Applied Mathematics Letters 26 (2013) 264–269

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Symmetric differentiation on time scales

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ARTICLE INFO

Article history:
Received 30 July 2012
Received in revised form 7 September 2012
Accepted 10 September 2012

Keywords:
Symmetric derivative
Quantum calculus
Time scales

ABSTRACT

We define a symmetric derivative on an arbitrary nonempty closed subset of the real numbers and derive some of its properties. It is shown that real-valued functions defined on time scales that are neither delta nor nabla differentiable can be symmetric differentiable.

1. Introduction

Symmetric properties of functions are very useful in a large number of problems. Particularly in the theory of trigonometric series, applications of such properties are well known [1]. Differentiability is one of the most important properties in the theory of functions of real variables. However, even simple functions such as

$$f(t) = |t|, \qquad g(t) = \begin{cases} t \sin \frac{1}{t}, & t \neq 0 \\ 0, & t = 0, \end{cases} \qquad h(t) = \frac{1}{t^2}, \quad t \neq 0, \tag{1.1}$$

do not have a (classical) derivative at $t = 0$. Authors like Riemann, Schwarz, Peano, Dini, and de la Vallée-Poussin extended the classical derivative in different ways, depending on the purpose [1]. One of those notions is the symmetric derivative:

$$f^s(t) = \lim_{h \to 0} \frac{f(t+h) - f(t-h)}{2h}. \tag{1.2}$$

While the functions in (1.1) do not have ordinary derivatives at $t = 0$, they have symmetric derivatives: $f^s(0) = g^s(0) = h^s(0) = 0$. For a deeper understanding of the symmetric derivative and its properties, we refer the reader to the specialized monograph [2]. Here we note that the symmetric quotient $(f(t+h) - f(t-h))/(2h)$ has, in general, better convergence properties than the ordinary difference quotient [3], leading naturally to the so-called h-symmetric quantum calculus [4]. A more recent theory is the general time scale calculus. In 1988, Hilger introduced the calculus on time scales as a generalization of continuous and discrete time theories, obviating the need for separate proofs and highlighting the differences between them [5,6]. Here we introduce the notion of symmetric derivative on time scales, initiating the corresponding theory and putting into context some of the recent results found in the literature.

The article is organized as follows. In Section 2 we review the necessary concepts and we fix notations. The results are then given in Section 3, where we define the time scale symmetric derivative and derive some of its properties. Applications are found in the context of quantum calculus [4]. Finally, we show in Section 4 that the new symmetric derivative is a generalization of the diamond-α derivative [7], which brings new insights.

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doi:10.1016/j.aml.2012.09.005

2. Preliminary notions and notations

A nonempty closed subset of $\mathbb{R}$ is called a time scale and is denoted by $\mathbb{T}$. We assume that a time scale has the topology inherited from $\mathbb{R}$ with the standard topology. Two jump operators are considered: the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M), and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ with $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m). A point $t \in \mathbb{T}$ is said to be right-dense, right-scattered, left-dense or left-scattered if $\sigma(t) = t, \sigma(t) > t, \rho(t) = t$ or $\rho(t) < t$, respectively. A point $t \in \mathbb{T}$ is dense if it is right and left dense; isolated if it is right and left scattered. If $\sup \mathbb{T}$ is finite and left-scattered, then we define $\mathbb{T}^\kappa := \mathbb{T} \setminus \{\sup \mathbb{T}\}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$; if $\inf \mathbb{T}$ is finite and right-scattered, then $\mathbb{T}_\kappa := \mathbb{T} \setminus \{\inf \mathbb{T}\}$, otherwise $\mathbb{T}_\kappa := \mathbb{T}$. We set $\mathbb{T}^\kappa_\kappa := \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$ and we denote $f \circ \sigma$ by f^σ and $f \circ \rho$ by f^ρ.

3. Main results

In quantum calculus, the h-symmetric difference and the q-symmetric difference, $h > 0$ and $0 < q < 1$, are defined by

$$\tilde{D}_h(t) = \frac{f(t+h) - f(t-h)}{2h} \tag{3.1}$$

and

$$\tilde{D}_q(t) = \frac{f(qt) - f\left(q^{-1}t\right)}{\left(q - q^{-1}\right)t}, \quad t \neq 0, \tag{3.2}$$

respectively [4]. Here we propose a general notion of symmetric derivative on time scales that encompasses all the three definitions (1.2), (3.1) and (3.2).

