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A bound on the size of a graph with given order and bondage number

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Abstract

The *domination number* of a graph is the minimum number of vertices in a set S such that every vertex of the graph is either in S or adjacent to a member of S. The *bondage number* of a graph G is the cardinality of a smallest set of edges whose removal results in a graph with domination number greater than that of G. We prove that a graph with p vertices and bondage number b has at least $p(b+1)/4$ edges, and for each b there is at least one p for which this bound is sharp.

Keywords: Domination number; Bondage number

1. Introduction and previous results

For a vertex x in a graph $G=(V,E)$, the *closed neighbourhood* of x is the set $N[x]$ consisting of x together with all the vertices of G adjacent to x. The set of neighbours of x is the *open neighbourhood* $N(x)$. The set $A \subseteq V$ is a *dominating set* of G if A has a nonempty intersection with $N[u]$ for each $u \in V$. If, from among all dominating sets of G, A has minimum cardinality, we call A a *γ-set* of G and its cardinality $|A|$ is the *domination number* $\gamma(G)$ of G.

It is immediate that $\gamma(H) \geqslant \gamma(G)$ when H is a spanning subgraph of G. Every connected graph G has a spanning tree T with $\gamma(G)=\gamma(T)$ and so, in general, a graph will have nonempty sets of edges $F \subseteq E$ for which $\gamma(G-F)=\gamma(G)$. Such a set F will be called an *inessential set* of edges in G. However, many graphs also possess single edges e for which $\gamma(G-e)>\gamma(G)$. In [6], the present authors give a structural

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characterization of the class of trees in which every single edge is inessential while later, in [5], a linear recognition algorithm and an alternate characterization is presented.

One measure of the stability of the domination number of G under edge removal is the *bondage number* $b(G)$ defined in [4] (although actually presented at the SIAM meeting on Discrete Math held at M.I.T. in 1982 but appearing in print much later; this concept was also called the *domination line-stability* in [1]). In particular, $b(G)$ is the largest positive integer k so that every subset of edges $F \subseteq E$ with $|F| = k - 1$ is inessential. Fink et al. [4] determined the bondage numbers of cycles, paths and complete multipartite graphs and showed that $b(T) \leqslant 2$ for any tree T. The previously mentioned result in [6] can thus be interpreted as a characterization of trees T with $b(T) = 2$.

Along with the exact values for $b(G)$ computed in [4] several general upper bounds were also derived. In particular, the following theorem was proved.

Theorem A (Bauer et al. [1]; Fink et al. [4]). *If G is a nonempty graph, then*

$$b(G) \leqslant \min_{uv \in E(G)} (\deg(u) + \deg(v) - 1).$$

In [7], this result is improved and, in [11], Wang, by careful consideration of the nature of the edges from the neighbours of u and v, further refines this bound.

In [4], the authors conjectured that if G is a nonempty graph, then $b(G) \leqslant \Delta(G) + 1$, where $\Delta(G) = \max_{x \in V} \deg(x)$. As noted in [1], if F_v is the set of edges incident to v, then $\gamma(G - F_v) > \gamma(G)$ unless it is the case that $\gamma(G - v) = \gamma(G) - 1$. This tended to support this conjecture. Additional support was given by Chvátal and Cook [3], where it is shown that $b^*(G) \leqslant \Delta(G)$ where $b^*(G)$ (the *fractional bondage number* of G) is the linear programming relaxation of an integer linear program that gives the bondage number.

However, this conjecture was shown to be false by Teschner [8] as well as by the present authors [7]. In [8], the graph $K_3 \times K_3$ was given as a counterexample while in [7] the more general theorem given below was proved.

Theorem B (Hartnell and Rall [7]). *For a positive integer $n \geqslant 3$, let G_n be the cartesian product $K_n \times K_n$. Then $b(G_n) = 3(n-1) = \frac{3}{2}\Delta(G_n)$.*

A proof of this theorem also appears later in [10]. In [9], this bound is shown to be sharp for graphs with domination number 3 or smaller.

Still another upper bound (see [7]) is the following result, in terms of the maximum degree and the edge connectivity of a graph G.

Theorem C (Hartnell and Rall [7]). *If G has edge connectivity k, then $b(G) \leqslant \Delta(G) + k - 1$.*

In what follows, we will establish a lower bound on the number of *edges* in a graph with a specified bondage number and a fixed number of vertices. In doing so we also show that the bondage number itself is bounded above by twice the average degree of the vertices in the graph minus one. This, in general, improves the bound of twice the maximum degree minus one which follows immediately from Theorem A.

2. New bounds

Before we address the question on bondage we make a general observation regarding the degrees of vertices that are within distance 2 of each other.

For any connected graph G, let $\deg_a(G)$ represent the value of the expression $\sum_{v\in V(G)} \deg(v)/|V(G)|$.

Lemma 1. *For any connected graph G, there exists a pair of vertices, say u and v, that are either adjacent or at distance 2 from each other, with the property that* $\deg(u)+\deg(v)\leqslant 2\deg_a(G)$.

Proof. Assume that the lemma is false and let G be a graph where the result does not hold.

Let the vertices of degree less than or equal to $\deg_a(G)$ be $S=\{s_1,s_2,\ldots,s_m\}$ and the vertices of degree strictly greater than $\deg_a(G)$ be $F=\{f_1,f_2,\ldots,f_n\}$.

