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Factors in graphs with odd-cycle property*

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Abstract

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We present some conditions for the existence of a (g,f)-factor or a (g,f)-parity factor in a graph G with the odd-cycle property that any two odd cycles of G either have a vertex in common or are joined by an edge.

1. Definitions and notations

We consider finite graphs G with vertex set $V(G)$ and edge set $E(G)$. For any $v \in V(G)$, we denote by $d_G(v)$ the degree of v in G. For any $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $G-S$ the subgraph of G obtained from G by deleting the vertices in S together with their incident edges. Similarly, for any $M \subseteq E(G)$, we write $G-M$ for the subgraph of G obtained from G by deleting the edges in M. If S and T are subsets of $V(G)$, we denote by $e_G(S,T)$ the number of edges of G joining a vertex in S to a vertex in T. Write

$$Q_j(G) = \{v \mid v \in V(G) \text{ and } d_G(v) = j\},$$

and put $q_j(G) = |Q_j(G)|$. In particular, let $i(G) = q_0(G)$, i.e. $i(G)$ is the number of the isolated vertices of G. If G is a bipartite graph with bipartition (X, Y), then write $G = (X, Y; E(G))$. Let Z denote the set of nonnegative integers.

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Let G be a graph, and $g, f\colon V(G) \to Z$ such that

$$g(x) \leqslant f(x) \quad \text{for all } x \in V(G). \tag{1}$$

Then a spanning subgraph F of G is called a *(g,f)-factor* if $d_F(x) \in \{g(x), g(x)+1, \ldots, f(x)\}$ for all $x \in V(G)$. Furthermore, if

$$g(x) \equiv f(x) \pmod 2 \quad \text{for all } x \in V(G), \tag{2}$$

and $d_F(x) \in \{g(x), g(x)+2, \ldots, f(x)\}$, then the spanning subgraph F is called a *(g,f)-parity factor*. Particularly, both (g,f)-factors and (g,f)-parity factors are *f-factors* or *k-factors* according as $g(x)=f(x)$ or $g(x)=f(x)=k$ for all $x \in V(G)$. It is clear that if G has a (g,f)-factor, then either

$$\sum_{x \in V(G)} f(x) \equiv 0 \pmod 2 \tag{3}$$

or

$$g(y) < f(y) \quad \text{for some } y \in V(G), \tag{4}$$

and if G has a (g,f)-parity factor, then (3) must hold. In addition, G is said to have *the odd-cycle property* if any two odd cycles of G either have a vertex in common or are joined by an edge, and is said to have *the (g,f)-inequality property* if the vertex subset $\{x \mid x \in V(G) \text{ and } g(x) < f(x)\}$ is a clique of G (it may be the empty subset). Trivially, G has the odd-cycle property if G is a bipartite graph. For any nonnegative integer a, we can also define an (a,f)-factor, an (a,f)-parity factor and the (a,f)-inequality property similarly as above. Some other definitions and known results related to ours can be found in [1, 2, 7, 11].

2. Results on (g, f)-factors

The criterion for a general graph to have a (g,f)-factor is due to Lovász as follows.

Proposition 1 (Lovász [8]). *Let G be a graph, and $g, f\colon V(G) \to Z$. Then G has a (g,f)-factor if and only if*

$$\delta_G(S,T) := \sum_{s \in S} f(s) + \sum_{t \in T} (d_{G-S}(t) - g(t)) - h_G(S,T) \geqslant 0$$

for all $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, where $h_G(S,T)$ denotes the number of components C of $G-(S \cup T)$ such that $g(v)=f(v)$ for all $v \in V(C)$ and

$$J(C,T) := \sum_{v \in V(C)} f(v) - e_G(V(C), T) \equiv 1 \pmod 2.$$

The purpose of this section is to give some conditions which are simpler than that of Proposition 1 for the existence of a (g, f)-factor in a graph with special properties.