Definition 3.1. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is symmetric continuous at $t \in \mathbb{T}$ if, for any $\varepsilon > 0$, there exists a neighborhood $U_t \subset \mathbb{T}$ of t such that for all $s \in U_t$ for which $2t - s \in U_t$ one has $|f(s) - f(2t-s)| \leqslant \varepsilon$.

It is easy to see that continuity implies symmetric continuity.

Proposition 3.2. *Let $\mathbb{T}$ be a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is a continuous function, then f is symmetric continuous.*

Proof. Since f is continuous at $t \in \mathbb{T}$, then, for any $\varepsilon > 0$, there exists a neighborhood U_t of t such that $|f(s) - f(t)| < \frac{\varepsilon}{2}$ and $|f(2t-s) - f(t)| < \frac{\varepsilon}{2}$ for all $s \in U_t$ for which $2t - s \in U_t$. Thus, $|f(s) - f(2t-s)| \leqslant |f(s) - f(t)| + |f(t) - f(2t-s)| < \varepsilon$. $\square$

The next example shows that symmetric continuity does not imply continuity.

Example 3.3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 0, & \text{if } t \neq 0, \\ 1, & \text{if } t = 0. \end{cases}$$

Function f is symmetric continuous at 0: for any $\varepsilon > 0$, there exists a neighborhood U_t of $t = 0$ such that $|f(s) - f(-s)| = 0 < \varepsilon$ for all $s \in U_t$ for which $-s \in U_t$. However, f is not continuous at 0.

Definition 3.4. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa_\kappa$. The symmetric derivative of f at t, denoted by $f^\diamondsuit(t)$, is the real number (provided it exists) with the property that, for any $\varepsilon > 0$, there exists a neighborhood $U \subset \mathbb{T}$ of t such that

$$\left|[f^\sigma(t) - f(s) + f(2t-s) - f^\rho(t)] - f^\diamondsuit(t)[\sigma(t) + 2t - 2s - \rho(t)]\right| \leqslant \varepsilon \left|\sigma(t) + 2t - 2s - \rho(t)\right|$$

for all $s \in U$ for which $2t - s \in U$. A function f is said to be symmetric differentiable provided $f^\diamondsuit(t)$ exists for all $t \in \mathbb{T}^\kappa_\kappa$.

Some useful properties of the symmetric derivative are given in Theorem 3.5.

Theorem 3.5. *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa_\kappa$. The following holds:*

(i) *Function f has at most one symmetric derivative at t.*
(ii) *If f is symmetric differentiable at t, then f is symmetric continuous at t.*
(iii) *If f is continuous at t and t is not dense, then f is symmetric differentiable at t with $f^\diamondsuit(t) = \frac{f^\sigma(t) - f^\rho(t)}{\sigma(t) - \rho(t)}$.*
(iv) *If t is dense, then f is symmetric differentiable at t if and only if the limit $\lim_{s \to t} \frac{f(2t-s) - f(s)}{2t - 2s}$ exists as a finite number. In this case $f^\diamondsuit(t) = \lim_{s \to t} \frac{f(2t-s) - f(s)}{2t - 2s} = \lim_{h \to 0} \frac{f(t+h) - f(t-h)}{2h}$.*
(v) *If f is symmetric differentiable and continuous at t, then $f^\sigma(t) = f^\rho(t) + f^\diamondsuit(t)[\sigma(t) - \rho(t)]$.*

Proof. (i) Suppose that f has two symmetric derivatives at t, $f_1^{\diamondsuit}(t)$ and $f_2^{\diamondsuit}(t)$. Then, there exists a neighborhood U_1 of t such that

$$\left|[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]-f_1^{\diamondsuit}(t)[\sigma(t)+2t-2s-\rho(t)]\right|\leqslant\frac{\varepsilon}{2}|\sigma(t)+2t-2s-\rho(t)|$$

for all $s\in U_1$ for which $2t-s\in U_1$, and a neighborhood U_2 of t such that

$$\left|[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]-f_2^{\diamondsuit}(t)[\sigma(t)+2t-2s-\rho(t)]\right|\leqslant\frac{\varepsilon}{2}|\sigma(t)+2t-2s-\rho(t)|$$

