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Counting Rational Points on K3 Surfaces

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For any algebraic variety X defined over a number field K, and height function H_D on X corresponding to an ample divisor D, one can define the counting function $N_{X,D}(B) = \#\{P \in X(K) \mid H_D(P) \leqslant B\}$. In this paper, we calculate the counting function for hyperelliptic $K3$ surfaces X which admit a generically two-to-one cover of $\mathbf{P}^1 \times \mathbf{P}^1$ branched over a singular curve. In particular, we effectively construct a [illegible] amounts to proving that Y is the first layer of the arithmetic stratification of X. We prove a more precise result in the special case where X is a Kummer surface whose associated Abelian surface is a product of elliptic curves. © 2000 Academic Press

Key Words: rational points; K3 surfaces; height; Kummer surfaces; Abelian surfaces.

1. INTRODUCTION

Counting rational points on algebraic varieties is one of the fundamental questions of number theory. However, if an algebraic variety contains infinitely many rational points one must define the question more precisely. The most natural way to do this is to define a notion of density on the set of rational points. This density is calculated with respect to a height H, which assigns a real number to a rational point P. Thus, for a variety X defined over a number field K and an ample divisor D on X, we study the counting function:

$$N_{X,D}(B) = \mathrm{card}\{P \in X(K) \mid H_D(P) \leqslant B\}$$

and investigate the properties of $N_{X,D}(B)$ as B gets arbitrarily large. This function may be radically different for different choices of D. Also, the precise definition of H_D depends not simply on the geometric choice of D, but on a choice of metrisations of certain line bundles as well. Happily, these choices will not affect our results, so they can be made arbitrarily.

In the case of $K3$ surfaces, this question has been investigated by many people. Silverman [10] introduced a canonical height on $K3$ surfaces

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embedded in $\mathbf{P}^2 \times \mathbf{P}^2$, analogous to the canonical height on an elliptic curve. Baragar [1] extended Silverman's results to other $K3$ surfaces. Although both authors obtain theorems about the distribution of rational points in orbits of certain group actions, neither was able to obtain estimates of the global counting function. Billard [3] has recently extended their results still further, and gives an estimate for $N_{X,D}(B)$ in a certain case.

Another approach was taken by Tschinkel [11], who develops a theory of finite heights to obtain estimates of $N_{X,D}(B)$ for some rational surfaces, and upper bounds on $N_{X,D}(B)$ for some $K3$ and Enriques surfaces. King and Todorov [6] use the results of [11] to estimate $N_{X,D}(B)$ for a certain class of Kummer surfaces admitting a double cover of a del Pezzo surface.

In this paper, the particular $K3$ surfaces we will study are hyperelliptic $K3$ surfaces, which admit a generically two-to-one map to $\mathbf{P}^1 \times \mathbf{P}^1$, branched over a singular (4, 4) curve. We define a certain cone $\mathscr{C}$ of ample divisors in the Néron-Severi lattice of X, and calculate the value of $N_{X,D}(B)$ with respect to an arbitrary divisor in $\mathscr{C}$. In the generic case, this cone is of full dimension in $NS_{\mathbf{R}}(X)$. More specifically, if we measure heights with respect to a divisor $D \in \mathscr{C}$, we will show that $N_{X,D}(B)$ is asymptotically equal to $N_{Y,D}(B)$, where Y is the union of all rational curves of minimal D-degree on X. We also calculate explicitly which curves lie in C.

Batyrev and Manin [4] have introduced a refinement of the counting function called the *arithmetic stratification*. Roughly speaking, a subset Y of X is said to be *accumulating* with respect to an ample divisor D if most of the rational points of X lie on Y, where heights are measured with respect to D. That is, if $\lim_{B\to\infty} N_{X,D}(B)/N_{Y,D}(B) = 1$. The arithmetic stratification of a variety X with respect to D is an ascending chain of Zariski closed subsets $Y_1 \subset Y_2 \subset Y_3 \subset \cdots$ with the property that $Y_i - Y_{i-1}$ is an accumulating subset of $X - Y_{i-1}$ with respect to D. Y_i is said to be the ith layer of the arithmetic stratification. Since layers in the arithmetic stratification are typically finite unions of rational curves, the value of $N_{X,D}(B)$ will immediately follow from Schanuel's theorem [8], which calculates the counting function for $\mathbf{P}^n$.

Given a divisor D in $\mathscr{C}$, Corollary 2.2 explicitly identifies the first layer of the arithmetic stratification of X with respect to D. The number of rational points lying on any given rational curve on X can be easily calculated from Schanuel's theorem; the hard part comes from Theorem 2.1, which estimates the number of rational points on the complement of the union of these curves. By comparing the counting functions for certain rational curves constructed on X with the counting function for the complement U of the union of these curves, the structure of the top layers of the arithmetic stratification is revealed.