Theorem 1. *Let G be a connected graph with the odd-cycle property and $g, f\colon V(G)\to Z$ satisfy* (1) *and either* (3) *or* (4). *Then G has a (g, f)-factor if*

$$\gamma_G(S,T):=\sum_{s\in S} f(s)+\sum_{t\in T}(d_{G-S}(t)-g(t))\geqslant \eta_1$$

for all $S, T\subseteq V(G)$ with $S\cap T=\emptyset$ and $S\cup T\neq\emptyset$, where

$$\eta_1=\begin{cases}1 & \text{if } G \text{ has odd cycles, and there exist } x, y\in V(G)\\ & \text{such that } g(x)=f(x) \text{ and } g(y)<f(y);\\ 0 & \text{otherwise.}\end{cases}$$

Note that if $g(x)<f(x)$ for all $x\in V(G)$, then Proposition 1 implies a stronger result than that of Theorem 1 since $h_G(S,T)=0$ for all $S, T\subseteq V(G)$ with $S\cap T=\emptyset$ in this case. So we may consider Theorem 1 with the condition that there exists at least one vertex x of G such that $g(x)=f(x)$.

Proof of Theorem 1. Let $S, T\subseteq V(G)$ with $S\cap T=\emptyset$. It is sufficient by Proposition 1 to show that $\delta_G(S,T)\geqslant 0$. By the hypotheses, we may assume that $S\cup T\neq\emptyset$ and $h_G(S,T)\geqslant 1$. Write $U=V(G)-(S\cup T)$, and let $C_1,\ldots,C_r$ be the components of $G[U]$. Then

$$\begin{aligned}\delta_G(S,T)&=\sum_{s\in S} f(s)-\sum_{t\in T} g(t)+2|E(G[T])|+e_G(T,U)-h_G(S,T)\\ &=\sum_{s\in S} f(s)-\sum_{t\in T} g(t)+2|E(G[T])|\\ &\quad+\sum_{j=1}^{r}\Bigl(-J(C_j,T)+\sum_{x\in V(C_j)} f(x)\Bigr)-h_G(S,T)\\ &=\sum_{v\in V(G)} f(v)-\sum_{t\in T}(f(t)+g(t))+2|E(G[T])|\\ &\quad-\sum_{j=1}^{r} J(C_j,T)-h_G(S,T) &&(5)\\ &\geqslant\sum_{v\in V(G)} f(v)-\sum_{t\in T}(f(t)+g(t))+2|E(G[T])|\\ &\quad-\sum_{j=1}^{r} 2\lceil \tfrac{1}{2} J(C_j,T)\rceil. &&(6)\end{aligned}$$

Let $C=(A,B;\, E(C))$ be a bipartite component of $G[U]$ which is counted in $h_G(S,T)$. Then since $f(x)=g(x)$ for all $x\in A\cup B$, and $E(G[A])=E(G[B])=\emptyset$, we have by (5) that

$$\begin{aligned}
&\delta_G(S\cup B, T\cup A)-\delta_G(S,T)\\
&\quad=-\sum_{x\in A}(f(x)+g(x))+2|E(G[A])|+2e_G(A,T)\\
&\qquad+J(C,T)-h_G(S\cup B, T\cup A)+h_G(S,T)\\
&\quad=2\left\{-\sum_{x\in A}f(x)+e_G(A,T)+\lceil \tfrac{1}{2}J(C,T)\rceil\right\}\ (\equiv 0\ (\mathrm{mod}\ 2)).
\end{aligned}\tag{7}$$

Clearly, (7) still holds when A and B are interchanged. Hence by (7) we obtain that

$$\begin{aligned}
&[\delta_G(S\cup B, T\cup A)-\delta_G(S,T)]+[\delta_G(S\cup A, T\cup B)-\delta_G(S,T)]\\
&\quad=-2\sum_{x\in V(C)}f(x)+2e_G(V(C),T)+4\lceil \tfrac{1}{2}J(C,T)\rceil\\
&\quad=-2J(C,T)+4\lceil \tfrac{1}{2}J(C,T)\rceil\\
&\quad=2
\end{aligned}$$

since $J(C,T)$ is odd. Combining it with (7), we can claim that either $\delta_G(S\cup B, T\cup A)\leqslant \delta_G(S,T)$ or $\delta_G(S\cup A, T\cup B)\leqslant\delta_G(S,T)$. Therefore, by the odd-cycle property, we can choose $S^*, T^*\subseteq V(G)$ with $S^*\cap T^*=\emptyset$ and $S^*\cup T^*\neq\emptyset$ such that $\delta_G(S^*,T^*)\leqslant\delta_G(S,T)$, and $h_G(S^*,T^*)\leqslant 1$ and $h_G(S^*,T^*)=1$ only if G has odd cycles. Thus it follows from the hypotheses that

$$\delta_G(S,T)\geqslant\delta_G(S^*,T^*)\geqslant\eta_1-h_G(S^*,T^*)\geqslant -1.$$