for all $s\in U_2$ for which $2t-s\in U_2$. Therefore, for all $s\in U_1\cap U_2$ for which $2t-s\in U_1\cap U_2$,

$$\begin{aligned}
|f_1^{\diamondsuit}(t)-f_2^{\diamondsuit}(t)| &= \left|\left[f_1^{\diamondsuit}(t)-f_2^{\diamondsuit}(t)\right]\frac{\sigma(t)+2t-2s-\rho(t)}{\sigma(t)+2t-2s-\rho(t)}\right|\\
&= \frac{1}{|\sigma(t)+2t-2s-\rho(t)|}\,\Big|\,[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]\\
&\quad -f_2^{\diamondsuit}(t)[\sigma(t)+2t-2s-\rho(t)]\\
&\quad -[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]+f_1^{\diamondsuit}(t)[\sigma(t)+2t-2s-\rho(t)]\Big|\\
&\leqslant \frac{1}{|\sigma(t)+2t-2s-\rho(t)|}\Bigg(\Big|\,[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]\\
&\quad -f_2^{\diamondsuit}(t)[\sigma(t)+2t-2s-\rho(t)]\Big|\\
&\quad +\Big|[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]-f_1^{\diamondsuit}(t)[\sigma(t)+2t-2s-\rho(t)]\Big|\Bigg)\\
&\leqslant \varepsilon.
\end{aligned}$$

(ii) From the hypothesis, for any $\epsilon^*>0$, there exists a neighborhood U of t such that

$$\left|[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]-f^{\diamondsuit}(t)[\sigma(t)+2t-2s-\rho(t)]\right|\leqslant\varepsilon^*|\sigma(t)+2t-2s-\rho(t)|$$

for all $s\in U$ for which $2t-s\in U$. Therefore, for all $s\in U\cap]t-\varepsilon^*,t+\varepsilon^*[$,

$$\begin{aligned}
|f(2t-s)-f(s)| &\leqslant \left|[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]-f^{\diamondsuit}(t)[\sigma(t)+2t-2s-\rho(t)]\right|\\
&\quad+\left|[f^{\sigma}(t)-f^{\rho}(t)]-f^{\diamondsuit}(t)[\sigma(t)+2t-2s-\rho(t)]\right|\\
&\leqslant \left|[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]-f^{\diamondsuit}(t)[\sigma(t)+2t-2s-\rho(t)]\right|\\
&\quad+\left|[f^{\sigma}(t)-f(t)+f(t)-f^{\rho}(t)]-f^{\diamondsuit}(t)[\sigma(t)+2t-2t-\rho(t)]\right|+2\left|f^{\diamondsuit}(t)\right||t-s|\\
&\leqslant \varepsilon^*|\sigma(t)+2t-2s-\rho(t)|+\varepsilon^*|\sigma(t)+2t-2t-\rho(t)|+2\left|f^{\diamondsuit}(t)\right||t-s|\\
&\leqslant \varepsilon^*|\sigma(t)-\rho(t)|+2\varepsilon^*|t-s|+\varepsilon^*|\sigma(t)-\rho(t)|+2\left|f^{\diamondsuit}(t)\right||t-s|\\
&= 2\varepsilon^*|\sigma(t)-\rho(t)|+2\left(\varepsilon^*+\left|f^{\diamondsuit}(t)\right|\right)|t-s|\\
&\leqslant 2\varepsilon^*|\sigma(t)-\rho(t)|+2\left(\varepsilon^*+\left|f^{\diamondsuit}(t)\right|\right)\varepsilon^*\\
&= 2\varepsilon^*\left[|\sigma(t)-\rho(t)|+\varepsilon^*+\left|f^{\diamondsuit}(t)\right|\right],
\end{aligned}$$

proving that f is symmetric continuous at t.