We conclude by proving some more precise results in the case that X is a Kummer surface whose associated Abelian surface is isomorphic to a product of elliptic curves. In this case, the cone $\mathscr{C}$ is 18-dimensional, and is of full dimension exactly when the corresponding pair of elliptic curves is non-isogenous.

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2. GENERAL HYPERELLIPTIC $K3$ SURFACES

Algebraic $K3$ surfaces lie in countably infinitely many 19-dimensional families, one family for each positive integer. The nth such family consists of the $K3$ surfaces containing a smooth irreducible curve C with $C^2 = 2n$, and no smooth irreducible curves of positive self-intersection less than $2n$.

Fix a number field K. In this paper, we will be concerned with smooth, K-rational $K3$ surfaces which admit a generically two-to-one K-rational morphism to $\mathbf{P}^1_K \times \mathbf{P}^1_K$, branched over a singular curve of type $(4, 4)$. Such surfaces are all of type $n = 1$; that is, they contain a curve of self-intersection 2. This curve is explicitly calculated below.

Let X be a smooth $K3$ surface with a generically two-to-one map $f\colon X \to \mathbf{P}^1 \times \mathbf{P}^1$ branched over a singular $(4, 4)$-curve. The singularities of the branch curve will correspond to the one-dimensional fibres of f; assume that K is large enough so that all of these singular points are K-rational. Assume further that the one-dimensional fibres of f are simple and irreducible, and denote them by $E_1, ..., E_m$. By composition of f with the projections, we have two morphisms $\pi_i\colon X \to \mathbf{P}^1$. Hence, we may define divisor classes F_1 and F_2 on X, corresponding to the fibres of the maps π_1 and π_2, respectively.

For each $i = 1, ..., m$, define a divisor $A_i = F_1 + F_2 - E_i$. Together with F_1 and F_2, these define an $(m+2)$-dimensional subspace of the realification $NS_{\mathbf{R}}(X)$ of the Néron-Severi group of X. For a generic surface X of this type, $NS_{\mathbf{R}}(X)$ is of dimension $m+2$, so these divisors form a basis. Note also that $|A_m|$ contains curves on X with self-intersection 2.

Choose a divisor D, and assume it can be written in the form $D = \sum a_i A_i + c_1 F_1 + c_2 F_2$ for some rational numbers a_i and c_i. (In the generic case, this will always be possible.) Computation shows that $F_i^2 = 0$, $E_i^2 = -2$, $F_i.E_j = 0$, $F_1.F_2 = 2$, and $E_i.\ E_j = 0$ if $i \neq j$. Also, define divisors $L_{ij} = F_i - E_j$, with $E_i.\ L_{jk} = 0$ if $i \neq k$, $E_j.L_{ij} = 2$, $F_i.L_{ij} = 0$, and $F_i.E_{jk} = 2$ if $i \neq j$. We are now ready to state the main theorem.

THEOREM 2.1. *Let* $D=\sum a_i A_i + c_1 F_1 + c_2 F_2$ *be an ample divisor on* X, *and write* $a=\sum a_i$. *Define* $U = X - \bigcup E_i - \bigcup L_{ij}$. *Then we have the following bound for the number of K-rational points of bounded D-height on* U,

$$N_{U,D}(B) = \#\{P \in U(K) \mid H_D(P) \leqslant B\} = O(B^{\alpha}\log B).$$

where $\alpha = \max\{4/(a+c_1+c_2), 2/(a+c_1), 2/(a+c_2)\}$. *If* $a \neq |c_1 - c_2|$, *then:*

$$N_{U,D}(B) = O(B^{\alpha}).$$

If furthermore we have $c_i \geqslant 0$, *then we may take* $\alpha = 6/(2a + 3\min\{c_i\})$ *and prove that* $N_{U,D}(B) = O(B^{\alpha}\log B)$.

COROLLARY 2.2. *Let X be a K3 surface as described above, and let* $D = \sum a_i A_i + c_1 F_1 + c_2 F_2$ *be an ample divisor. Define e to be the minimum D-degree of a curve disjoint from U, and assume that* $e\alpha < 2$. *Then the counting function for* $X(K)$ *is given by* $N_{X,D}(B) = cB^{2/e} + E(B)$, *where c is a constant depending only on K, X, and the choice of height function* H_D, *and* $E(B) = O(B^q)$ *is an error term with an easily calculable* $q < 2/e$.

More precisely, the main term measures the number of rational points lying on the union of rational curves of minimal D-degree, which all must be components of some E_i or L_{ij}, and $E(B)$ bounds the number of rational points not lying on such curves. $E(B)$ is the maximum of Schanuel's error term for rational points on $\mathbf{P}^1$, the estimate from Theorem 2.1 for $N_{U,D}(B)$, and the number of rational points lying on curves outside U, but of non-minimal D-degree.