Suppose that $\delta_G(S,T)=-1$, then $\eta_1=0$ and $h_G(S^*,T^*)=1$. So we must have that $g(x)=f(x)$ for all $x\in V(G)$ by the definition of η_1 and the choice of (S^*,T^*). Since in this case both (3) and the equality in (6) hold, we have $\delta_G(S,T)\equiv 0$ (mod 2), and hence $\delta_G(S,T)\geqslant 0$. Thus, the proof is complete. □

Corollary 1.1 (Folkman and Fulkerson [3]). *Let $G=(X,Y;\, E(G))$ be a bipartite graph and $g,f\colon V(G)\to Z$ satisfy* (1). *Then G has a (g,f)-factor if and only if*

$$\gamma_G(A,B)\geqslant 0 \quad \text{and} \quad \gamma_G(B,A)\geqslant 0 \tag{8}$$

for all $A\subseteq X$ and $B\subseteq Y$.

Proof. The necessity is trivial. To show the sufficiency, we first note that for any component C of G, either there exists a vertex v of C such that $g(v)<f(v)$ or

$$\sum_{x\in V(C)\cap X}f(x)=\sum_{y\in V(C)\cap Y}f(y)$$

by setting $A=V(C)\cap X$ and $B=V(C)\cap Y$ in (8). Thus either (4) or (3) holds for any component C, and so we may assume that G is connected. Hence it suffices by Theorem 1 to show that $\gamma_G(S,T)\geqslant 0$ for any $S,T\subseteq V(G)$ with $S\cap T=\emptyset$. In fact

$$\gamma_G(S,T)=\gamma_G(S\cap X,T\cap Y)+\gamma_G(S\cap Y,T\cap X)\geqslant 0$$

by (8). Thus, the proof is complete. □

The following corollary was originally proved by using integer programming techniques.

Corollary 1.2 (Fulkerson et al. [4]). *Let G be a graph with the odd-cycle property and $f\colon V(G)\to Z$ satisfy* (3). *Then G has an f-factor if and only if*

$$\sum_{s\in S} f(s)+\sum_{t\in T}(d_{G-S}(t)-f(t))\geqslant 0 \tag{9}$$

for all $S,T\subseteq V(G)$ with $S\cap T=\emptyset$.

Proof. The necessity is trivial. To show the sufficiency by using Theorem 1 (with $\eta_1=0$), it suffices to show that

$$\sum_{v\in V(C)} f(v)\equiv 0 \pmod 2 \tag{10}$$

for any component C of G, by which we may assume that G is connected. In fact, if C is bipartite, then we obtain (10) from (9) by the same conclusion as used in the proof of Corollary 1.1; if C is not bipartite, since G has at most one such component, we still have (10) by the condition (3). □

Theorem 2. *Let a be an integer, G be a connected graph with the odd-cycle property, and $f\colon V(G)\to Z$ satisfy $f(x)\geqslant a$ for all $x\in V(G)$ and either* (3) *or $f(y)>a$ for some $y\in V(G)$. Then G has an (a,f)-factor if*

$$\sum_{j=0}^{a-1}(a-j)q_j(G-S)\leqslant\sum_{x\in S} f(x)-\eta_1 \tag{11}$$

for all $S\subseteq V(G)$.