(iii) Suppose that $t\in\mathbb{T}_{\kappa}^{\kappa}$ is not dense and f is continuous at t. Then,

$$\lim_{s\to t}\frac{f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)}{\sigma(t)+2t-2s-\rho(t)}=\frac{f^{\sigma}(t)-f^{\rho}(t)}{\sigma(t)-\rho(t)}.$$

Hence, for any $\varepsilon>0$, there exists a neighborhood U of t such that

$$\left|\frac{f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)}{\sigma(t)+2t-2s-\rho(t)}-\frac{f^{\sigma}(t)-f^{\rho}(t)}{\sigma(t)-\rho(t)}\right|\leqslant\varepsilon$$

for all $s\in U$ for which $2t-s\in U$. It follows that

$$\left|[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]-\frac{f^{\sigma}(t)-f^{\rho}(t)}{\sigma(t)-\rho(t)}[\sigma(t)+2t-2s-\rho(t)]\right|$$
$$\leqslant \varepsilon[\sigma(t)+2t-2s-\rho(t)],$$

which proves that $f^{\diamondsuit}(t)=(f^{\sigma}(t)-f^{\rho}(t))/(\sigma(t)-\rho(t))$.

(iv) Assume that f is symmetric differentiable at t and t is dense. Let $\varepsilon>0$ be given. Then, there exists a neighborhood U of t such that

$$\left|[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]-f^{\diamondsuit}(t)[\sigma(t)+2t-2s-\rho(t)]\right|\leqslant\varepsilon|\sigma(t)+2t-2s-\rho(t)|$$

for all $s\in U$ for which $2t-s\in U$. Since t is dense, $\left|[-f(s)+f(2t-s)]-f^{\diamondsuit}(t)[2t-2s]\right|\leqslant\varepsilon|2t-2s|$ for all $s\in U$ for which $2t-s\in U$. It follows that $\left|\frac{f(2t-s)-f(s)}{2t-2s}-f^{\diamondsuit}(t)\right|\leqslant\varepsilon$ for all $s\in U$ with $s\neq t$. Therefore, we get the desired result: $f^{\diamondsuit}(t)=\lim_{s\to t}\frac{f(2t-s)-f(s)}{2t-2s}$. Conversely, let us suppose that t is dense and the limit $\lim_{s\to t}\frac{f(2t-s)-f(s)}{2t-2s}=:L$ exists. Then, there exists a neighborhood U of t such that $\left|\frac{f(2t-s)-f(s)}{2t-2s}-L\right|\leqslant\varepsilon$ for all $s\in U$ for which $2t-s\in U$. Because t is dense, we have $\left|\frac{f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)}{\sigma(t)+2t-2s-\rho(t)}-L\right|\leqslant\varepsilon$. Therefore,

$$|[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)]-L[\sigma(t)+2t-2s-\rho(t)]|\leqslant\varepsilon|\sigma(t)+2t-2s-\rho(t)|,$$

which leads us to the conclusion that f is symmetric differentiable and $f^{\diamondsuit}(t)=L$. Note that if we use the substitution $s=t+h$, then $f^{\diamondsuit}(t)=\lim_{h\to 0}\frac{f(t+h)-f(t-h)}{2h}$.

(v) If t is a dense point, then $\sigma(t)=\rho(t)$ and $f^{\sigma}(t)=f^{\rho}(t)+f^{\diamondsuit}(t)[\sigma(t)-\rho(t)]$. If t is not dense, and since f is continuous, then $f^{\diamondsuit}(t)=\frac{f^{\sigma}(t)-f^{\rho}(t)}{\sigma(t)-\rho(t)}\Leftrightarrow f^{\sigma}(t)=f^{\rho}(t)+f^{\diamondsuit}(t)[\sigma(t)-\rho(t)]$. □

Example 3.6. Let $\mathbb{T}=\mathbb{R}$. Then our symmetric derivative coincides with the classic symmetric derivative (1.2): $f^{\diamondsuit}=f^{s}$.

Example 3.7. Let $\mathbb{T}=h\mathbb{Z}$, $h>0$. Then the symmetric derivative is the symmetric difference operator (3.1): $f^{\diamondsuit}=\tilde{D}_h$.

Example 3.8. Let $\mathbb{T}=\overline{q^{\mathbb{Z}}}$, $0<q<1$. Then the symmetric derivative coincides with the q-symmetric difference operator (3.2): $f^{\diamondsuit}=\tilde{D}_q$.

Remark 3.9. Independently of the time scale $\mathbb{T}$, the symmetric derivative of a constant is zero and the symmetric derivative of the identity function is one.

Remark 3.10. An alternative way to define the symmetric derivative of f at $t\in\mathbb{T}^{\kappa}_{\kappa}$ consists in saying that the limit $f^{\diamondsuit}(t)=\lim_{s\to t}\frac{f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)}{\sigma(t)+2t-2s-\rho(t)}=\lim_{h\to 0}\frac{f^{\sigma}(t)-f(t+h)+f(t-h)-f^{\rho}(t)}{\sigma(t)-2h-\rho(t)}$ exists.

