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Zeros of polynomials orthogonal with respect to a signed weight

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Abstract

In this paper we consider the monic polynomial sequence $(P_n^{\alpha,q}(x))$ that is orthogonal on $[-1, 1]$ with respect to the weight function $x^{2q+1}(1-x^2)^\alpha(1-x), \alpha > -1, q \in \mathbb{N}$; we obtain the coefficients of the tree-term recurrence relation(TTRR) by using a different method from the one derived in Atia et al. (2002) [2]; we prove that the interlacing property does not hold properly for $(P_n^{\alpha,q}(x))$; and we also prove that, if $x_{n,n}^{\alpha+i,q+j}$ is the largest zero of $P_n^{\alpha+i,q+j}(x)$, $x_{2n-2j,2n-2j}^{\alpha+j,q+j} < x_{2n-2i,2n-2i}^{\alpha+i,q+i}$, $0 \le i < j \le n-1$.

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Contents

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1. Introduction

It is a well known fact that if (p_n) is orthogonal with respect to a (real) weight function, namely $w(x)$, and such weight function is positive on $[a, b]$, then the zeros of p_n are real, distinct, interlace, and lie inside $]a, b[$, but such an interlacing property is no longer valid when the weight is a signed function. In fact, Perron [9] proved that when the $w(x)$ changes sign once then one of zeros can lie outside of $[a, b]$.

In this paper we prove that such zero can lie into one of the endpoints of the interval, a or b. We consider the monic polynomial sequence $(P_n^{\alpha,q}(x))$ that is orthogonal on $[-1, 1]$ with respect to the weight function $x^{2q+1}(1-x^2)^\alpha(1-x)$, $\alpha > -1$, $q \in \mathbb{N}$ that changes sign once, at $x = 0$, and we prove that all the zeros are real, non interlacing, and that one of the zeros is the endpoint $a = -1$.

The sequence of monic orthogonal polynomials $(P_n^{\alpha,q}(x))$ satisfies for $n \geq 0$ the following TTRR [4]:

$$P_{n+2}^{\alpha,q}(x) = (x - \beta_{n+1}^{\alpha,q}) P_{n+1}^{\alpha,q}(x) - \gamma_{n+1}^{\alpha,q} P_n^{\alpha,q}(x), \tag{1}$$

with initial conditions $P_0^{\alpha,q}(x) = 1$, $P_1^{\alpha,q}(x) = x - \beta_0^{\alpha,q}$, being $(\beta_n^{\alpha,q})$ and $(\gamma_n^{\alpha,q})$ the coefficients of the recurrence relation. They were calculated in [2] by using the Laguerre–Freud equations, and, later on, an explicit expression for $P_n^{\alpha,q}(x)$ was given in [1]. The main aim of this paper is to keep studying these polynomials, more precisely, the behavior of the zeros of $(P_n^{\alpha,q}(x))$.

People working on zeros of orthogonal polynomials know how difficult it is to explore this area. In fact, even in the case of Jacobi polynomials results on zeros are presented as conjectures (see [5,6]).

In order to do this study, we use generalized Gegenbauer polynomials $(GG_n^{\alpha,\mu})$ that are orthogonal on $[-1, 1]$ with respect to the weight function $|x|^\mu(1-x^2)^\alpha$, $\alpha > -1$, $\mu > -1$. Actually, $GG_{2n}^{\alpha,\mu}(x) = J_n^{\alpha,\frac{\mu-1}{2}}(x^2)$ $\left(GG_n^{\alpha,\mu}(x) = x^{\frac{1-(-1)^n}{2}} J_{[\frac{n}{2}]}^{\alpha,\frac{\mu-(-1)^n}{2}}(x^2)\right)$ where the J_n are the Jacobi polynomials on the interval $[0, 1]$ with weight $x^{\frac{\mu-1}{2}}(1-x)^\alpha$ and are given by the classical formula

$$J_n^{\alpha,\frac{\mu-1}{2}}(x) = x^{-\frac{\mu-1}{2}}(1-x)^{-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (x^{n+\frac{\mu-1}{2}}(1-x)^{n+\alpha}).$$

Some properties of GG-polynomials can be found in [10,3].

The structure of the paper is the following: in Section 2 we present basic definitions, some notations, and a few preliminary results, in Section 3 we obtain some algebraic relations between $(P_n^{\alpha,q}(x))$ and the GG-polynomials as well as the recurrence coefficients of the TTRR fulfilled by $(P_n^{\alpha,q}(x))$, and in Section 4 some results regarding zeros of $(P_n^{\alpha,q}(x))$ are given.