Proof. Let $S,T\subseteq V(G)$ with $S\cap T=\emptyset$ and $S\cup T\neq\emptyset$. Put

$$Q=\bigcup_{j=0}^{a-1} Q_j(G-S).$$

Then we have

$$\begin{aligned}\gamma_G(S,T) &:= \sum_{s\in S} f(s) + \sum_{t\in T} (d_{G-S}(t)-a)\\ &\geqslant \sum_{s\in S} f(s) + \sum_{t\in Q} (d_{G-S}(t)-a)\\ &= \sum_{s\in S} f(s) + \sum_{j=0}^{a-1} (j-a)q_j(G-S)\\ &\geqslant \eta_1\end{aligned}$$

by (11). So the theorem is proved by Theorem 1. □

Note that Theorem 2 can be strengthened in some cases, in analogy to that of Theorem 1. For example, if G is a bipartite graph or $f(x)=a$ for all $x\in V(G)$, then (11) is also a necessary condition, and the connectivity condition of G can be omitted since $\eta_1=0$, and it can be showed that either (3) or (4) holds for any component of G in these cases. In particular, we can obtain from Theorem 2 the following criterion for a graph with the odd-cycle property to have a 1-factor, which is also an easy consequence of the theorem due to Tutte [12] that a graph G has a $\{K_2, C_j \mid j\geqslant 3\}$-factor if and only if $i(G-S)\leqslant |S|$ for all $S\subseteq V(G)$.

Corollary 2.1. *A graph G of even order with the odd-cycle property has a 1-factor if and only if*

$$i(G-S)\leqslant |S| \quad \text{for all } S\subseteq V(G).$$

Theorem 3. *Let G be a connected graph with the odd-cycle property, and $g, f\colon V(G)\to Z$ satisfy* (1), *either* (3) *or* (4), *and $g(x)<d_G(x)$ and $f(x)>0$ for all $x\in V(G)$. Then G has a (g,f)-factor if*

$$\frac{g(x)}{d_G(x)}\leqslant\frac{f(y)}{d_G(y)} \tag{12}$$

for any two adjacent vertices x and y of G.

Proof. By Theorem 1, it suffices to show that $\gamma_G(S,T)\geqslant\eta_1$ for any $S, T\subseteq V(G)$ with $S\cap T=\emptyset$ and $S\cup T\neq\emptyset$. Let $C_1,\ldots,C_m$ be all the components of $G-(S\cup T)$. Then by (12) we have that

$$\begin{aligned}\gamma_G(S,T) &= \sum_{s\in S} d_G(s)\frac{f(s)}{d_G(s)} + \sum_{t\in T} d_G(t)\left(1-\frac{g(t)}{d_G(t)}\right) - e_G(S,T)\\ &\geqslant \sum_{s,s'\in S} e_G(s,s')\frac{f(s)}{d_G(s)} + \sum_{t,t'\in T} e_G(t,t')\left(1-\frac{g(t)}{d_G(t)}\right)\end{aligned}$$

$$+\sum_{j=1}^{m}\left[\sum_{s\in S} e_G(s, V(C_j))\frac{f(s)}{d_G(s)}+\sum_{t\in T} e_G(t, V(C_j))\left(1-\frac{g(t)}{d_G(t)}\right)\right]$$

$$+\sum_{s\in S}\sum_{t\in T}\left[e_G(s,t)\frac{f(s)}{d_G(s)}+e_G(s,t)\left(1-\frac{g(t)}{d_G(t)}\right)-e_G(s,t)\right]$$

$$\geqslant\sum_{s\in S}\sum_{t\in T}\left[e_G(s,t)\frac{f(s)}{d_G(s)}+e_G(s,t)\left(1-\frac{g(t)}{d_G(t)}\right)-e_G(s,t)\right]\geqslant 0,$$

where for brevity we have written s instead of $\{s\}$, etc. Note that $\gamma_G(S,T)=0$ only if $V(G)=S\cup T$ and neither $G[S]$ nor $G[T]$ has an edge, i.e. G is a bipartite graph with bipartite (S,T). So it follows $\gamma_G(S,T)\geqslant\eta_1$. Thus, the theorem is proved. □

Kano and Saito [5] presented that if $g(x)<f(x)$ and $g(x)\leqslant\theta d_G(x)\leqslant f(x)$ for all $x\in V(G)$, then G has a (g,f)-factor, where θ is some real number such that $0\leqslant\theta\leqslant 1$. Our Corollary 3.1 is a similar result for the special graphs and functions f, g without the constraint that $g(x)<f(x)$ for all $x\in V(G)$, and from the proofs of the next three theorems we shall see that the result is very useful.