Put another way, the first layer of the arithmetic stratification (as defined by Batyrev and Manin [4]) of X with respect to an ample divisor D is the union of all smooth rational curves of minimal D-degree, provided that D satisfies $e\alpha < 1$. Moreover, these curves are all of the form E_i or L_{ij}.

Note also that Theorem 2.1 is still true in the case that S is smooth. Moreover, in that case, there are no exceptional curves E_i, and therefore $U = X$, so Theorem 2.1 directly gives an estimate for $N_{X,D}(B)$. However, this estimate is quite poor, and the fact that $U = X$ makes the definition of e in Corollary 2.2 nonsensical in that case.

Proof of Corollary 2.2. Consider the union of curves of minimal D-degree in $X - U$. By Schanuel's theorem [8], the counting function for those curves is $cB^{2/e} + O(B^{2/e - 2/Ne})$, where $N = [K : \mathbf{Q}]$. (In the special case $K = \mathbf{Q}$, the error term must be modified to $O(B^{1/e}\log B)$.) By Theorem 2.1, it suffices to show that $2/e > \alpha$, which is to say that $e\alpha < 2$. This is true by assumption.

Finally, we must establish that there are no rational curves of minimal D-degree not contained in $X - U$. But by the estimate of $N_{U,D}(B)$, there are simply not enough rational points in U for it to contain an open subset of a rational curve of minimal D-degree. ∎

Proof of Theorem 2.1. The key idea is to estimate the height function H_D in terms of the the height functions H_{F_i}, which are easily computable. Write $D=\sum a_i A_i + c_1 F_1 + c_2 F_2$; since D is ample, we must have $a_i > 0$ and $a + c_i - a_j > 0$ for each i and j, where $a = \sum a_i$. Write $H_i = H_{F_i}$. The basic height estimate follows from the following general lemma:

LEMMA 2.3. *Let V be a normal algebraic variety defined over a number field K. Let Γ_1, Γ_2, and Δ be divisors on V such that $\Gamma_i - \Delta$ is effective for $i=1, 2$. Let W be the union of the fixed loci of $|\Gamma_1 - \Delta|$ and $|\Gamma_2 - \Delta|$, and let U be the complement $V - W$ of W. Write $\Gamma = \Gamma_1 + \Gamma_2 - \Delta$. Then for any point $P \in U(K)$, we have*

$$H_\Gamma(P) \gg \max\{H_{\Gamma_1}(P), H_{\Gamma_2}(P)\} \tag{1}$$

for any choice of height functions H_Γ, H_{Γ_1}, and H_{Γ_2}.

Proof of Lemma. The lemma follows immediately from the effectivity of $\Gamma_i - \Delta$. ∎

Lemma 2.3, together with elementary properties of height functions (see for example [9]) gives the estimate

$$H_D(P) \gg H_1(P)^{c_1} H_2(P)^{c_2} \max\{H_1(P), H_2(P)\}^a \tag{2}$$

for any point P in the set $U(K)$, where $U = X - (\bigcup E_i \cup L_{ij})$. We wish to compute $N_{U,D}(B)$. Since P is determined up to a finite choice by fixing $\pi_1(P)$ and $\pi_2(P)$, it suffices to count the number of pairs $(P_1, P_2) \in \mathbf{P}^1 \times \mathbf{P}^1$ corresponding to points of D-height at most B in U.

Set $x = H_1(P)^2$ and $y = H_2(P)^2$. If $H_D(P) \leqslant B$, then 2.2 implies that

$$\max\{x^{a+c_1} y^{c_2}, x^{c_1} y^{a+c_2}\} \leqslant B^2.$$

Consider the function $H_i^2\colon U(K) \to \mathbf{R}$. By Schanuel's Theorem, its image G_i has the property that there is some constant C_i such that for any $B > 0$, the set $\{x \in G_i \mid x < B\}$ has cardinality at most $C_i B$. By increasing C_i slightly, and by decreasing the elements of G_i slightly, we may retain all these properties while also demanding that G_i be a subset of $(1/C_i)\,\mathbf{Z}$. Hence, for the purposes of our calculation, we may assume that x and y are each elements of $(1/C_i)\,\mathbf{Z}$.

Thus, $N_{U,D}(B)$ is bounded above by a constant factor times the number of integer lattice points contained in the plane region R defined by the inequalities:

$$R = \{(x, y) \in \mathbf{R}^2 \mid x \geqslant 1,\ y \geqslant 1,\ x^{a+c_1}y^{c_2} \leqslant B^2, \text{ and } x^{c_1}y^{a+c_2} \leqslant B^2\}.$$

This is asymptotically equal to the area of this region (again, up to an irrelevant constant factor), plus two extra terms counting lattice points lying on the boundary lines $x = 1$ and $y = 1$. This may be computed as follows.

