$$

and hence

$$d(y_0^2, V^+) \geqslant d(q_{-t}^2, V^+) - d(y_0^2, q_{-t}^2) > \gamma^{t+1} r_n - C(n).$$

By Claim 3(ii) we have

$$d(y_{-i}^2, V^+) \leqslant \lambda^i d(y_0^2, V^+) \leqslant \lambda d(y_0^2, V^+) \quad (i \geqslant 1),$$

from which

$$d(y_{-i}^2, y_0^2) \geqslant d(y_0^2, V^+) - d(y_{-i}^2, V^+)$$
$$> (1-\lambda) d(y_0^2, V^+)$$
$$> (1-\lambda)(\gamma^{t+1} r_n - C(n))$$
$$> C(n)^{1/(1+\alpha)} \quad (\text{if } n \text{ is large}).$$

Thus we have (i).

Let A be as in the proof of Claim 1. Then we have

$$d(f^s(q_{-t}^2), y_0^2) \geqslant d(y_0^2, f^s(y_0^2)) - d(f^s(q_{-t}^2), f^s(y_0^2))$$
$$\geqslant d(y_0^2, f^s(y_0^2)) - A^s d(q_{-t}^2, y_0^2)$$
$$\geqslant d(y_0^2, f^s(y_0^2)) - A^s C(n),$$

and by Proposition 1

$$\gamma^s d(f^s(y_0^2), V^+) \leqslant d(y_0^2, V^+),$$

from which

$$d(f^s(y_0^2), V^+) \leqslant \gamma^{-s} d(y_0^2, V^+) \leqslant \gamma^{-s}(r_n/2 + C(n)) < r_{n-1}$$

if n is large. Since $d(f^s(y_0^2), V^-) = 0$, we have $f^s(y_0^2) \in V_{n-1}$ $(1 \leqslant s \leqslant t)$. By Proposition 2

$$d(y_0^2, f^s(y_0^2)) \geqslant d(f^s(y_0^2), V^+) - d(y_0^2, V^+)$$
$$\geqslant (\lambda^{-s} - 1) d(y_0^2, V^+)$$
$$\geqslant (\lambda^{-s} - 1)(\gamma^{t+1} r_n - C(n)),$$

from which

$$d(f^s(q^2_{-t}), y^2_0) \geq (\lambda^{-s} - 1)(\gamma^{t+1} r_n - C(n)) - A^s C(n)$$
$$> C(n)^{1/(1+\alpha)} \quad \text{(if } n \text{ is large)}.$$

Thus we obtain (ii).

For $x \in B(C(n)^{1/(1+\alpha)}, y^2_0)$

$$d(x, V^+) \leq d(y^2_0, V^+) + d(x, y^2_0)$$
$$\leq r_n/2 + C(n) + C(n)^{1/(1+\alpha)} < r_n$$

if n is large. On the other hand, since $y^2_0 \in V^-$, we have $d(x, V^-) \leq C(n)^{1/(1+\alpha)} < r_n$. Therefore $x \in V_n$ and so we obtain (iii).

(iv) is easily checked by (iii), and (v) follows from the fact

$$d(z, V^-) \leq d(z, y^2_0) \leq C(n)^{1/(1+\alpha)} < r_n/4$$

for every $z \in B(C(n)^{1/(1+\alpha)}, y^2_0)$.

Choose $c > 0$ such that $0 < c < \alpha$ and $(1+\alpha)(1+c) < 1 + \delta$. Let $\mathscr{U}(f)$ be a neighborhood of f in $C^1(M)$. Then there exists a neighborhood $\mathscr{N}$ of the identity in the C^1-topology such that $\mathscr{N} \circ f \subset \mathscr{U}(f)$. To obtain the conclusion we need the following lemma.