Corollary 3.1. *Let G be a connected graph with the odd-cycle property, and g,f: $V(G)\to Z$ satisfy either* (3) *or* (4). *If there exists a real number θ with $0<\theta<1$ such that*

$$g(x)\leqslant\theta d_G(x)\leqslant f(x)$$

for all $x\in V(G)$, then G has a (g,f)-factor.

Proof. This is an immediate consequence of Theorem 3. □

Theorem 4. *Every r-regular graph G with the odd-cycle property has a k-factor, where $0\leqslant k\leqslant r$ and $k|V(G)|\equiv 0$* (mod 2).

Proof. Since G has at most one component which is not bipartite, it is easy to show that $k|V(C)|\equiv 0$ (mod 2) for each component C of G. Thus we may assume that $0<k<r$ and G is connected. Set $\theta=k/r$ and $g(x)=f(x)=k$ for all $x\in V(G)$. Then the conditions in Corollary 3.1 are satisfied, and so the theorem is proved. □

Before stating the next two theorems, let us recall two definitions. A graph G is called an $[a,b]$-graph if $a\leqslant d_G(x)\leqslant b$ for all $x\in V(G)$, and said to be $[a,b]$-factorable if G can be decomposed into some edge-disjoint $[a,b]$-factors, where $a\leqslant b$.

Theorem 5. *Let G be a graph with the strong odd-cycle property that any two odd cycles of G have a vertex in common, and $k\geqslant 1$ such that $k|V(G)|\equiv 0$* (mod 2). *Then G is k-factorable if and only if G is a km-regular graph for some $m\geqslant 1$.*

Proof. The necessity is trivial. We show the sufficiency by induction on $m \geqslant 2$. By Theorem 4, G has a k-factor F. Then $G - E(F)$ is a $k(m-1)$-regular graph having at most one component which is not bipartite. It is easy to see by the hypotheses that each component C of $G - E(F)$ also has the strong odd-cycle property and satisfies $k|V(C)| \equiv 0 \pmod 2$. Thus, every C can be decomposed into $m-1$ k-factors by induction, and so G can be decomposed into m k-factors. $\square$

Note that Theorem 5 is an extension of the theorem due to König [6] that a bipartite graph is 1-factorable if and only if it is regular. Furthermore, if G is a bipartite graph, then Theorem 5 can be strengthened to give the next result due to de Werra (see [7]), whose short argument using Corollary 3.1 is given here.

Theorem 6 (D. de Werra). *Let $0 \leqslant a \leqslant b$ and G be a bipartite graph. Then G is $[a,b]$-factorable if and only if G is an $[am, bm]$-graph for some $m > 0$.*

Proof. The necessity is trivial. Conversely, we shall show the sufficiency by induction on m. Without loss of generality, we may assume that $m \geqslant 2$ and G is connected. Put $\theta = 1/m$ and define functions g and f on $V(G)$ as follows:

$$\begin{cases} f(x) = g(x) = a & \text{if } d_G(x) = am, \\ f(x) - 1 = g(x) = \lfloor (1/m) d_G(x) \rfloor & \text{if } am < d_G(x) < bm, \\ f(x) = g(x) = b & \text{if } d_G(x) = bm, \end{cases}$$

for all $x \in V(G)$. Note that when $U := \{v \in V(G) \mid am < d_G(v) < bm\} = \emptyset$, the function f satisfies (3). In fact, let $G = (X, Y; E(G))$, and set

$$X_1 = \{x \in X \mid d_G(x) = am\} \quad \text{and} \quad X_2 = \{x \in X \mid d_G(x) = bm\},$$

$$Y_1 = \{y \in Y \mid d_G(y) = am\} \quad \text{and} \quad Y_2 = \{y \in Y \mid d_G(y) = bm\},$$

then it is easy to show that

$$a(|X_1| + |Y_1|) + b(|X_2| + |Y_2|) \equiv 0 \pmod 2.$$

Thus, the conditions in Corollary 3.1 are satisfied, and so G has a (g, f)-factor F. Put $F' = G - E(F)$. We observe that for any $v \in U$, it follows that

$$a < \frac{1}{m} d_G(v) < b \quad \text{and} \quad a(m-1) < \left(1 - \frac{1}{m}\right) d_G(v) < b(m-1).$$