Case I. $c_2 > 0$. Define $\delta = 2(a + c_1 + c_2)^{-1}$. The two curves $x^{a+c_1}y^{c_2} = B^2$ and $x^{c_1}y^{a+c_2} = B^2$ intersect at the point (B^δ, B^δ). Thus, the number of lattice points inside R may be computed by

$$\begin{aligned}
&\int_1^{B^\delta} (B^2 x^{-c_1})^{1/(a+c_2)}\, dx \\
&\qquad + \int_{B^\delta}^{B^{2/(a+c_1)}} (B^2 x^{-a-c_1})^{1/c_2}\, dx + B^{2/(a+c_1)} + B^{2/(a+c_2)} \\
&= -\left(\frac{a+c_2}{a+c_2-c_1}\right)(B^{2/(a+c_2)} - B^{2\delta}) \\
&\quad -\left(\frac{c_2}{a+c_1-c_2}\right)(B^{2/(a+c_1)} - B^{2\delta}) + B^{2/(a+c_1)} + B^{2/(a+c_2)} \\
&= O(B^\alpha)
\end{aligned}$$

unless $a + c_1 = c_2$ or $a + c_2 = c_1$, in which case obvious modifications to the computation will give the desired result. Note that $a + c_i > 0$ for all i by the positivity of $\deg_D L_{ij}$.

Case II. $c_2 = 0$. Retain the notation of the previous case. The number of lattice points lying inside R is now bounded by:

$$\begin{aligned}
&\int_1^{B^{2/(a+c_1)}} (B^2 x^{-c_1})^{1/a}\, dx + B^{2/(a+c_1)} + B^{2/a} \\
&\qquad = -\left(\frac{a}{a-c_1}\right)(B^{2/a} - B^{2/(a+c_1)}) + B^{2/(a+c_1)} + B^{2/a} \\
&\qquad = O(B^\alpha).
\end{aligned}$$

Note that the ampleness of D ensures that $a \neq c_1$.

Case III. $c_2 < 0$. Again retaining the notation of the previous cases, we may compute the number of lattice points lying inside R by

$$\int_1^{B^\delta} (B^2 x^{-c_1})^{1/(a+c_2)} - \max\{(B^2 x^{-a-c_1})^{1/c_2}, 1\}\, dx + B^\delta + B^{2/(a+c_2)}$$

$$\leqslant \int_1^{B^\delta} (B^2 x^{-c_1})^{1/(a+c_2)}\, dx + B^\delta + B^{2/(a+c_2)}$$

$$= -\left(\frac{a+c_2}{a+c_2-c_1}\right)(B^{2/(a+c_2)} - B^{2\delta}) + B^{2/(a+c_1)} + B^{2/(a+c_2)}$$

$$= O(B^\alpha)$$

again with obvious modifications in the case that $c_1 = c_2 + a$.

Now assume that $c_i \geqslant 0$, and write $F = \frac{2}{3}(F_1 + F_2)$. Then we may write $D = \sum a_i A_i + \frac{3}{2} \min\{c_i\}\, F + E$ for some effective divisor E. The following lemma will give us the estimate we need.

Lemma 2.4. *Let V be a normal algebraic variety defined over a number field K, and let $D_1, ..., D_n$ be Weil divisors on V. Let U be an arbitrary subset of V, and assume that the counting functions $N_{U, D_i}(B)$ are well defined (i.e., finite) for K-rational points. Assume without loss of generality that $N_{U, D_i}(B) \ll N_{U, D_1}(B)$ for all i, and write $D = \sum D_i$. If $N_{U, D_1}(B) = O(f(B))$ for some increasing real-valued function $f(B)$, then*

$$N_{U, D}(B) = O(f(B^{1/n}))$$

Proof of Lemma. Take any K-rational point $P \in U(K)$, and assume $H_D(P) \leqslant B$. We have $H_D(P) = \prod H_{D_i}(P) \leqslant B$, so by the Pigeonhole Principle there must be some i for which $H_{D_i}(P) \leqslant B^{1/n}$. For each i, the number of K-rational points in $U(K)$ with $H_{D_i}(P) \leqslant B^{1/n}$ is $N_{U, D_i}(B) \ll N_{U, D_1}(B) = O(f(B^{1/n}))$. Therefore, there are at most $nO(f(B^{1/n})) = O(f(B^{1/n}))$ points in $U(K)$ of D-height at most B, as desired. ▮

For each i, the divisor A_i is the pullback of $\mathcal{O}(1)$ via a certain morphism from X to $\mathbf{P}^2$ (see [7, (5.1)]). Therefore, we have $N_{U, A_i}(B) = O(B^3)$, by Schanuel's Theorem for $\mathbf{P}^2$. The divisor $F_1 + F_2$ is the pullback of the class $(1, 1)$ via a morphism from X to $\mathbf{P}^1 \times \mathbf{P}^1$, so we have $N_{U, F_1+F_2}(B) = O(B^2 \log B)$, and hence $N_{U, F}(B) = O(B^3 \log B)$.