Theorem 3.11. *Let $f,g:\mathbb{T}\to\mathbb{R}$ be two symmetric differentiable functions at $t\in\mathbb{T}^{\kappa}_{\kappa}$ and $\lambda\in\mathbb{R}$. The following holds:*

(i) *Function $f+g$ is symmetric differentiable at t with $(f+g)^{\diamondsuit}(t)=f^{\diamondsuit}(t)+g^{\diamondsuit}(t)$.*
(ii) *Function λf is symmetric differentiable at t with $(\lambda f)^{\diamondsuit}(t)=\lambda f^{\diamondsuit}(t)$.*
(iii) *If f and g are continuous at t, then fg is symmetric differentiable at t with $(fg)^{\diamondsuit}(t)=f^{\diamondsuit}(t)g^{\sigma}(t)+f^{\rho}(t)g^{\diamondsuit}(t)$.*
(iv) *If f is continuous at t and $f^{\sigma}(t)f^{\rho}(t)\neq 0$, then $1/f$ is symmetric differentiable at t with $\left(\frac{1}{f}\right)^{\diamondsuit}(t)=-\frac{f^{\diamondsuit}(t)}{f^{\sigma}(t)f^{\rho}(t)}$.*
(v) *If f and g are continuous at t and $g^{\sigma}(t)g^{\rho}(t)\neq 0$, then f/g is symmetric differentiable at t with $\left(\frac{f}{g}\right)^{\diamondsuit}(t)=\frac{f^{\diamondsuit}(t)g^{\rho}(t)-f^{\rho}(t)g^{\diamondsuit}(t)}{g^{\sigma}(t)g^{\rho}(t)}$.*

Proof. (i) For $t\in\mathbb{T}^{\kappa}_{\kappa}$ we have

$$\begin{aligned}(f+g)^{\diamondsuit}(t)&=\lim_{s\to t}\frac{(f+g)^{\sigma}(t)-(f+g)(s)+(f+g)(2t-s)-(f+g)^{\rho}(t)}{\sigma(t)+2t-2s-\rho(t)}\\&=\lim_{s\to t}\frac{f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)}{\sigma(t)+2t-2s-\rho(t)}+\lim_{s\to t}\frac{g^{\sigma}(t)-g(s)+g(2t-s)-g^{\rho}(t)}{\sigma(t)+2t-2s-\rho(t)}\\&=f^{\diamondsuit}(t)+g^{\diamondsuit}(t).\end{aligned}$$

(ii) Let $t\in\mathbb{T}^{\kappa}_{\kappa}$ and $\lambda\in\mathbb{R}$. Then,

$$\begin{aligned}(\lambda f)^{\diamondsuit}(t)&=\lim_{s\to t}\frac{(\lambda f)^{\sigma}(t)-(\lambda f)(s)+(\lambda f)(2t-s)-(\lambda f)^{\rho}(t)}{\sigma(t)+2t-2s-\rho(t)}\\&=\lambda\lim_{s\to t}\frac{f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)}{\sigma(t)+2t-2s-\rho(t)}=\lambda f^{\diamondsuit}(t).\end{aligned}$$

(iii) Let us assume that $t \in \mathbb{T}_\kappa^\kappa$ and f and g are continuous at t. If t is dense, then

$$\begin{aligned}(fg)^\diamond(t) &= \lim_{h\to 0}\frac{(fg)(t+h)-(fg)(t-h)}{2h}\\ &= \lim_{h\to 0}\frac{f(t+h)-f(t-h)}{2h}g(t+h)+\lim_{h\to 0}\frac{g(t+h)-g(t-h)}{2h}f(t-h)\\ &= f^\diamond(t)g^\sigma(t)+f^\rho(t)g^\diamond(t).\end{aligned}$$

If t is not dense, then

$$\begin{aligned}(fg)^\diamond(t) &= \frac{(fg)^\sigma(t)-(fg)^\rho(t)}{\sigma(t)-\rho(t)} = \frac{f^\sigma(t)-f^\rho(t)}{\sigma(t)-\rho(t)}g^\sigma(t)+\frac{g^\sigma(t)-g^\rho(t)}{\sigma(t)-\rho(t)}f^\rho(t)\\ &= f^\diamond(t)g^\sigma(t)+f^\rho(t)g^\diamond(t)\end{aligned}$$

proving the intended equality.