Hence

$$\begin{aligned} a - 1 < \frac{1}{m} d_G(v) - 1 \leqslant f(v) - 1 = g(v) \leqslant d_F(v) \leqslant f(v) \\ = g(v) + 1 \leqslant \frac{1}{m} d_G(v) + 1 < b + 1 \end{aligned}$$

and

$$a(m-1)-1<\left(1-\frac{1}{m}\right)d_G(v)-1\leqslant d_G(v)-g(v)-1=d_G(v)-f(v)\leqslant d_{F'}(v)$$

$$\leqslant d_G(v)-g(v)=d_G(v)-f(v)+1\leqslant\left(1-\frac{1}{m}\right)d_G(v)+1<b(m-1)+1.$$

Therefore, F is an $[a,b]$-factor and F' is an $[a(m-1),b(m-1)]$-factor of G. By induction, F' can be decomposed into $m-1$ $[a,b]$-factors, and so G can be decomposed into m $[a,b]$-factors. □

3. Results on (g,f)-parity factors

The criterion for a general graph to have an f-factor was found by Tutte as follows.

Proposition 2 (Tutte [12]). *Let G be a graph and $f\colon V(G)\to Z$ satisfy* (3). *Then G has an f-factor if and only if*

$$\sum_{s\in S} f(s)+\sum_{t\in T}(d_{G-S}(t)-f(t))-h'_G(S,T)\geqslant 0$$

for all $S,T\subseteq V(G)$ with $S\cap T=\emptyset$, where $h'_G(S,T)$ is the number of components C of $G-(S\cup T)$ with

$$\sum_{v\in V(C)} f(v)-e_G(V(C),T)\equiv 1 \pmod 2.$$

A similar criterion for a general graph to have a (g,f)-parity factor can be derived from Proposition 2 or some other results due to Lovász [10]. That is

Proposition 3 [1, 10]. *Let G be a graph, and $g,f\colon V(G)\to Z$ satisfy* (1), (2) *and* (3). *Then G has a (g,f)-parity factor if and only if*

$$\sum_{s\in S} f(s)+\sum_{t\in T}(d_{G-S}(t)-g(t))-h'_G(S,T)\geqslant 0$$

for all $S,T\subseteq V(G)$ with $S\cap T=\emptyset$.

Here we shall present some simpler conditions than that of Proposition 3 for the existence of a (g,f)-parity factor in a graph with special properties, which are similar to that given in Theorems 1–3.

Theorem 7. *Let G be a connected graph with the odd-cycle and (g,f)-inequality properties, where $g,f\colon V(G)\to Z$ satisfy* (1), (2) *and* (3). *Then G has a (g,f)-parity factor if*

$$\gamma_G(S,T):=\sum_{s\in S} f(s)+\sum_{t\in T}(d_{G-S}(t)-g(t))\geqslant\eta_2$$

for all $S, T \subseteq V(G)$ *with* $S \cap T = \emptyset$ *and* $S \cup T \neq \emptyset$, *where*

$$\eta_2 = \begin{cases} 1 & \text{if } G \text{ has odd cycles and there exists } x \in V(G) \\ & \text{such that } g(x) < f(x); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first construct a new graph H from G by joining $\frac{1}{2}(f(v)-g(v))$ new loops to each vertex v of G. Then for all $v \in V(H) = V(G)$,

$$d_H(v) = d_G(v) + f(v) - g(v). \tag{13}$$

Clearly, G has a (g,f)-parity factor if and only if H has an f-factor. Thus, it suffices by Proposition 2 to show that

$$\delta'_H(S,T) := \sum_{s \in S} f(s) + \sum_{t \in T} (d_{H-S}(t) - f(t)) - h'_H(S,T) \geqslant 0$$