We have $D = \sum_{i=1}^m a_i A_i + (\frac{3}{2} \min\{c_i\})\, F + E$ for some effective divisor E. Write $D' = D - E$; it follows that $N_{U, D}(B) \ll N_{U, D'}(B)$. The lemma, using $D_i = A_j$ or F, gives that $N_{U, D'}(B) = O(B^{3/(a + (3/2) \min\{c_i\})} \log B)$. The proof of Theorem 2.1 is complete. ▮

3. THE GEOMETRY OF KUMMER SURFACES

In this section, we specialise to the case in which X is a Kummer surface whose associated Abelian surface is a product of elliptic curves, and use slightly more refined techniques to estimate $N_{U,D}(B)$. In particular, we will be able to say much more about the set of rational curves on X, and hence about the relations between height functions on X.

Let C_1 and C_2 be elliptic curves defined over some number field K, such that all points of order 1 and 2 on the curve are also defined over K. Let A be the product $C_1 \times C_2$. Let $i: A \to A$ be the involution $i(x, y) = (-x, -y)$, and let V be the quotient of A by i. Then there is a 2-to-1 map $q: A \to V$ which is ramified at 16 points; namely, the points (a, b), where a and b are points of order 1 or 2. It turns out that these 16 points are rational double points of V, which is smooth away from them.

By blowing up these 16 points, one constructs a smooth surface $p: X \to V$, which is a $K3$ surface defined over K [5]. This construction can be done with an arbitrary Abelian surface A, and the resulting $K3$ surface is called the Kummer surface associated to the Abelian surface A.

Let $\tilde{\pi}_i: A \to C_i$ be the projection maps, and let F'_i be the algebraic equivalence class of fibres of $\tilde{\pi}_i$. This induces a pair of algebraic equivalence classes $F_i = p^* q_* F'_i$ on X. Since algebraic and linear equivalence are identical on a $K3$ surface [5], these are divisor classes on X. Thus, the maps $\tilde{\pi}_i$ descend to maps $\pi_i: X \to \mathbf{P}^1$.

Denote the 16 singular points of V by (a_i, b_j), $1 \leqslant i, j \leqslant 4$, where a_i and b_j denote the 2-division points on C_1 and C_2, respectively, and let E_{ij} denote the corresponding exceptional divisors on S. For each i, $1 \leqslant i \leqslant 4$, the divisor $B_i = p^* q_* \pi_1^* a_i$ is the union of the four curves E_{ij}, $1 \leqslant j \leqslant 4$, and the strict transform of $q_*(\{a_i\} \times E_2)$. By the theory of singular fibres of elliptic surfaces [5], it follows that this strict transform is a double curve, which is smooth and rational in its induced reduced structure. Thus, we may write $F_1 \equiv B_i = \sum_{j=1}^{4} E_{ij} + 2L_i$, where L_i is a smooth rational curve. Similarly, we may write $F_2 \equiv \sum_{i=1}^{4} E_{ij} + 2M_j$, where M_j is a smooth rational curve.

Using the adjunction formula and elementary properties of intersection theory, it is not hard to verify the following intersection numbers:

$$L_i^2 = M_i^2 = E_{ij}^2 = -2 \qquad\qquad L_i M_j = 0$$

$$F_1 L_i = F_2 M_i = 0 \qquad\qquad F_1 M_i = F_2 L_i = 1$$

$$L_i L_j = M_i M_j = 0 \qquad (\text{if} \quad i \neq j).$$

Let S and T be non-empty subsets of $N_4 = \{1, 2, 3, 4\}$. Define divisors

$$A_{S,T} = (\mathrm{card}(S))\, F_1 + (\mathrm{card}(T))\, F_2 - \sum_{i \in S,\, j \in T} E_{ij}.$$

These divisors, together with F_1 and F_2, span a rank 18 sublattice of Pic(X), and therefore an 18-dimensional subspace of the vector space $NS_{\mathbf{R}}(X) = \mathrm{Pic}(X) \otimes \mathbf{R}$. For a generic choice of C_1 and C_2, $NS_{\mathbf{R}}(X)$ has dimension 18 [5], so the divisors $A_{S,T}$, and F_i span all of $NS_{\mathbf{R}}(X)$ for such X.