2. Basic definitions and preliminary results

The *Pochhammer symbol*, or shifted factorial, is defined as

$$(\alpha)_0 = 1, \qquad (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \quad n \geq 1. \tag{2}$$

The *Gauss's hypergeometric function* is

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} z^n, \quad \alpha, \beta \in \mathbb{C}; \gamma \in \mathbb{C} \setminus \mathbb{Z}_-; |z| < 1. \tag{3} $$

When α, or β, is a negative integer the hypergeometric series (3) terminates, i.e. it reduces to a polynomial (of degree α, or β resp.).

Observe that after straightforward calculation one gets

$$ {}_2F_1(-m, \beta; \gamma; z) = \sum_{n=0}^{m} \frac{(-m)_n(\beta)_n}{n!(\gamma)_n} z^n = \sum_{n=0}^{m} \binom{m}{n} \frac{(\beta)_n}{(\gamma)_n} (-z)^n. \tag{4} $$

Denoting by

$$ {}_2F_1(\alpha, \beta; \gamma; z) \equiv F, \qquad {}_2F_1(\alpha \pm 1, \beta; \gamma; z) \equiv F(\alpha \pm 1), $$
$$ {}_2F_1(\alpha, \beta \pm 1; \gamma; z) \equiv F(\beta \pm 1), \qquad {}_2F_1(\alpha, \beta; \gamma \pm 1; z) \equiv F(\gamma \pm 1). $$

The functions $F(\alpha \pm 1)$, $F(\beta \pm 1)$, and $F(\gamma \pm 1)$ are said to be *contiguous* of F [7, p. 242]. Among the relations of this type we cite the following ones:

$$ (\alpha - \beta)F - \alpha F(\alpha + 1) + \beta F(\beta + 1) = 0, \tag{5} $$
$$ (\alpha - \beta)(1 - z)F + (\gamma - \alpha)F(\alpha - 1) - (\gamma - \beta)F(\beta - 1) = 0. \tag{6} $$

Definition 2.1. For $n \geq 0$, GG polynomials are given by

$$ GG_{2n}^{\alpha,\mu}(x) = x^{2n}\, {}_2F_1\left(-n, -n - \frac{\mu}{2} + \frac{1}{2}; -2n - \alpha - \frac{\mu}{2} + \frac{1}{2}; \frac{1}{x^2}\right) \tag{7} $$
$$ GG_{2n+1}^{\alpha,\mu}(x) = x^{2n+1}\, {}_2F_1\left(-n, -n - \frac{\mu}{2} - \frac{1}{2}; -2n - \alpha - \frac{\mu}{2} - \frac{1}{2}; \frac{1}{x^2}\right). \tag{8} $$

3. Algebraic relations between $(P_n^{\alpha,q}(x))$ and the GG-polynomials

Proposition 3.1. *For any $n \geq 0$, and any integer q the following identities hold:*

$$ P_{2n}^{\alpha,q}(x) = GG_{2n}^{\alpha,2q+2}(x), \tag{9} $$
$$ P_{2n+1}^{\alpha,q}(x) = (1 + x)GG_{2n}^{\alpha+1,2q+2}(x). \tag{10} $$

Remark 3.1. Observe that with (9), we can write the last equation as

$$ P_{2n+1}^{\alpha,q}(x) = (1 + x)P_{2n}^{\alpha+1,q}(x), \tag{11} $$

one should point out that we have α in the left hand side whereas we have $\alpha + 1$ in the right hand side.

Proof. One can easily show that

$$ \int_{-1}^{1} x^{2q+1}(1 - x^2)^{\alpha}(1 - x)x^k GG_{2n}^{\alpha,2q+2}(x)\mathrm{d}x = 0, \quad 0 \leq k \leq 2n - 1. $$

They all follow from the orthogonality property of GG_{2n}, except for $k = 0$, where the oddity of x^{2q+1} is used in addition. It can also be shown that

$$\int_{-1}^{1} x^{2q+1}(1-x^2)^{\alpha}(1-x)x^k(1+x)GG_{2n}^{\alpha+1,2q+2}(x)\mathrm{d}x = 0, \quad 0 \le k \le 2n.$$

They all follow from the orthogonality property of GG_{2n}, except for $k = 2n$, where again the oddity of x^{2q+1} is important.

These identities immediately imply that the polynomials

$$P_{2n}^{\alpha,q} = GG_{2n}^{\alpha,2q+2}, \qquad P_{2n+1}^{\alpha,q} = (1+x)GG_{2n}^{\alpha+1,2q+2}$$

from a sequence of polynomials, orthogonal with respect to the weight $x^{2q+1}(1-x^2)^{\alpha}(1-x)$. □

Remark 3.2. By using Eq. (7) we get for $n \ge 0$ the hypergeometric representation for $P_{2n}^{\alpha,q}(x)$:

$$P_{2n}^{\alpha,q}(x) = x^{2n}\,{}_2F_1\left(-n, -n-q-\frac{1}{2}; -2n-\alpha-q-\frac{1}{2}; \frac{1}{x^2}\right), \tag{12}$$

that we can be written as [8, V1 p. 40 (23)]

$$P_{2n}^{\alpha,q}(x) = \frac{(-1)^n\left(q+\frac{3}{2}\right)_n}{\left(n+q+\alpha+\frac{3}{2}\right)_n}\,{}_2F_1\left(-n, n+q+\alpha+\frac{3}{2}; q+\frac{3}{2}; x^2\right). \tag{13}$$

Once we have got these algebraic relations we can compute the recurrences coefficients associated to the polynomial sequence $(P_n^{\alpha,q}(x))$.