for any $S, T \subseteq V(H)$ with $S \cap T = \emptyset$ and $S \cup T \neq \emptyset$. Similarly as in the proof of Theorem 1, we can choose $S^*, T^* \subseteq V(H)$ with $S^* \cap T^* = \emptyset$ and $S^* \cup T^* \neq \emptyset$ such that $\delta'_H(S^*, T^*) \leqslant \delta'_H(S,T)$ and $h'_H(S^*, T^*) \leqslant \eta_2 + 1$. Thus, by (13) and the hypotheses, we have that

$$\begin{aligned} \delta'_H(S,T) &\geqslant \sum_{s \in S^*} f(s) + \sum_{t \in T^*} (d_{H-S^*}(t) - f(t)) - h'_H(S^*, T^*) \\ &\geqslant \sum_{s \in S^*} f(s) + \sum_{t \in T^*} (d_{G-S^*}(t) - g(t)) - \eta_2 - 1 \\ &\geqslant -1, \end{aligned}$$

and hence $\delta'_H(S,T) \geqslant 0$ since it can be shown that $\delta'_H(S,T) \equiv 0 \pmod 2$ similarly as the equality in (6). Consequently, the theorem is proved. □

Corollary 7.1. *Let* $G = (X, Y; E(G))$ *be a bipartite graph, and* $g, f: V(G) \to Z$ *satisfy* (1), (2) *and* (3). *Suppose that there exists at most one vertex or one pair of adjacent vertices such that* (4) *holds. Then* G *has a* (g,f)*-parity factor if and only if*

$$\gamma_G(A,B) \geqslant 0 \quad \text{and} \quad \gamma_G(B,A) \geqslant 0 \tag{14}$$

for all $A \subseteq X$ *and* $B \subseteq Y$.

Proof. The proof is analogous to that of Corollary 1.1 by using Theorem 7 instead of Theorem 1. □

Note that Corollary 7.1 is sharp in the sense that the condition 'there exists at most one vertex or one pair of adjacent vertices such that (4) holds' cannot be replaced by that 'there exist at most two vertices such that (4) holds'. This fact will be shown by the graph G and the functions g and f given in Fig. 1, where the integers beside

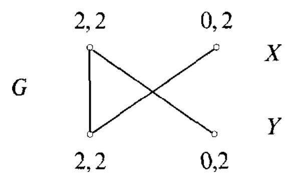

Fig. 1.

the vertices v denote $g(v)$ and $f(v)$. Obviously, G has no (g, f)-parity factor, but the conditions (1), (2), (3) and (14) are satisfied, and G has only two vertices such that (4) holds.

To end this paper, we finally state another two theorems without proofs since they can be proved analogously to Theorems 2 and 3 by applying Theorem 7 instead of Theorem 1.

Theorem 8. *Let a be an integer and G be a connected graph with the odd-cycle and (a, f)-inequality properties, where f: $V(G) \to Z$ satisfies* (3), *and $f(x) \geqslant a$ and $f(x) \equiv a$* (mod 2) *for all $x \in V(G)$. Then G has an (a, f)-parity factor if*

$$\sum_{j=0}^{a-1} (a-j) q_j(G-S) \leqslant \sum_{x \in S} f(x) - \eta_2$$

for all $S \subseteq V(G)$.

Theorem 9. *Let G be a connected graph with the odd-cycle and (g, f)-inequality properties, where g, f: $V(G) \to Z$* satisfy (1), (2), (3) *and $g(x) < d_G(x)$ and $f(x) > 0$ for all $x \in V(G)$. Then G has a (g, f)-parity factor if*

$$\frac{g(x)}{d_G(x)} \leqslant \frac{f(y)}{d_G(y)}$$

for any two adjacent vertices x and y of G.

Corollary 9.1. *Let G be a connected graph with the odd-cycle and (g, f)-inequality properties, where g, f: $V(G) \to Z$* satisfy (2) *and* (3). *If there exists a real number θ with $0 < \theta < 1$ such that*

$$g(x) \leqslant \theta d_G(x) \leqslant f(x)$$

for all $x \in V(G)$, then G has a (g, f)-parity factor.

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