Lemma 6 (cf. [3]). *Given a constant $c > 0$ and a neighborhood $\mathscr{N}$ of the identity, there exists $R > 0$ such that for $0 < r \leq R$ and $x, y \in M$ with $d(x, y) \leq r^{1+c}$ there is $h \in \mathscr{N}$ satisfying that $h(x) = y$ and $h(z) = z$ for all z outside of $B(r, x)$.*

Choose a sufficiently large n such that $\max\{r_n^{1+\alpha}, K(n)^{1/(1+\alpha)}, C(n)^{1/(1+\alpha)}\} < R$. If $d(q^1, V^-) > r_n/2$, then there exists $y^1_0 \in V^+ \cap B(r_n^{1+\delta}, q^1)$ such that Claim 1 holds. Since $r_n^{1+\alpha} > r_n^{(1+\delta)/(1+c)}$, as in Lemma 6 there exists $h_1 \in \mathscr{N}$ such that

(1-i) $h_1(q^1) = y^1_0$,

(1-ii) $h_1 = \mathrm{id}$ on $M \backslash B(r_n^{1+\alpha}, q^1)$.

Let $q^2_{-t} \in V_n$ and $y^2_0 \in V^-$ as above. Then we have $q^2_{-t} \in B(C(n), y^2_0)$ and so there exists $h_2 \in \mathscr{N}$ such that

(2-i) $h_2(y^2_0) = q^2_{-t}$,

(2-ii) $h_2 = \mathrm{id}$ on $M \backslash B(C(n)^{1/(1+c)}, y^2_0) \supset M \backslash B(C(n)^{1/(1+\alpha)}, y^2_0)$.

By Claim 1(ii) and Claim 4(v)

$$B(r_n^{1+\alpha}, q^1) \cap B(C(n)^{1/(1+\alpha)}, y^2_0) = \emptyset$$

from which $h_1 \circ h_2 \in \mathscr{N}$. Define $g \in \mathscr{U}(f)$ by $g = h_1 \circ h_2 \circ f$. Then it is easily checked that $W^s(\Lambda, g) \cap W^u(\Lambda, g) \backslash \Lambda \neq \emptyset$ by Claims 1 and 4.

Similarly, we obtain the conclusion for the case $d(q^1, V^-) \leq r_n/2$ by Claims 2 and 4. We proved Theorem A for the case (a).

If (b) is satisfied, then there exists an $(\boldsymbol{x}^k, n+1)$-string σ_1 such that $\sigma \cap V_n = \emptyset$ for every $(\boldsymbol{x}^k, 0)$-string σ with $\sigma \neq \sigma_1$. Let q^2 be the first point of $\sigma_1 \cap V_n$ and put

$q^2 = x_l^k$ for some $-m_k \leqslant l < 0$. Then we have (i)–(iv) of Claim 4 by the same way as the case (a). Since $x_i^k \notin B(C(n)^{1/(1+\alpha)}, y_0^2)$ for every $l \leqslant i \leqslant 0$ by Claim 4(iii), there exists $g \in \mathscr{U}(f)$ such that $g = f$ on $M \setminus B(C(n)^{1/(1+\alpha)}, y_0^2)$ and $g^l(y_0^2) = x_0^k$. Therefore $x_0^k \in W^u(\Lambda, g)$.

The proof of Theorem A is completed.

Theorem C is proved by using Proposition 4. For the details see the proof of Theorem D in [3].

References

[1] N. Aoki and M. Oka, Homoclinic points C^r-created under hypotheses by probability measures, in: Probability Measures on Groups X (Plenum, New York, 1991) 1–10.
[2] M. Hirsch, J. Palis, C. Pugh and M. Shub, Neighborhoods of hyperbolic sets, Invent. Math. 9 (1970) 121–134.
[3] R. Mañé, On the creation of homoclinic points, Publ. Math. IHES 66 (1987) 139–159.
[4] R. Mañé, A proof of the C^1 stability conjecture, Publ. Math. IHES 66 (1987) 161–210.
[5] F. Oliveira, On the generic existence of homoclinic points, Ergodic Theory Dynamical Systems 7 (1987) 567–595.
[6] D. Pixton, Planar homoclinic points, J. Differential Equations 44 (1982) 365–382.
[7] F. Przytycki, Anosov endomorphisms, Studia Math. 58 (1976) 249–285.
[8] F. Przytycki, On Ω-stability and structural stability of endomorphisms satisfying Axiom A, Studia Math. 60 (1977) 61–77.
[9] C. Robinson, Closing stable and unstable manifolds on the two-sphere, Proc. Amer. Math. Soc. 41 (1973) 299–303.
[10] F. Takens, Homoclinic points in conservative systems, Invent. Math. 18 (1972) 267–292.