Remark 3.3. Notice that due the expression of the integrals and the weight functions we can find a link between the polynomials $(P_n^{\alpha,q})$ and GG-polynomials, which was not possible to do with Laguerre–Freud equation [2] or with the explicit representation of $P_n^{\alpha,q}(x)$ [1].

Proposition 3.2. *The recurrence coefficients of the monic polynomial sequence* $(P_n^{\alpha,q}(x))$ *fulfills (1) are given by*

$$\beta_n^{\alpha,q} = (-1)^{n+1}, \quad n \ge 0 \tag{14}$$

$$\gamma_{2n}^{\alpha,q} = -2\frac{n(2n+2q+1)}{(4n+2\alpha+2q+1)(4n+2\alpha+2q+3)}, \quad n \ge 1 \tag{15}$$

$$\gamma_{2n+1}^{\alpha,q} = -2\frac{(n+\alpha+1)(2n+2\alpha+2q+3)}{(4n+2\alpha+2q+3)(4n+2\alpha+2q+5)}, \quad n \ge 0. \tag{16}$$

Proof. These recurrence coefficients follow from the contiguity relations between hypergeometric functions. □

4. Zeros of $(P_n^{\alpha,q})$

Using (9) and (10) we can state the following result:

Theorem 4.1. *The following statements hold:*

- *All the zeros of* $P_n^{\alpha,q}(x)$ *are real.*
- *The Perron's zero is* -1.
- *The zeros of* $P_{2n}^{\alpha,q}(x)$ *and the zeros of* $P_{2n+1}^{\alpha,q}(x)$ *do not interlace.*

Proof. By (9) and (10), the first two statements follow. To prove the third one, it is sufficient to point out that the zeros of $GG_{2n}^{\alpha,\mu}$ and $GG_{2n}^{\alpha+1,\mu}$ are non-interlacing. However, both polynomials are even of the same degree and non-zero at the origin. It is straightforward that two such polynomials cannot have interlacing zero sets. The zeros do not coincide because the parameter α becomes $\alpha+1$ and μ is unchanged. □

Proposition 4.2. *If we denote by* $x_{n,n}^{\alpha+i,q+j}$ *the largest zero of* $P_n^{\alpha+i,q+j}$, *then*

$$x_{2n-2j,2n-2j}^{\alpha+j,q+j} < x_{2n-2i,2n-2i}^{\alpha+i,q+i}, \quad 0 \le i < j \le n-1. \tag{17}$$

Proof. $P_{2n}^{\alpha,q}(x) = GG_{2n}^{\alpha,2q+2}(x) = J_n^{\alpha,\frac{2q+1}{2}}(x^2)$, then zeros of $J_n^{\alpha,\frac{2q+1}{2}}$ are the same as those of $K_n^{\alpha,\frac{2q+1}{2}}$ with $K_n^{\alpha,\frac{2q+1}{2}}(x) = (\frac{\mathrm{d}}{\mathrm{d}x})^n(x^{n+\frac{2q+1}{2}}(1-x)^{\alpha+n})$ thus

$$K_n^{\alpha,\frac{2q+1}{2}}(x) = \frac{\mathrm{d}}{\mathrm{d}x}K_{n-1}^{\alpha+1,\frac{2q+3}{2}}(x)$$

this implies that between two consecutive zeros of $K_n^{\alpha,\frac{2q+1}{2}}$ there exists one zero of $K_{n-1}^{\alpha+1,\frac{2q+3}{2}}$ and then the largest zero of $P_{2n}^{\alpha,q}$ is greater than $P_{2(n-1)}^{\alpha+1,q+1}$ and so on for the largest zero of $P_{2(n-1)}^{\alpha+1,q+1}$, $P_{2(n-2)}^{\alpha+2,q+2}, \ldots$. □

Remark 4.1. Using Proposition 4.2. and the relation

$$x_{2n+1,m}^{\alpha+k,q+l} = x_{2n,m-1}^{\alpha+k+1,q+l}; \quad k,l \in \mathbb{N}, 2 \le m \le 2n+1,$$

with $x_{2n+1,1}^{\alpha+k,q+l} = -1$, we obtain

$$x_{2n-2j+1,2n-2j+1}^{\alpha+j-1,q+j} < x_{2n-2i+1,2n-2i+1}^{\alpha+i-1,q+i}, \quad 0 \le i < j \le n.$$

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