Moreover, for any ample divisor D, write $D = d_1 F_1 + d_2 F_2 + \sum e_{ij} E_{ij}$. Since $D.E_{ij} > 0$ and $E_{ij}^2 = -2$, we must have $e_{ij} < 0$. Therefore, it follows that any ample divisor D on X can be written as

$$D = \sum_{S,T} a_{S,T} A_{S,T} + c_1 F_1 + c_2 F_2, \tag{3}$$

where $a_{S,T} \geqslant 0$. Note that this representation is not unique, unlike in the previous, more general case, since the divisors $A_{S,T}$, F_1, and F_2 are not linearly independent. Different representations of the same divisor D will lead to different estimates of $N_{U,D}(B)$ from Theorem 4.1. In such cases, since Theorem 0.0 gives an upper bound, the lowest estimate can be inferred. If D is written in the form of (3), then we may assume without loss of generality that $a_{N_4, N_4} = \min\{e_{ij}\}$. This will be assumed to be true in all that follows.

4. THE MAIN THEOREM FOR KUMMER SURFACES

We are now ready to state the main theorem for Kummer surfaces. All counting functions are defined with respect to the height associated to the divisor D.

THEOREM 4.1. *Let D be an ample divisor on X written as in* (3). *Assume that $a_{S,T}$ are non-negative rational numbers, and c_i are rational numbers. Define*:

$$\gamma_1 = \sum_{S,T} \mathrm{card}(S)\, a_{S,T}, \; \gamma_2 = \sum_{S,T} \mathrm{card}(T)\, a_{S,T}$$

$$\alpha = \max\left\{\frac{2\gamma_1 + 2\gamma_2}{\gamma_1\gamma_2 + \gamma_2 c_1 + \gamma_1 c_2}, \frac{2}{\gamma_1 + c_1}, \frac{2}{\gamma_2 + c_2}\right\}.$$

Define $U = X - \bigcup R$, where R ranges over all smooth rational curves on X of the form E_{ij}, L_i, or M_i. Assume that $\gamma_1\gamma_2 + \gamma_2 c_1 + \gamma_1 c_2 > 0$. Then:

(i) *If $\alpha = (2\gamma_1 + 2\gamma_2)/(\gamma_1\gamma_2 + \gamma_2 c_1 + \gamma_1 c_2)$ and either $c_1 = \gamma_2 + c_2$ or $c_2 = \gamma_1 + c_1$, then $N_{U,D}(B) = O(B^\alpha \log B)$.*

(ii) *If $\alpha = 2/(\gamma_1 + c_1)$ and $c_2 = \gamma_1 + c_1$, then $N_{U,D}(B) = O(B^\alpha \log B)$.*

(iii) *If* $\alpha = 2/(\gamma_2 + c_2)$ *and* $c_1 = \gamma_2 + c_2$, *then* $N_{U,D}(B) = O(B^{\alpha} \log B)$.

(iv) *If none of the previous three cases occur, then* $N_{U,D}(B) = O(B^{\alpha})$.

COROLLARY 4.2. *Let X be a $K3$ surface as described above, and let D be an ample divisor, written as in* (3). *Write* $A = \min\{D.E_{ij}, D.L_i, D.M_i\}$, *and assume that the following inequality holds*:

$$A(\gamma_1 + \gamma_2) < \gamma_1\gamma_2 + \gamma_2 c_1 + \gamma_1 c_2. \tag{4}$$

Then the counting function for $X(K)$ is given by $N_{X,D}(B) = cB^{8/A} + E(B)$, *where c is a constant depending only on K, X, and the choice of height function H_D, and $E(B) = O(B^q)$ is an error term with an easily calculable $q < 8/A$. Moreover, the main term measures the number of rational points lying on the union of rational curves of minimal D-degree, which must all be of the form E_{ij}, L_i, or M_i, and $E(B)$ bounds the number of rational points not lying on such curves.*

More precisely, we prove that the first term in the above expression represents the number of rational points lying on smooth rational curves of minimal D-degree on X. The error term represents the combination of Schanuel's error term, the estimate from Theorem 4.1 for $N_{U,D}(B)$, and the number of rational points lying on the curves E_{ij}, L_i, and M_i of non-minimal D-degree.

Put another way, the first layer of the arithmetic stratification (as defined by Batyrev and Manin [4]) of X with respect to an ample divisor D is the union of all smooth rational curves of minimal D-degree, provided that D can be expressed in a form for which inequality (4) is satisfied. Moreover, these curves are all of the form E_{ij}, L_i, or M_i.