Observe that no pair of vertices of S can be joined by an edge. Hence, each $s_i\in S$ has only vertices in F as neighbours. Also note that each $f_j\in F$ has at most one vertex of S as a neighbour otherwise, if there were two, they would contradict our assumption.

Now we proceed to sum the degrees of all vertices in the graph as follows. For each $s_i\in S$ we consider a neighbour $f_j\in F$ and take $\deg(s_i)+\deg(f_j)$. Observe that $\deg(s_i)+\deg(f_j)>2\deg_a(G)$. Furthermore, by the above remarks, these neighbours in F must be distinct. After adding m such pairs (to exhaust S), the degrees of any remaining members of F are included. But the total sum of degrees is greater than $|V(G)|\deg_a(G)$ which is impossible. The lemma follows. □

We now turn our attention to the main result.

First, recall [1,4] that for any graph G, if u and v are adjacent, then $b(G)\leqslant \deg(u)+\deg(v)-1$.

We extend this result to include distance 2 vertices.

Theorem 2. *If u and v are vertices of G such that the distance between them is at most 2, then* $b(G)\leqslant \deg(u)+\deg(v)-1$.

Proof. By [1,4], if u and v are adjacent, the result holds. Now consider u and v non-adjacent but with w, say, the common neighbour of u and v. Delete all edges meeting u and all edges incident with v except for the one to w. Consider a γ-set of the resulting graph G'. Since v is a leaf in G' we may, without loss of generality, assume that w is in the γ-set of G'. But then all neighbours of u are dominated by vertices other than

u (in this γ-set) and hence, in G, we could use the same set of vertices, but without u itself, to dominate G. The theorem follows. $\square$

We are now in a position to establish the following.

Theorem 3. *For any connected graph G, with p vertices and bondage number $b(G)$, the number of edges is at least $(p/4)(b(G)+1)$.*

Proof. Let G be a graph satisfying the hypothesis. Let $\deg_a(G)$ represent the average degree. By Lemma 1 we know there is at least one pair of vertices, say u and v, that are either adjacent or at distance 2 from each other, with the property that $\deg(u)+\deg(v)\leqslant 2\deg_a(G)$. In either case by Theorem 2 we have

$$b(G)+1\leqslant \deg(u)+\deg(v)\leqslant 2(\deg_a(G)). \tag{1}$$

But

$$2|E(G)| = p(\deg_a(G)) \Rightarrow 4|E(G)| = 2p(\deg_a(G)) \geqslant p(b(G)+1) \quad \text{(by (1))}$$

$$\Rightarrow |E(G)| \geqslant \frac{p}{4}(b(G)+1). \qquad \square$$

Corollary 4. $b(G)\leqslant 2(\deg_a(G))-1$.

We observe that for each value of $b(G)$, the lower bound given in the theorem is sharp for some values of p.

If $b(G)=1$, simply take $p=2$ (necessary for G to be connected) and G isomorphic to K_2.

If $b(G)=2$, consider $p=4$ and G isomorphic to a path on four vertices.

For $b(G)=k$, $k>2$, let G be the graph on $p=4m$ vertices constructed as follows. Start with the Harary graph (see [2]) on $2m$ vertices. In particular, for k even take a cycle on $2m$ vertices and then join vertices i and j if $i-\lfloor k/2\rfloor\leqslant j\leqslant i+\lfloor k/2\rfloor$ (working modulo $2m$). For k odd add the main diagonals. Observe that each vertex is of degree $k-1$. Now attach a leaf to each of the $2m$ vertices to form G.

It was shown in [1] (and later in [6]) that the star is the unique graph with the property that the bondage number is 1 and the deletion of *any* edge results in the domination number increasing. We conclude by determining when this very special property holds for higher bondage number. Let us call a graph *uniformly bonded* if it has bondage number b and the deletion of *any* b edges results in a graph with increased domination number.

Theorem 5. *The only uniformly bonded graphs with bondage number 2 are C_3 and P_4. The unique graph with bondage number 3 that is uniformly bonded is the graph C_4. There are no such graphs for bondage number greater than 3.*

Proof. We first observe that if one deletes a single edge from a uniformly bonded graph with bondage number 2 then the resulting graph must be either a star or a collection of stars (since only one edge removed in fact there can be at most two stars).

Say after one edge is deleted, we obtain one star. If this star has 3 or more leaves, then the original graph has bondage number 1. If this star has two leaves, then we started with C_3. We cannot obtain K_2 nor K_1.

Say after one edge is deleted, we obtain two stars. If one of these stars has three or more vertices and the other at least two, then the original graph must have had bondage number 1 (delete edge to a leaf in star of order 3 or more). If one star has order 1 and the other order 4 or more, then the original graph again must have had bondage number 1. If one star has order 1 and the other 3, then the original graph must be P_4 if the bondage number is 2. The original graph could not be any smaller.

Now consider uniformly bonded graphs with bondage number 3. Deleting any single edge must yield C_3 or P_4 or a collection of these. But more than one component would not be possible as then one could delete one further edge from each without affecting the domination number. So we must obtain either C_3 (impossible) or P_4 by deleting one edge. It is easy to verify that the original graph must be C_4.

Next consider a graph with bondage number 4 that is uniformly bonded. But then deleting a single edge results in C_4 (as more than one component would allow two edges to be deleted from one and one edge from another). But C_4 with a diagonal does not have bondage number 4 so there is no such graph. Hence there are none for higher values of b. □

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