(iv) Using the relation $\left(\frac{1}{f}\times f\right)(t)=1$ we can write that $0=\left(\frac{1}{f}\times f\right)^\diamond(t)=f^\diamond(t)\left(\frac{1}{f}\right)^\sigma(t)+f^\rho(t)\left(\frac{1}{f}\right)^\diamond(t)$. Therefore, $\left(\frac{1}{f}\right)^\diamond(t)=-\frac{f^\diamond(t)}{f^\sigma(t)f^\rho(t)}$.

(v) Let $t\in\mathbb{T}_\kappa^\kappa$. Then,

$$\begin{aligned}\left(\frac{f}{g}\right)^\diamond(t) &= \left(f\times\frac{1}{g}\right)^\diamond(t)=f^\diamond(t)\left(\frac{1}{g}\right)^\sigma(t)+f^\rho(t)\left(\frac{1}{g}\right)^\diamond(t)\\ &= \frac{f^\diamond(t)}{g^\sigma(t)}+f^\rho(t)\left(-\frac{g^\diamond(t)}{g^\sigma(t)g^\rho(t)}\right)=\frac{f^\diamond(t)g^\rho(t)-f^\rho(t)g^\diamond(t)}{g^\sigma(t)g^\rho(t)}.\quad\square\end{aligned}$$

Example 3.12. The symmetric derivative of $f(t)=t^2$ is $f^\diamond(t)=\sigma(t)+\rho(t)$.

Example 3.13. The symmetric derivative of $f(t)=1/t$ is $f^\diamond(t)=-\frac{1}{\sigma(t)\rho(t)}$.

4. Particular cases

In Section 3 we introduced the symmetric derivative on a time scale $\mathbb{T}$ and derived some of its properties. It has been shown that the new notion unifies the symmetric derivatives of classical analysis [2] and quantum calculus [4]. Here we note that our symmetric derivative is different from the delta and nabla derivatives considered in the time scale literature [5,6,8]. A simple example of a function that is neither delta nor nabla differentiable, in the sense of time scales, but that is symmetric differentiable, is the absolute value function.

Example 4.1. Let $\mathbb{T}$ be a time scale with $0\in\mathbb{T}_\kappa^\kappa$ and $f:\mathbb{T}\to\mathbb{R}$ be defined by $f(t)=|t|$. This function is not differentiable at point $t=0$ in the sense of time scales [5–8]. However, the symmetric derivative is always well defined:

$$f^\diamond(0)=\lim_{h\to 0}\frac{f^\sigma(0)-f(0+h)+f(0-h)-f^\rho(0)}{\sigma(0)-2h-\rho(0)}=\lim_{h\to 0}\frac{\sigma(0)+\rho(0)}{\sigma(0)-2h-\rho(0)},$$

so that $f^\diamond(0)=0$ if 0 is dense, and $f^\diamond(0)=(\sigma(0)+\rho(0))/(\sigma(0)-\rho(0))$ otherwise.

In the particular case a function is simultaneously delta and nabla differentiable [6,8], Proposition 4.2 shows that a relation can be done between our symmetric derivative and the recent diamond-α derivative [7, Corollary 4.4].

Proposition 4.2. *If f is delta and nabla differentiable, then f is symmetric differentiable and, for each $t\in\mathbb{T}_\kappa^\kappa$, $f^\diamond(t)=\gamma(t)f^\Delta(t)+(1-\gamma(t))f^\nabla(t)$, where*

$$\gamma(t)=\lim_{s\to t}\frac{\sigma(t)-s}{\sigma(t)+2t-2s-\rho(t)}. \tag{4.1}$$

Proof. Note that

$$\begin{aligned}f^\diamond(t) &= \lim_{s\to t}\frac{f^\sigma(t)-f(s)+f(2t-s)-f^\rho(t)}{\sigma(t)+2t-2s-\rho(t)}\\ &= \lim_{s\to t}\left(\frac{\sigma(t)-s}{\sigma(t)+2t-2s-\rho(t)}\frac{f^\sigma(t)-f(s)}{\sigma(t)-s}+\frac{(2t-s)-\rho(t)}{\sigma(t)+2t-2s-\rho(t)}\frac{f(2t-s)-f^\rho(t)}{(2t-s)-\rho(t)}\right)\\ &= \lim_{s\to t}\left(\frac{\sigma(t)-s}{\sigma(t)+2t-2s-\rho(t)}f^\Delta(t)+\frac{(2t-s)-\rho(t)}{\sigma(t)+2t-2s-\rho(t)}f^\nabla(t)\right).\end{aligned}$$