Proof of Corollary 4.2. The following curves have the following degrees,

$$\deg_D(E_{mn}) = 2 \sum_{S \ni m,\, T \ni n} a_{S,T}$$

$$\deg_D(L_n) = c_2 + \sum_{S \not\ni n} \sum_{T} \operatorname{card}(T)\, a_{S,T}$$

$$\deg_D(M_n) = c_1 + \sum_{S} \sum_{T \not\ni n} \operatorname{card}(S)\, a_{S,T}$$

and by Schanuel's Theorem for a smooth rational curve C of degree d in projective space, we have $N_{C,\mathcal{O}(d)}(B) = cB^{2/d} + O(B^{2/d - 1/Nd})$, where $N = [K:\mathbf{Q}] > 1$ and c is a complicated constant, calculated explicitly by Schanuel. (In the special case $K = \mathbf{Q}$, the error term must be replaced by $O(B^{1/d} \log B)$.)

It suffices to show that $N_{U,D}(B) < N_{E_{ij},D}(B)$ for sufficiently high B, where E_{ij} is the exceptional curve of lowest degree. By Schanuel's Theorem, we have $N_{E_{ij},D}(B) = cB^{2/A} + O(B^{2/A-2/NA})$, where c is a constant depending only on K, X, and D, and $N = [K:\mathbf{Q}]$. (If $K = \mathbf{Q}$, the error term must be appropriately modified.) By Theorem 4.1, then, it suffices to show that $\frac{2}{A} > \alpha$.

If $\alpha = (2\gamma_1 + 2\gamma_2)/(\gamma_1\gamma_2 + \gamma_2 c_1 + \gamma_1 c_2)$, then the desired inequality follows immediately from Eq. (4).

Assume that $\alpha = 2/(c_1 + \gamma_1)$. Since $\deg_D M_n > 0$ for $n = 1, 2, 3, 4$, it follows that

$$\begin{aligned}
0 &< c_1 + \sum_S \sum_{T \not\ni n} \mathrm{card}(S)\, a_{S,T} \\
&< c_1 + A + \sum_{S \neq N_4 \neq T} \mathrm{card}(S)\, a_{S,T} \\
&< c_1 - A + \gamma_1
\end{aligned}$$

which implies immediately that $\frac{2}{A} > \alpha$, as desired. Similarly, if $\alpha = 2/(c_2 + \gamma_2)$, then $\frac{2}{A} > \alpha$ follows from the positivity of $\deg L_n$.

Finally, we must establish that there are no rational curves of minimal D-degree other than those of the form E_{ij}, L_i, or M_i. Assume there exists such a curve C of minimal D-degree. Then $C \cap U$ is a dense open subset of U, so $N_{C,D}(B) \ll N_{U,D}(B)$. But for a curve of minimal D-degree, we have just established that $N_{C,D}(B) \not\ll N_{U,D}(B)$. The corollary follows. ∎

Proof of Theorem 4.1. The key idea is to estimate an arbitrary height function H_L in terms of the height functions H_{F_i}, which are easily computed. The first step is to note that the divisors $A_{S,T}$, F_1, and F_2 span a rank 18 sublattice of $\mathrm{Pic}(X)$. This can be proven by explicit calculation. For non-isogenous elliptic curves C_1 and C_2 (as is generally the case), $\mathrm{Pic}(X)$ is a free $\mathbf{Z}$-module of rank 18. Therefore, height calculations with respect to a general ample sheaf L can be reduced to calculations with respect to the divisors $A_{S,T}$, F_1, and F_2. Write $H_1 = H_{F_1}$ and $H_2 = H_{F_2}$.

From Lemma 2.3, we get the following inequalities:

$$H_1(P)^{c_1} H_2(P)^{c_2} \prod_{S,T} \max\{H_1(P)^{\mathrm{card}(S)}, H_2(P)^{\mathrm{card}(T)}\}^{a_{S,T}} \ll H_D(P)$$

$$H_D(P) \ll H_1(P)^{\gamma_1 + c_1} H_2(P)^{\gamma_2 + c_2}.$$

These estimates are enough to prove Theorem 4.1. The first inequality above implies for any point $P \in U(K)$ (since $a_{S,T} \geqslant 0$):

$$H_D(P) \gg \max\{H_1(P)^{\gamma_1 + c_1} H_2(P)^{c_2}, H_1(P)^{c_1} H_2(P)^{\gamma_2 + c_2}\}.$$

The number of points of points of height at most B on U is therefore bounded by the number of integer lattice points contained in a certain plane region times a constant factor (which is immaterial to the result of the theorem). By Schanuel's Theorem, there are $\ll B^2$ points of height at most B on $\mathbf{P}^1$ with respect to the height attached to $\mathcal{O}(1)$, so that if $H_1(P) \leqslant B$, then there are $\ll B^2$ choices for $\pi_1(P)$, and similarly for $H_2(P)$. Therefore, set $x = H_1(P)^2$ and $y = H_2(P)^2$. If $H_D(P) \leqslant B$, then we get

$$\max\{x^{\gamma_1 + c_1} y^{c_2}, x^{c_1} y^{\gamma_2 + c_2}\} \leqslant B^2.$$

Thus, $N_{U, D}(B)$ is bounded above by a constant factor times the number of lattice points contained in the plane region R defined by the inequalities:

$$R = \{(x, y) \in \mathbf{R}^2 \mid x \geqslant 1,\ y \geqslant 1,\ x^{\gamma_1 + c_1} y^{c_2} \leqslant B^2, \text{ and } x^{c_1} y^{\gamma_2 + c_2} \leqslant B^2\}$$

This is asymptotically equal to the area of this region (again, up to an irrelevant constant factor), plus two extra terms counting lattice points lying on the boundary lines $x = 1$ and $y = 1$. This may be computed as follows.

Case I. $c_2 > 0$. Define $\delta = 2(\gamma_1\gamma_2 + \gamma_2 c_1 + \gamma_1 c_2)^{-1}$. The two curves $x^{\gamma_1 + c_1} y^{c_2} = B^2$ and $x^{c_1} y^{\gamma_2 + c_2} = B^2$ intersect at the point $(B^{\delta\gamma_2}, B^{\delta\gamma_1})$. Thus, the number of lattice points inside R may be computed by

$$\begin{aligned}
&\int_1^{B^{\delta\gamma_2}} (B^2 x^{-c_1})^{1/(\gamma_2 + c_2)}\, dx \\
&\qquad + \int_{B^{\delta\gamma_2}}^{B^{2/(\gamma_1 + c_1)}} (B^2 x^{-\gamma_1 - c_1})^{1/c_2}\, dx + B^{2/(\gamma_1 + c_1)} + B^{2/(\gamma_2 + c_2)} \\
&= -\left(\frac{\gamma_2 + c_2}{\gamma_2 + c_2 - c_1}\right)(B^{2/(\gamma_2 + c_2)} - B^{\delta(\gamma_1 + \gamma_2)}) \\
&\qquad - \left(\frac{c_2}{\gamma_1 + c_1 - c_2}\right)(B^{2/(\gamma_1 + c_1)} - B^{\delta(\gamma_1 + \gamma_2)}) + B^{2/(\gamma_1 + c_1)} + B^{2/(\gamma_2 + c_2)} \\
&= O(B^{\alpha})
\end{aligned}$$

unless $\gamma_1 + c_1 = c_2$ or $\gamma_2 + c_2 = c_1$, in which case obvious modifications to the computation will give the desired result. (Note that $\gamma_i + c_i > 0$ for all i by the positivity of $\deg_D L_n$ and $\deg_D M_n$.)

Case II. $c_2 = 0$. Retain the notation of the previous case. The number of lattice points lying inside R is now bounded by:

$$\int_1^{B^{2/(\gamma_1+c_1)}} (B^2 x^{-c_1})^{1/\gamma_2}\,dx + B^{2/(\gamma_1+c_1)} + B^{2/\gamma_2}$$

$$= -\left(\frac{\gamma_2}{\gamma_2 - c_1}\right)(B^{2/\gamma_2} - B^{2/(\gamma_2+c_1)}) + B^{2/(\gamma_1+c_1)} + B^{2/\gamma_2}$$

$$= O(B^\alpha),$$

again with the obvious modifications in the case that $c_1 = \gamma_2$.

Case III. $c_2 < 0$. Again retaining the notation of the previous cases, we may compute the number of lattice points lying inside R by

$$\int_1^{B^{\delta\gamma_2}} (B^2 x^{-c_1})^{1/(\gamma_2+c_2)} - \max\{(B^2 x^{-\gamma_1-c_1})^{1/c_2}, 1\}\,dx + B^{\gamma_2\delta} + B^{2/(\gamma_2+c_2)}$$

$$\leqslant \int_1^{B^{\delta\gamma_2}} (B^2 x^{-c_1})^{1/(\gamma_2+c_2)}\,dx + B^{\delta\gamma_2} + B^{2/(\gamma_2+c_2)}$$

$$= -\left(\frac{\gamma_2+c_2}{\gamma_2+c_2-c_1}\right)(B^{2/(\gamma_2+c_2)} - B^{\delta(\gamma_1+\gamma_2)}) + B^{2/(\gamma_1+c_1)} + B^{2/(\gamma_2+c_2)}$$

$$= O(B^\alpha),$$

again with obvious modifications in the case that $c_1 = c_2 + \gamma_2$.

Thus, the proof of Theorem 4.1 is complete. ∎

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