For each $t \in \mathbb{T}$, define $\gamma(t) := \lim_{s\to t} \frac{\sigma(t)-s}{\sigma(t)+2t-2s-\rho(t)}$ and $\tilde{\gamma}(t) := \lim_{s\to t} \frac{(2t-s)-\rho(t)}{\sigma(t)+2t-2s-\rho(t)}$. It is clear that $\gamma(t)+\tilde{\gamma}(t) = 1$. Note that if $t \in \mathbb{T}$ is dense, then

$$\gamma(t) = \lim_{s\to t} \frac{\sigma(t)-s}{\sigma(t)+2t-2s-\rho(t)} = \lim_{s\to t} \frac{t-s}{2t-2s} = \frac{1}{2}$$

and, therefore, $\tilde{\gamma}(t) = 1/2$. On the other hand, if $t \in \mathbb{T}$ is not dense, then

$$\gamma(t) = \lim_{s\to t} \frac{\sigma(t)-s}{\sigma(t)+2t-2s-\rho(t)} = \frac{\sigma(t)-t}{\sigma(t)-\rho(t)}$$

and $\tilde{\gamma}(t) = \frac{t-\rho(t)}{\sigma(t)-\rho(t)}$. Hence, functions $\gamma, \tilde{\gamma} : \mathbb{T} \to \mathbb{R}$ are well defined and, if f is delta and nabla differentiable, then $f^{\diamondsuit}(t) = \gamma(t)f^{\Delta}(t) + \tilde{\gamma}(t)f^{\nabla}(t) = \gamma(t)f^{\Delta}(t) + (1-\gamma(t))f^{\nabla}(t)$. □

Remark 4.3. Functions $\gamma, \tilde{\gamma} : \mathbb{T} \to \mathbb{R}$ are bounded and nonnegative: $0 \leqslant \gamma(t), \tilde{\gamma}(t) \leqslant 1$. This is due to the fact that $\rho(t) \leqslant t \leqslant \sigma(t)$ for every $t \in \mathbb{T}$.

Corollary 4.4. *If f is delta and nabla differentiable and if function $\gamma(\cdot)$ in (4.1) is a constant, $\gamma(t) \equiv \alpha$, then the symmetric derivative coincides with the diamond-α derivative: $f^{\diamondsuit}(t) = \alpha f^{\Delta}(t) + (1-\alpha)f^{\nabla}(t)$.*

In the classical case $\mathbb{T} = \mathbb{R}$ it can be proved that "A continuous function is necessarily increasing in any interval in which its symmetric derivative exists and is positive" [2]. We note that this result is not valid for the symmetric derivative on time scales. For instance, consider the time scale $\mathbb{T} = \mathbb{N}$ and function $f(n) = n$ if n is odd and $f(n) = 10n$ if n is even. The symmetric derivative of f is given by $f^{\diamondsuit}(n) = \frac{f^{\sigma}(n)-f^{\rho}(n)}{\sigma(n)-\rho(n)} = \frac{10(n+1)-10(n-1)}{(n+1)-(n-1)} = 10$ for n odd, while for n even one has $f^{\diamondsuit}(n) = \frac{f^{\sigma}(n)-f^{\rho}(n)}{\sigma(n)-\rho(n)} = \frac{(n+1)-(n-1)}{(n+1)-(n-1)} = 1$. Clearly, function f is non-increasing although its symmetric derivative is always positive. In this example, the symmetric derivative coincides with the diamond-α derivative, $\alpha = 1/2$, showing that there is an inconsistency in Corollary 2.1 of [9].

Acknowledgments

This work was supported by *FEDER* funds through *COMPETE*—Operational Programme Factors of Competitiveness ("Programa Operacional Factores de Competitividade") and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. AMCBdC is also supported by FCT through the Ph.D. fellowship SFRH/BD/33634/2009. The authors are grateful to the referees for their valuable comments and helpful suggestions.

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